Aequat. Math. 98 (2024), 661–686

0001-9054/24/030661-26
published online May 8, 2024
https://doi.org/10.1007/s00010-024-01076-y

Aequationes Mathematicae

Gromov hyperbolicity of Johnson and Kneser graphs

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Abstract. The concept of Gromov hyperbolicity is a geometric concept that leads to a rich general theory. Johnson and Kneser graphs are interesting combinatorial graphs defined from systems of sets. In this work we compute the precise value of the hyperbolicity constant of every Johnson graph. Also, we obtain good bounds on the hyperbolicity constant of every Kneser graph, and in many cases, we even compute its precise value.

Mathematics Subject Classification. Primary 05C75; Secondary 05C12.

Keywords. Johnson graphs, Kneser graphs, Gromov hyperbolicity, Geodesics.

1. Introduction

Johnson and Kneser graphs are classes of graphs defined from systems of sets. In both classes of graphs the adjacency of their vertices depends on the intersection of their respective sets, however, these cases can be seen as opposites, because, while in Johnson graphs the condition for two vertices to be adjacent is that their respective sets differ in only one element, in Kneser graphs it is that they differ completely.

The vertices of the Johnson graph $J(n, k)$ are the k-element subsets of an n-element set; two vertices are adjacent when the intersection of the two vertices (subsets) contains $(k-1)$-elements. Note that if $k = 1$ then $J(n, 1)$ is the complete graph K_n. The Johnson graph for $n = 5$ and $k = 2$ is the complement of the Petersen graph, i.e., $J(5, 2)$ is the line graph of K_5. In general, $J(n, 2)$ is the line graph of the complete graph K_n.

The *Kneser graph* $K(n, k)$ is the graph whose vertices correspond to the k-element subsets of an n-element set A^n; two vertices are adjacent when the intersection of their corresponding sets is empty. The Kneser graph $K(n, 1)$ is the complete graph with n vertices K_n. The Petersen graph is the Kneser

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graph for $k = 2$, $n = 5$, $K(5, 2)$. Also, the *odd graph* O_n, a generalization of the Petersen graph, is the Kneser graph $O_n = K(2n - 1, n - 1)$.

Hyperbolicity in Gromov's sense leads to a very rich general theory. This term comes into use after the seminal work of Gromov [12]. This concept allows to understand the properties of spaces such as classical hyperbolic spaces, Riemannian manifolds of negative sectional curvature and discrete spaces such as trees and Cayley graphs of many finitely generated groups. Hyperbolicity is a very useful tool that allows us to understand the relationships between graphs and varieties. It is known that graphs can model manifolds and many metric spaces (this is an interesting fact, as it allows us to move from working with a continuous structure that could be complicated to dealing with a discrete structure, see, for example, [11,12]). In [19,22,27] the equivalence between the hyperbolicity of many surfaces and the hyperbolicity of simpler graphs is proved.

Today, the mathematical properties and applications of Gromov hyperbolicity are topics of growing interest in graph theory (see [1–7,9,10,13–16,19–21,27,29] and the references therein).

Gromov hyperbolicity has also been applied to computer science in areas such as automatic groups (see, for example, [18]), networks and algorithms (see [17] and its references), random graphs (see, for example, [23–25]), etc.

There are several definitions of Gromov hyperbolicity which are equivalent. We will work with the definition given by Rips' condition (see the definition in Sect. 2) for its geometrical meaning.

In Sect. 3 we compute the precise value of $\delta(J(n, k))$ for every n and k.

In Sect. 4 we obtain good bounds of $\delta(K(n, k))$ in relation of n and k, and in many cases, we even compute its precise value. In order to do that we obtain some results about Kneser graphs which are interesting by themselves, as Propositions 4.7 and 4.15.

2. Background and previous results

We collect in this section some previous definitions and results which will be useful later in the paper.

If X is a metric space, we say that a curve $\gamma : [a, b] \to X$ is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where $\gamma|_{[t,s]}$ denotes the restriction of the curve γ to the interval $[t, s]$, and L denotes the length in X (then γ is equipped with an arc-length parametrization). The metric space X is said to be *geodesic* if for every pair of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by uv.

Throughout this paper, $G = (V, E) = (V(G), E(G))$ denotes an undirected (finite or infinite) simple (without loops and multiple edges) graph (not necessarily connected) such that $V \neq \emptyset$ and every edge has length 1. In order to consider a connected graph G as a geodesic metric space, identify (by an isometry) any edge $uv \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge uv (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. We denote by d_G or d this distance. If x, y are in different connected components of G, we define $d_G(x, y) = \infty$. These properties guarantee that any connected component of any graph is a geodesic metric space.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the *vertices of the triangle* T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ-*thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, i.e., $\delta(T) := \inf\{\delta \geq 0 \mid T \text{ is} \delta - \text{thin}\}$.

Definition 2.1. A metric space X is *Gromov* δ-*hyperbolic* or just δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e.,

$$\delta(X) := \sup\{\delta(T) \mid T \text{ is a geodesic triangle in } X\}.$$

We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If X has connected components $\{X_i\}_{i \in I}$, then we define $\delta(X) := \sup_{i \in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

If we have a triangle with two identical vertices, we call it a "*bigon*". Obviously, every bigon in a δ-hyperbolic space is δ-thin.

It is clear by the definition that to compute the precise value of $\delta(X)$ is a difficult task. Hence, it is very useful to obtain bounds of the hyperbolicity constant of X as those that appear in Lemmas 2.2 and 2.5.

In the classical references on this subject (see, e.g., [11]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ).

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [8]).

For any connected graph G, we define, as usual,

$$\operatorname{diam} V(G) := \sup \{ d_G(v,w) \mid v,w \in V(G) \},$$
$$\operatorname{diam} G := \sup \{ d_G(x,y) \mid x,y \in G \},$$

i.e., $\operatorname{diam} V(G)$ is the diameter of the set of vertices of G, and $\operatorname{diam} G$ is the diameter of the whole graph G (recall that in order to have a geodesic metric space, G must contain both the vertices and the points in the interior of any edge of G).

Lemma 2.2. *In any graph G the following inequality holds*

$$\delta(G) \le \frac{1}{2} \operatorname{diam} G \le \frac{1}{2} (\operatorname{diam} V(G) + 1).$$

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Given a graph G, we denote by $VM(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $VM(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following result, which appears in [1, Theorems 2.5, 2.6 and 2.7], will be used throughout the paper.

Theorem 2.3. *For every graph G we have $\delta_1(G) = \delta(G)$. Furthermore, if G is hyperbolic, then $\delta(G)$ is an integer multiple of $1/4$ and there exists $T \in \mathbb{T}_1$ with $\delta(T) = \delta(G)$.*

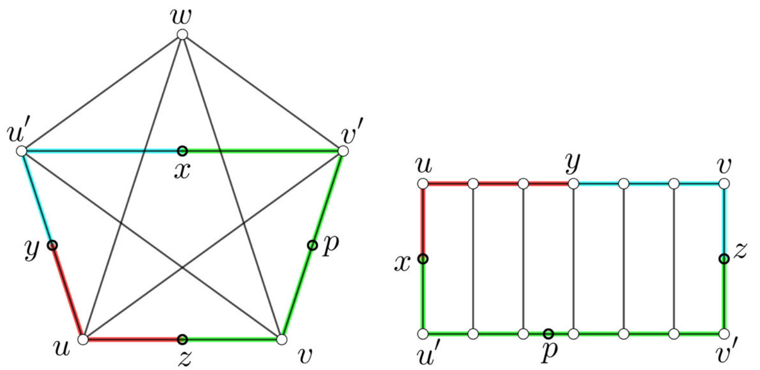
Hence, by Theorem 2.3, in order to compute the hyperbolicity constant of a graph, we can consider just geodesic triangles such that the vertices of the triangle are vertices of the graph or midpoints of edges of the graph.

Lemma 2.4. *Let G and H be isomorphic graphs. Then, $\delta(G) = \delta(H)$.*

A subgraph H of G is said *isometric* if $d_H(x,y) = d_G(x,y)$ for every $x, y \in H$. Note that this condition is equivalent to $d_H(u,v) = d_G(u,v)$ for every vertices $u, v \in V(H)$.

The following results appear in [3, Lemma 9] and [21, Theorem 11].

Lemma 2.5. *If H is an isometric subgraph of G, then $\delta(H) \le \delta(G)$.*



(a) $G = K_5$. Note that $\operatorname{diam} V(G) = 1$ and $\operatorname{diam} G = 2$. $T = \{x, y, z\}$ where x, y and z are midpoints of the edges $u'v'$, $u'u$ and uv respectively.

(b) $G = P_7 \times P_2$. Note that $\operatorname{diam} G = \operatorname{diam} V(G) = 7$. $T = \{x, y, z\}$ where y is a vertex of the graph and x and z are midpoints of the edges uu' and vv', respectively.

FIGURE 1. Metric graphs: in figure **a** the complete graph K_5, in figure **b** the Cartesian product $P5 \times P2$. In both cases $p \in [xz]$ and $\delta(G) = \delta(T) = d(p, [xy] \cup [yz])$. Note that in figure **a** $\delta(K_5) = \operatorname{diam}(K_5)/2$ while in figure **b** $\delta(P_7 \times P_2) = 3/2 < 7 = \operatorname{diam}(P_7 \times P_2)$. In general $\operatorname{diam}(P_n \times P_2) = n$ while $\delta(P_n \times P_2) = 3/2$ for each $n \geq 4$ so the hyperbolicity constant δ is not necessarily close to the diameter of the graph

Theorem 2.6. *The following graphs have these precise values of* δ.
- *If* P_n *is a* path graph, *then* $\delta(P_n) = 0$ *for all* $n \geq 1$.
- *If* C_n *is a* cycle graph, *then* $\delta(C_n) = \frac{1}{4} L(C_n) = \frac{n}{4}$ *for all* $n \geq 3$, *see Fig. 1a.*
- *If* K_n *is a* complete graph, *then* $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$ *and* $\delta(K_n) = 1$ *for all* $n \geq 4$.
- *If* P *is the* Petersen graph, *then* $\delta(P) = 3/2$.

Recall that $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the lower integer part and upper integer part, respectively, of the real number t, i.e., the greatest integer less than or equal to t and the least integer that is greater than or equal to t, respectively.

We denote by v_j the vertex of $J(n, k)$ associated to the set V_j. We use the same notation for the vertices of $K(n, k)$ and their respective sets.

3. Johnson graphs

The distance between two vertices of the Johnson graph is given by the half of the cardinal of the symmetric difference of their respective sets, so we have that $\operatorname{diam} V\big(J(n, k)\big) = \min\{k, n - k\}$.

The following results are well known.

Lemma 3.1. *If $J(n,k)$ is a Johnson graph, then*

$$\operatorname{diam} V\big(J(n,k)\big) = \min\{k, n-k\}.$$

Lemma 3.2. *If $J(n,k)$ is a Johnson graph, then $J(n,k)$ is isomorphic to $J(n,n-k)$.*

Remark 3.3. By Lemma 3.2, it suffices to consider the Johnson graph $J(n,k)$ with $2k \le n$.

We are going to compute the precise value of $\delta(J(n,k))$ for every n and k.

Theorem 3.4. *Let $J(n,k)$ be a Johnson graph. If $n \ge 2k+2$, then*

$$\delta(J(n,k)) = \frac{k+1}{2}.$$

Proof. Lemmas 2.2 and 3.1 give that

$$\delta\big(J(n,k)\big) \le \frac{\operatorname{diam} V\big(J(n,k)\big)+1}{2} = \frac{\min\{k,n-k\}+1}{2} = \frac{k+1}{2}.$$

So it suffices to prove that there exists a geodesic triangle T such that $\delta(T) = \frac{k+1}{2}$.

Case A. We suppose that $k \le n/3$.

We define the sets $V_j = \{j+1, j+2, \dots, j+k\}$, $0 \le j \le 2k$ and $V_{2k+i} = \{1, 2, \dots, i, 2k+i+1, 2k+i+2, \dots, 3k\}$, $1 \le i \le k-1$. By construction $v_0 v_{3k-1}, v_i v_{i+1} \in E\big(J(n,k)\big)$, $0 \le i < 3k-1$. Let x, y be the midpoints of $v_0 v_{3k-1}$ and $v_k v_{k+1}$ respectively and $z = v_{2k}$. We consider the geodesics $P = [xy]$, $P' = [yz]$ and $P'' = [xz]$ such that $P \cap V\big(J(n,k)\big) = \{v_0, v_1, \dots, v_k\}$, $P' \cap V\big(J(n,k)\big) = \{v_{k+1}, v_{k+2}, \dots, v_{2k}\}$ and $P'' \cap V\big(J(n,k)\big) = \{v_{2k}, v_{2k+1}, \dots, v_{3k-1}\}$. We consider the geodesic triangle $T = P \cup P' \cup P''$.

Let p be the midpoint of P, then

$$\delta\big(J(n,k)\big) \ge d(p, [xz] \cup [yz]) = d(p, \{x,y\}) = \frac{k+1}{2}.$$

Case B. We suppose that $n/3 < k < n/2$, therefore $\min\{k, n-k\} = k$. Since $k < n/2$ there exists $r \ge 1$ such that $2k + r = n$. We consider the sets $V_j = \{j+1, j+2, \dots, j+k\}$, $0 \le j \le k+r$ and $V_{k+r+i} = \{1, \dots, i, k+r+i+1, k+r+i+2, \dots, n\}$, $1 \le i \le k-1$. By construction $v_0 v_{n-1}, v_i v_{i+1} \in E\big(J(n,k)\big)$, $0 \le i < n-1$. Let x and y be the midpoints of $v_0 v_{n-1}$ and $v_k v_{k+1}$ respectively. Consider a path P^* joining x and y such that $P^* \cap V\big(J(n,k)\big) = \{v_{k+1}, v_{k+2}, \dots, v_{n-1}\}$ and let z be the midpoint of P^*.

We consider the geodesics $P = [xy]$, $P' = [yz]$ and $P'' = [xz]$ such that $P \cap V\big(J(n,k)\big) = \{v_0, v_1, \dots, v_k\}$, $P' \cap V\big(J(n,k)\big) = \{v_{k+1}, v_{k+2}, \dots, v_{k+\lfloor \frac{n-k}{2} \rfloor}\}$ and $P'' \cap V\big(J(n,k)\big) = \{v_{k+\lceil \frac{n-k}{2} \rceil}, v_{2k+1}, \dots, v_{n-1}\}$. We consider the geodesic triangle $T = P \cup P' \cup P''$.

Let p be the midpoint of P, then

$$\delta\big(J(n,k)\big) \geq d(p, [xz] \cup [yz]) = d(p, \{x, y\}) = \frac{k+1}{2}.$$

□

Theorem 3.5. *If $J(n,k)$ is a Johnson graph and $n = 2k+1$, then*

$$\delta(J(n,k)) = \frac{2k+1}{4}.$$

Proof. Note that for every vertex v there exist $k+1$ vertices v' such that $d(v,v') = k$. If v_1' and v_2' are vertices such that $d(v,v_1') = d(v,v_2') = k$ and p is the midpoint of $v_1'v_2'$, then $d(v,p) = k + 1/2$. On the other hand, for every $u \in V\big(J(n,k)\big)$ such that $uv \in E(J(n,k))$, we have that $d(u, \{v_1', v_2'\}) = k-1$; therefore, if q is the midpoint of uv, then $d(p,q) = k$. So we have that $\operatorname{diam} J(n,k) = k + 1/2$ and $\delta(J(n,k)) \leq (k+1/2)/2 = (2k+1)/4$.

Consider the sets $V_i = \{i, i+1, \ldots, i+k-1\}$, $1 \leq i \leq k+2$ and $V_{k+j} = \{1, \ldots, j-2, k+j, \ldots, 2k+1\}$, $3 \leq j \leq k+1$. Let x be the vertex v_1 and y the midpoint of $v_{k+1}v_{k+2}$. Let P and P^* be the geodesics joining x and y such that $P \cap V\big(J(n,k)\big) = \{v_1, v_2, \ldots, v_{k+1}\}$ and $P^* \cap V\big(J(n,k)\big) = \{v_1, v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\}$. Let z and p be the midpoints of P and P^*, respectively. If T is the geodesic triangle $T = P^* \cup [yz] \cup [xz]$, then

$$\delta\big(J(n,k)\big) \geq \delta(T) \geq d(p, [xz] \cup [yz]) = d(p, \{x, y\}) = \frac{k+1/2}{2} = \frac{2k+1}{4}.$$

□

Theorem 3.6. *If $J(n,k)$ is a Johnson graph, n is even and $n = 2k$, then*

$$\delta(J(n,k)) = \frac{k}{2}.$$

Proof. Note that for every vertex v there exists only one vertex u such that $d(u,v) = k$, so if p is the midpoint of the edge u_1u_2 and $v \in V(J(n,k))$ then $d(v,p) = \max\{d(v,u_1), d(v,u_2)\} - 1/2 \leq k - 1/2$ and so, $\operatorname{diam}(J(n,k)) = k$ and $\delta\big(J(n,k)\big) \leq k/2$ by Lemma 2.2.

Consider the sets $V_i = \{i, i+1, \ldots, i+k-1\}$, $1 \leq i \leq k+1$ and $V_{k+j} = \{1, \ldots, j-1, k+j, \ldots, 2k\}$ $2 \leq j \leq k$. Let x and y be the vertices v_1 and v_{k+1}, respectively; let P^* be the path joining x to y such that $P^* \cap V\big(J(n,k)\big) = \{v_1, v_{k+1}, v_{k+2}, \ldots, v_{2k}\}$ and let z be the midpoint of P^*. Consider the geodesics $P = [xy]$, $P' = [yz]$ and $P'' = [xz]$ such that $P \cap V\big(J(n,k)\big) = \{v_0, v_1, \ldots, v_{k+1}\}$, $P' \cap V\big(J(n,k)\big) = \{v_{k+1}, v_{k+2}, \ldots, v_{k+\lfloor \frac{k}{2} \rfloor}\}$ and $P'' \cap V\big(J(n,k)\big) = \{v_1, v_{k+\lceil \frac{k}{2} \rceil}, \ldots, v_{2k}\}$. Let T be the geodesic triangle $T = P \cup P' \cup P''$.

Let p be the midpoint of P, then

$$\delta\big(J(n,k)\big) \geq \delta(T) \geq d(p, [xz] \cup [yz]) = d(p, \{x, y\}) = \frac{k}{2}.$$

□

4. Kneser graphs

Let $K(n,k)$ be a Kneser graph. Note that if $n<2k$, then the Kneser graph is a set of isolated vertices. If $n=2k$, then the Kneser graph is a set of isolated edges, see Theorem 4.2. Therefore, we will consider $n=2k+m$ with $m\geq 1$.

The Kneser graph $K(n,1)$ is the complete graph with n vertices K_n. The Petersen graph is the Kneser graph for $n=2$, $m=5$, $K(5,2)$. Also, the *odd graph* O_k, a generalization of the Petersen graph, is the Kneser graph $O_k=K(2k-1,k-1)$.

We can see from the definition of Kneser graphs, that they are interesting combinatorial graphs.

In this section, we are going to compute good bounds of $\delta(K(2k+m,k))$ for every m and k, and in many cases, we even compute its precise value. In order to do that we obtain some results about Kneser graphs which are interesting by themselves, as Propositions 4.7 and 4.15.

If $u_i,v_j\in V\left(K\left(2k+m,k\right)\right)$, we denote by U_i, V_j their corresponding subsets of A^m, respectively.

The following results on $K(s,1)$ and $K(2k,k)$, i.e. $m=0$, are direct:

Theorem 4.1. *Let $K(s,1)$ be a Kneser graph with $s\geq 2$. The following statements hold:*

(1) If $s=2$, then $\delta\left(K(2,1)\right)=0$.
(2) If $s=3$, then $\delta\left(K(3,1)\right)=3/4$.
(3) If $s>3$, then $\delta\left(K(s,1)\right)=1$.

Theorem 4.2. *If $K(2k,k)$ is a Kneser graph with $k\geq 2$, then $\delta\left(K(2k,k)\right)=0$.*

Distances in the Kneser graphs were studied in [28]. The following results in [28] will be useful:

Theorem 4.3. *The diameter of vertices of the Kneser graph $K\left(2k+m,k\right)$ is given by*

$$\operatorname{diam} V\left(K\left(2k+m,k\right)\right)=\left\lceil\frac{k-1}{m}\right\rceil+1.$$

Theorem 4.4. *If $v_1,v_2\in V\left(K\left(2k+m,k\right)\right)$ and $s=|V_1\cap V_2|$, then*

$$d(v_1,v_2)=\min\left\{2\left\lceil\frac{k-s}{m}\right\rceil,2\left\lceil\frac{s}{m}\right\rceil+1\right\}.$$

The following results appear in [26] and [28].

Lemma 4.5. *Let V_1, V_2 be k-subsets of an $(2k+m)$-set A^{2k+m}. If there exists a path of length $2i$ joining v_1 with v_2 in $K\left(2k+m,k\right)$, then*

$$|V_1\cap V_2|\geq k-im.$$

Lemma 4.6. *Let V_1, V_2 be k-subsets of an $(2k+m)$-set A^{2k+m}. If there exists a path of length $2i+1$ joining v_1 with v_2 in $K(2k+m,k)$, then*

$$|V_1 \cap V_2| \le im.$$

Lemmas 4.6 and 4.5 have the following consequences:

Proposition 4.7. *Let $K(2k+m,k)$ be a Kneser graph. Consider $D = \lceil \frac{k-1}{m} \rceil + 1$, $r = \lfloor D/2 \rfloor$, $i \le r-1$ and $v_1, v_2 \in V(K(2k+m,k))$ with $|V_1 \cap V_2| = s$. The following statements hold:*

1. *If $d(v_1,v_2) < D$, then $d(v_1,v_2) = 2i+1$ if and only if $(i-1)m+1 \le s \le im$.*
2. *If $d(v_1,v_2) < D$, then $d(v_1,v_2) = 2i$ if and only if $k-im \le s \le k-(i-1)m-1$.*
3. *If $D = 2r+1$, then $d(v_1,v_2) = D$ if and only if $(r-1)m+1 \le s \le k-rm-1$.*
4. *If $D = 2r$, then $d(v_1,v_2) = D$ if and only if $(r-1)k+1 \le s \le k-(r-1)m-1$.*

Proof. Since $D \le 2r$ and $i \le r-1$, we have

$$\begin{aligned} 2\lceil \frac{im}{m} \rceil + 1 &= 2i+1 \le 2r-1 < 2r+2 \le 4r-2i \le 2\left(\lceil \frac{k-1}{m} \rceil - i\right) \\ &\le 2\lceil \frac{k-im}{m} \rceil. \end{aligned}$$

Let us prove now the four statements.

(1) Lemma 4.6 gives that if $d(v_1,v_2) = 2i+1$, then $(i-1)m+1 \le s \le im$. If $s = im$, then

$$d(v_1,v_2) = \min\left\{2\lceil \frac{k-im}{m} \rceil, 2\lceil \frac{im}{m} \rceil + 1\right\} = 2i+1.$$

If $s = im - j$, $0 \le j \le m-1$, then

$$\begin{aligned} d(v_1,v_2) &= \min\left\{2\lceil \frac{k-im+j}{m} \rceil, 2\lceil \frac{im-j}{m} \rceil + 1\right\} \\ &= \min\left\{2\lceil \frac{k+j}{m} \rceil - 2i, 2i+1\right\} = 2i+1. \end{aligned}$$

(2) Lemma 4.5 gives that if $d(v_1,v_2) = 2i$, then $k-im \le s \le k-(i-1)m-1$. If $s = k-im$, then

$$d(v_1,v_2) = \min\left\{2\lceil \frac{im}{m} \rceil,\ 2\lceil \frac{k-im}{m} \rceil + 1\right\} = 2i.$$

If $s = k-im+j$, $0 \le j \le m-1$, then

$$d(v_1,v_2) = \min\left\{2\lceil \frac{im-j}{m} \rceil,\ 2\lceil \frac{k-im+j}{m} \rceil + 1\right\} = 2i.$$

(3) If $s \le (r-1)m$, then Item (1) gives $d(v_1, v_2) \le 2r-1$. If $s = k-rm-1$, then

$$\begin{aligned} d(v_1, v_2) &= \min\left\{2\lceil\frac{rm+1}{m}\rceil,\ 2\lceil\frac{k-rm-1}{m}\rceil+1\right\} \\ &= \min\left\{2\,(r+1)\,, 2\lceil\frac{k-1}{m}\rceil - 2r+1\right\} \\ &= \min\{2r+2, 2r+1\} = 2r+1 = D. \end{aligned}$$

On the other hand, note that if $s = k - rm$, then

$$\begin{aligned} d(v_1, v_2) &= \min\left\{2\lceil\frac{rm}{m}\rceil,\ 2\lceil\frac{k-rm}{m}\rceil+1\right\} \\ &= \min\left\{2r, 2\lceil\frac{k}{m}\rceil - 2r+1\right\} = 2r = D-1. \end{aligned}$$

Therefore, if $d(v_1, v_2) = D = 2r+1$, then $(r-1)m \le s \le \min\{rm, k-rm-1\}$. Note that $k-rm-1 \le rm$ (the equality is obtained if $k-1$ is an even multiple of m). If $s = (r-1)m+j$ with $1 \le j \le k-(2r-1)m-1$, then

$$\begin{aligned} 2\lceil\frac{s}{m}\rceil+1 &\le 2\lceil\frac{(r-1)m+1}{m}\rceil+1 \le 2r+1 < 2r+2 \le 2\lceil\frac{k-1}{m}\rceil - 2(r-1) \\ &= 2\lceil\frac{k-(r-1)m-1}{m}\rceil \le 2\lceil\frac{k-s}{m}\rceil; \end{aligned}$$

and since $1 \le j \le k-(2r-1)m-1 \le m$, we conclude

$$d(v_1, v_2) = 2\lceil\frac{(r-1)m+j}{j}\rceil+1 = 2r+1.$$

(4) Since $D = 2r$, if $s \le (r-1)m$, then Item (1) gives $d(v_1, v_2) \le 2r-1$. If $k-(r-1)m \le s$, then Item (2) gives $d(v_1, v_2) \le 2r-2$. If $s = (r-1)m+j$ with $1 \le j \le k-2(r-1)m-1$, then

$$\begin{aligned} 2\lceil\frac{k-s}{m}\rceil &\le 2\lceil\frac{k-(r-1)m-1}{m}\rceil \le 2r < 2r+1 \le 2\lceil\frac{(r-1)m+1}{m}\rceil+1 \\ &= 2\lceil\frac{s}{m}\rceil+1 \end{aligned}$$

and

$$\begin{aligned} 2r &\ge 2\lceil\frac{k-s}{m}\rceil = 2\lceil\frac{k-(r-1)m-j}{m}\rceil \\ &= 2\lceil\frac{k-j}{m}\rceil - 2(r-1) \ge 2\lceil\frac{k-1}{m}\rceil - 2(r-1) = 2r. \end{aligned}$$

Therefore, $d(v_1, v_2) \le 2r = D$.

□

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a non-empty set. We define for $0 \le i \le n$:

$$A^i = \{a_1, \ldots, a_i\}, \qquad \overline{A^i} = A \backslash A^i, \qquad A^0 = \emptyset, \qquad \overline{A^n} = \emptyset.$$

4.1. The odd graphs

In this subsection we study the hyperbolicity of $K(n,k)$ with $n = 2k+1$ and $k \ge 3$.

In this case, we have that:

$$\operatorname{diam} V\left(K(2k+1,k)\right) = \lceil k-1 \rceil + 1 = k.$$

4.1.1. *k* is odd. If $v_1, v_2 \in V\left(K(2k+1,k)\right)$ with $d(v_1,v_2) = k$ and $|V_1 \cap V_2| = s$, taking into account that k is odd and $m = 1$, then

$$k = d(v_1, v_2) = \min\left\{2\lceil k-s \rceil, 2\lceil s \rceil + 1\right\} = 2s+1,$$

and we have

$$s = |V_1 \cap V_2| = \frac{k-1}{2}.$$

Thus, for the case k odd and $m = 1$, we can write $A^{2k+1} = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{a_1, a_2, a_3\}$, with $|A_i| = s$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. We have the following results:

Theorem 4.8. *If k is odd, then*

$$\operatorname{diam}\left(K(2k+1,k)\right) < \operatorname{diam} V\left(K(2k+1,k)\right) + 1 = k+1.$$

Proof. Note that $\operatorname{diam} K\left(2k+m,k\right) \le \operatorname{diam} V\left(K\left(2k+m,k\right)\right) + 1$. Seeking for a contradiction assume that there exist x, $y \in K\left(2k+1,k\right)$ such that $d(x,y) = \operatorname{diam} V(K\left(2k+1,k\right)) + 1 = k+1$. Therefore, x and y are midpoints of edges of $K\left(2k+1,k\right)$. Suppose that x is the midpoint of u_1v_1 and y is the midpoint of u_2v_2. Thus, $d(u_1,u_2) = d(u_1,v_2) = k = d(v_1,v_2) = d(v_1,u_2)$ and $|U_1 \cap V_2| = |V_1 \cap V_2| = s$.

Without loss of generality we can assume that:

$$U_1 = A_1 \cup A_2 \cup \{a_1\}, \quad V_1 = A_3 \cup A_4 \cup \{a_2\}, \quad V_2 = A_1 \cup A_2 \cup \{a_3\}.$$

Since $d(u_1,u_2) = d(v_1,u_2) = k$ and $d(u_2,v_2) = 1$, we have that $|U_1 \cap U_2| = |V_1 \cap U_2| = s$ and $V_2 \cap U_2 = \emptyset$. So, we can suppose that $U_2 = A_1 \cup A_3 \cup \{w\}$, but note that if $w \in A_2 \cup \{a_1\}$ (respectively, $A_4 \cup \{a_2\}$), then $|U_1 \cap U_2| = s+1$ (respectively, $|V_1 \cap U_2| = s+1$) and so, $w \neq a_3$. Thus, does not exist $u_2 \in V(K\left(2k+1,k\right))$ satisfying these conditions and we obtain the desired contradiction. □

Corollary 4.9. *If k is odd, then*

$$\operatorname{diam}\left(K(2k+1,k)\right) = \operatorname{diam} V\left(K(2k+1,k)\right)) + 1/2 = k + 1/2.$$

Proof. If we consider the sets U_1, V_1, V_2 and the point x as in the previous theorem, we have $d(x, v_2) = k + 1/2$. □

Theorem 4.10. *If k is odd, then*

$$\delta\big(K(2k+1,k)\big) = \frac{2k+1}{4}.$$

Proof. Lemma 2.2 and Corollary 4.9 give $\delta\big(K(2k+1,k)\big) \le \frac{2k+1}{4}$.

We can write $A^{2k+1} = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{a_1, a_2, a_3\}$, with $|A_i| = s = \frac{k-1}{2}$ and $A_i \cap A_j = \emptyset$ if $i \ne j$.

For $1 \le r \le s$, consider the following sets:

$$\begin{aligned}
V_0 &= A_1 \cup A_2 \cup \{a_1\},\\
V_{2k} &= A_3 \cup A_4 \cup \{a_2\},\\
V_k &= A_1 \cup A_3 \cup \{a_3\},\\
V_{2r-1} &= A_3 \cup \{a_3\} \cup \overline{A_4^{r-1}} \cup A_1^{r-1},\\
V_{2r} &= A_2 \cup \{a_1, a_2\} \cup \overline{A_1^{r}} \cup A_4^{r-1},\\
V_{k+2r} &= A_1 \cup \{a_3\} \cup \overline{A_3^{r}} \cup A_2^{r},\\
V_{k+2r-1} &= A_4 \cup \{a_1, a_2\} \cup A_3^{r-1} \cup \overline{A_2^{r}}.
\end{aligned}$$

Recall that we denote by v_i the corresponding vertex to the subset V_i.

Note that $v_0 v_{2k}$, $v_i v_{i+1} \in E\big(K(2k+1,k)\big)$ for $0 \le i \le 2k-1$. Let C be the subgraph induced by $\{v_i\}_{i=0}^{k}$. We are going to prove that C is an isometric subgraph of $K(2k+1,k)$. In order to do that, it suffices to prove that if $v_i, v_j \in V(C)$ with $i < j$, then $d(v_i, v_j) = d_C(v_i, v_j) = \min\{j-i, (2k+1)-(j-i)\}$. Since $v_i v_{i+1} \in E\left(K(2k+1,k)\right)$, we can assume that $i + 1 < j$.

We consider the following cases:

Case 1. $1 \le i < j \le k$.

Case 1-A. $i = 2r-1$, $j = 2\,l-1$, $1 \le r < l \le s$. Since $|V_{2r-1} \cap V_{2\,l-1}| = |A_3 \cup \{a_3\} \cup \overline{A_4^{l-1}} \cup A_1^{r-1}| = 2\,s + 1 - l + r = k - (l-r)$, so $d(v_{2r-1}, v_{2\,l-1}) = \min\{2\lceil k-(k+r-l)\rceil, 2\lceil k-(l-r)\rceil + 1\} = 2(l-r) = j - i$.

Case 1-B. $i = 2r$, $j = 2\,l$. It is similar to *Case A*.

Case 1-C. $i = 2r-1$, $j = 2\,l$, $r < l$. Thus, $|V_{2r-1} \cap V_{2\,l}| = \big|\big(\overline{A_4^{r-1}} \cap A_4^{l-1}\big) \cup \big(A_1^{r-1} \cap \overline{A_1^{l}}\big)\big| = l - r$ and $d(v_{2r-1}, v_{2\,l}) = \min\{2(k-(l-r)), 2(l-r)+1\} = 2(l-r)+1 = j - i$.

Case 1-D. $i = 2r$, $j = 2\,l-1$, $r < l$. $|V_{2r} \cap V_{2\,l-1}| = \big|\big(\overline{A_1^{r}} \cap A_1^{l-1}\big) \cup \big(A_4^{r-1} \cap \overline{A_4^{l-1}}\big)\big| = (l-1)-r$ and $d(v_{2r}, v_{2\,l-1}) = \min\{2\big(k-(l-1-r)\big), 2\big(l-1-r\big)+1\} = \min\{2(k-(l-r))+1, 2(l-r)-1\} = 2(l-r)-1 = j - i = d_C(v_{2r}, v_{2\,l-1})$.

Case 2. $k \le i < j \le 2k+1$. It is similar to *Case 1.*

Case 3. $1 \le i \le k \le j \le 2k+1$. We have the following cases:

Case 3-A. $i = 2r-1$, $j = k+2l-1$. $|V_{2r-1} \cap V_{k+2l-1}| = \left| A_3^{l-1} \cup \overline{A_4^{r-1}} \right| = (l-1)+(s-(r-1)) = s+l-r$ and $d(v_{2r-1}, v_{n+2l-1}) = \min\{2(k-(s+l-r)), 2(s+l-r)+1\} = \min\{k-2(l-r)+1, k+2l-2r\} = \min\{(2k+1)-(j-i), j-i\}$.

Case 3-B. $i = 2r$, $j = k+2l-1$. $|V_{2r} \cap V_{k+2l-1}| = \left| A_4^{r-1} \cup \overline{A_2^{l}} \cup \{a_1, a_2\} \right| = s+1+r-l$ and

$$\begin{aligned} d(v_{2r}, v_{k+2l-1}) &= \min\{2\left(k-(s+1+r-l)\right), 2\left(s+1+r-l\right)\} \\ &= \min\{k-1-2r-2l, k+2+2r-2l\} \\ &= \min\{j-i, 2k+1-(j-i)\} = d_C(v_{2r}, v_{k+2l-1}). \end{aligned}$$

Case 3-C. $i = 2r-1$, $j = k+2l$. $|V_{2r-1} \cap V_{k+2l}| = \left| A_1^{r-1} \cup \overline{A_3^{l}} \cup \{a_3\} \right| = (r-1)+(s-l)+1 = s-l+r$ and

$$\begin{aligned} d(v_{2r-1}, v_{k+2l}) &= \min\{2\left(k-(s-l+r)\right), 2\left(s-l+r\right)+1\} \\ &= \min\{2\left(k+1+2l-2r\right), 2\left(k-2l+2r\right)\} \\ &= \min\{j-i, (2k+1)-(j-i)\} = d_C(v_{2r-1}, v_{k+2l}). \end{aligned}$$

Case 3-D. $i = 2r$, $j = k+2l$. $|V_{2r} \cap V_{k+2l}| = \left| A_2^{l1} \cup \overline{A_1^{r}} \right| = (s-r)+l = s-r+l$ and $d(v_{2r}, v_{k+2l}) = \min\{2\left(k-(s-r+l)\right), 2\left(s-r+l\right)+1\} = \min\{k+1+2r+2l, k-2r+2l\} = \min\{(2k+1)-(j-i), j-i\} = d_C(v_{2r}, v_{k+2l})$.

Since C is an isometric subgraph of $K(2k+1,k)$, Theorem 2.6 and Lemma 2.5 give $\delta(C) = \frac{2k+1}{4} \le \delta\left(K(2k+1,k)\right)$. □

4.1.2. *k* is even. In this case we have that if $v_i, v_j \in V\big(K(2k+1,k)\big)$ and $d(v_i, v_j) = \operatorname{diam} V\left(K\left(2k+1,k\right)\right) = k$, then $|V_i \cap V_j| = s = k/2$ and $A^{2k+1} = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{a\}$.

Theorem 4.11. *If k is even, then*

$$\delta\left(K(2k+1,k)\right) = \frac{k+1}{2}.$$

Proof. Lemma 2.2 gives $\delta\left(K(2k+1,k)\right) \le \frac{k+1}{2}$. Let us prove the converse inequality. Consider the following sets:

$$W_j = \begin{cases} A_1 \cup \overline{A_2^i} \cup A_3^i & \text{if } j = 2i, 0 \le i \le k/2, \\ A_2^{i-1} \cup \overline{A_3^i} \cup A_4 \cup \{v\} & \text{if } j = 2i-1, 1 \le i \le k/2, \end{cases} \tag{1}$$

$$W'_j = \begin{cases} \overline{A_2^{k/2-i}} \cup A_3^{k/2-i} \cup A_4 & \text{if } j = 2i, 0 \le i \le k/2, \\ A_1 \cup A_2^{k/2-i} \cup \overline{A_3^{k/2-i+1}} \cup \{v\} & \text{if } j = 2i-1, 1 \le i \le k/2. \end{cases} \tag{2}$$

Let x and y be the midpoints of $w_0w'_0$ and $w_kw'_k$ respectively. Let P and P' two geodesics joining x to y such that $P \cap V(K(2k+1,k)) = \{w_0, w_1, \ldots, w_k\}$ and $P' \cap V(K(2k+1,k)) = \{w'_0, w'_1, \ldots, w'_k\}$. Let z and p be the midpoints of P' and P respectively, i.e., $z = w'_{k/2}$ and $p = w_{k/2}$. Consider the geodesic triangle $T = P \cup [yz] \cup [xz]$ in $K(2k+1,k)$, and let us prove that $\delta(T) \geq \frac{k+1}{2}$.

If $k/2$ is even, we have $W_{k/2} = W_{2(k/4)} = A_1 \cup \overline{A_2^{k/4}} \cup A_3^{k/4}$.

For $0 \leq i \leq k/2$ we have $\left|W_{k/2} \cap W'_{2i}\right| = \left|\overline{A_2^{k/4}} \cap \overline{A_2^{k/2-i}}\right| + \left|A_3^{k/4} \cap A_3^{k/2-i}\right|$.

For $1 \leq i \leq k/2$ we have $\left|W_{k/2} \cap W'_{2i-1}\right| = |A_1| + \left|\overline{A_2^{k/4}} \cap A_2^{k/2-i}\right| + \left|A_3^{k/4} \cap \overline{A_3^{k/2-i+1}}\right|$.

Therefore, we have

$$\left|W_{k/2} \cap W'_j\right| = \begin{cases} k/4 + i & \text{if } j = 2i, 0 \leq i < k/4, \\ k/2 & \text{if } j = 2i, i = k/4, \\ 3k/4 - i & \text{if } j = 2i, k/4 < i \leq k/2, \\ 3k/4 - i & \text{if } j = 2i-1, 1 \leq i < k/4, \\ k/2 & \text{if } j = 2i-1, k/4 \leq i \leq k/4+1, \\ k/4 - 1 + i & \text{if } j = 2i-1, k/4+1 < i \leq k/2. \end{cases} \tag{3}$$

Corollary 4.9 gives

$$d(w_{k/2}, w'_j) = \begin{cases} 2(k/4+i)+1 & \text{if } j = 2i, 0 \leq i < k/4, \\ k & \text{if } j = 2i, i = k/4, \\ 2(3k/4-i)+1 & \text{if } j = 2i, k/4 < i \leq k/2, \\ k - 2(k/4-i) & \text{if } j = 2i-1, 1 \leq i < k/4, \\ k & \text{if } j = 2i-1, k/4 \leq i \leq k/4+1, \\ k - 2(k/4-1+i-k/2) & \text{if } j = 2i-1, k/4+1 < i \leq k/2. \end{cases} \tag{4}$$

If $k/2$ is odd, then we have $W_{k/2} = W_{2(\lceil k/4 \rceil)-1} = A_2^{\lceil k/4 \rceil - 1} \cup \overline{A_3^{\lceil k/4 \rceil}} \cup A_4 \cup \{v\}$.

For $0 \leq i \leq k/2$ we have

$$\left|W_{k/2} \cap W'_{2i}\right| = \left|A_2^{\lceil k/4 \rceil - 1} \cap \overline{A_2^{k/2-i}}\right| + \left|\overline{A_3^{\lceil k/4 \rceil}} \cap A_3^{k/2-i}\right| + |A_4|.$$

For $1 \leq i \leq k/2$ we have

$$\left|W_{k/2} \cap W'_{2i-1}\right| = \left|A_2^{\lceil k/4 \rceil - 1} \cap A_2^{k/2-i}\right| + \left|\overline{A_3^{\lceil k/4 \rceil}} \cap \overline{A_3^{k/2-i+1}}\right| + 1.$$

$$\left|W_{k/2} \cap W'_j\right| = \begin{cases} k - (\lceil k/4 \rceil + i) & \text{if } j = 2i, 0 \leq i < \lfloor k/4 \rfloor, \\ k/2 & \text{if } j = 2i, \lfloor k/4 \rfloor \leq i \leq \lceil k/4 \rceil, \\ \lfloor k/4 \rfloor + i & \text{if } j = 2i, \lceil k/4 \rceil < i \leq k/2, \\ \lfloor k/4 \rfloor + i & \text{if } j = 2i-1, 1 \leq i < \lceil k/4 \rceil, \\ k/2 & \text{if } j = 2i-1, i = \lceil k/4 \rceil, \\ k/2 + \lceil k/4 \rceil - i & \text{if } j = 2i-1, \lceil k/4 \rceil < i \leq k/2. \end{cases} \tag{5}$$

$$d(w_{k/2}, w'_j) = \begin{cases} 2(\lceil k/4 \rceil + i) & \text{if } j = 2i, 0 \le i < \lfloor k/4 \rfloor, \\ k & \text{if } j = 2i, \lfloor k/4 \rfloor \le i \le \lceil k/4 \rceil, \\ k - 2(i - \lceil k/4 \rceil) & \text{if } j = 2i, \lceil k/4 \rceil < i \le k/2, \\ 2(\lfloor k/4 \rfloor + i) + 1 & \text{if } j = 2i-1, 1 \le i < \lceil k/4 \rceil, \\ k & \text{if } j = 2i-1, i = \lceil k/4 \rceil, \\ 2(k/2 + \lceil k/4 \rceil - i) + 1 & \text{if } j = 2i-1, \lceil k/4 \rceil < i \le k/2. \end{cases} \tag{6}$$

In both cases we have $d(w_{k/2}, w'_j) \ge k/2 + 1$ for $0 \le j \le k$ and

$$\delta(K(2k+1,k)) \ge \delta(T) \ge d(p, [xz] \cup [yz]) = \min\{d(p, \{w'_i\}_{i=0}^k), d(p, \{x,y\}\} = \min\{k/2+1, k/2+1/2\} = \frac{k+1}{2}.$$

□

4.2. Diameters 2 and 3

In this subsection we compute the precise value of $\delta(K(2k+m,k))$ when $m \ge k-1$ (i.e., $\operatorname{diam} V(K(2k+m,k)) = 2$), and we obtain good bounds of $\delta(K(2k+m,k))$ when $(k-1)/2 \le m < k-1$ (i.e., $\operatorname{diam} V(K(2k+m,k)) = 3$).

Theorem 4.12. *If $K(2k+m,k)$ is the Kneser graph with $m \ge k-1$ (i.e., $\operatorname{diam} V(K(2k+m,k)) = 2$), then*

$$\delta(K(2k+m,k)) = \frac{3}{2}.$$

Proof. Let A, B and C be pairwise disjoint sets such that $|A| = |B| = n$, $|C| = n-1$. Consider the following sets:

$$W_0 = A, \quad W_1 = B^1 \cup C, \quad W_2 = A^1 \cup \overline{B^1},$$
$$W'_0 = B, \quad W'_1 = A^1 \cup C, \quad W'_2 = B^1 \cup \overline{A^1}.$$

Let x and y be the midpoints of $w_0w'_0$ and $w_2w'_2$, respectively. Consider the geodesics P and P' joining x and y such that $P \cap V(K(2k+m,k)) = \{w_0, w_1, w_2\}$ and $P' \cap V(K(2k+m,k)) = \{w'_0, w'_1, w'_2\}$. Let $z = w_1$ and T be the geodesic triangle $T = \{[xz], [yz], P'\}$. We have

$$2 = d(w'_1, \{w_0, w_1, w_2\}) > d(w'_1, \{x,y\}) = 3/2,$$
$$\frac{3}{2} = d(w'_1, \{x,y\}) = d(w'_1, [xz] \cup [yz])$$
$$\le \delta(K(2k+m,k)) \le \frac{\operatorname{diam} V(K(2k+m,k)) + 1}{2} = \frac{3}{2}.$$

Therefore, we conclude

$$\delta(K(2k+m,k)) = \frac{3}{2}.$$

□

Theorem 4.13. *If* $(k-1)/2 \le m < k-1$ *(i.e.,* $\operatorname{diam} V(K(2k+m,k)) = 3$*), then*

$$\frac{7}{4} \le \delta(K(2k+m,k)) \le 2.$$

Proof. Lemma 2.2 gives the upper bound. Let A, B, C, U, V be pairwise disjoint sets such that $|A| = 1$, $|B| = k-m-1$, $|C| = m$, $|U| = k-1$, $|V| = m+1$. Consider the following sets:

$$W_0 = A \cup U, \quad W_1 = B \cup C^1 \cup \overline{V^1}, \quad W_2 = V^1 \cup U, \quad W_3 = A \cup B \cup C,$$
$$W_0' = B \cup V, \quad W_1' = A \cup C \cup U^{|u|-m}, \quad W_2' = U^{|u|-m} \cup V, \quad W_3' = W_3.$$

Let x be the midpoint of $w_0 w_0'$ and $y = w_3$. Consider the geodesics, P and P', joining x and y such that $P \cap V(K(2k+m,k)) = \{w_0, w_1, w_2, w_3\}$ and $P' \cap V(K(2k+m,k)) = \{w_0', w_1', w_2', w_3'\}$.

Note that $d(w_1, \{w_0', w_1', w_2', w_3'\}) = d(w_2, \{w_0', w_1', w_2'\}) = 2$ and so, we have $d(w_1, P') = d(w_1, w_0) + 1/2 = 3/2$ and $d(w_2, P') = d(w_2, w_3') = 1$.

Let z and p be the midpoints of $w_1' w_2'$ and P, respectively. Consider the geodesic triangle $T = \{[xz], [yz], P\}$. Note that $p \in [w_1 w_2]$ and

$$\begin{aligned} 2d(p, P') &= d(p, w_1) + d(w_1, P') + d(p, w_2) + d(w_2, P') \\ &= \operatorname{diam} V(K(2k+m,k)) + 1/2. \end{aligned}$$

Therefore,

$$\delta(K(2k+m,k)) \ge d(p, P') = \frac{\operatorname{diam} V(K(2k+m,k)) + 1/2}{2} = \frac{7}{4}.$$

□

4.3. Odd diameter

In this subsection we study the hyperbolicity constant of Kneser graphs with odd diameter D and $1 < m < (k-1)/2$. Unless otherwise specified we assume $D = 2r+1$. Note that it suffices to consider the case $1 < m < (k-1)/2$ (i.e., $D \ge 5$), since the case $r = 1$ was studied previously.

Theorem 4.14. *If* $\operatorname{diam} V(K(2k+m,k)) = D$ *is odd and* $1 < m < (k-1)/2$*, then*

$$\frac{D + 1/2}{2} \le \delta(K(2k+m,k)) \le \frac{D+1}{2}.$$

Proof. Lemma 2.2 gives the upper bound. Let us prove the lower bound.

Let $D = 2r+1$. If $w_i, w_j \in V(K(2k+m,k))$ satisfy $d(w_i, w_j) = D$, Proposition 4.7 gives $(r-1)m + 1 \le |W_i \cap W_j| \le k - rm - 1$.

Let A, B, C, U, V be pairwise disjoint sets such that $|A| = (r-1)m+1$, $|B| = k - rm - 1 = (r-1)m + l$, $|C| = m$, $|U| = k - (r-1)m - 1 = rm + l$, $|V| = rm + 1$, with $1 \le l \le m$.

If r is even, we consider the following sets:

$$W_j = \begin{cases} \overline{A^{im}} \cup U \cup V^{im} & \text{if } j = 2i, 0 \le i < r/2, \\ A^{(i-1)m} \cup B \cup C \cup \overline{V^{im}} & \text{if } j = 2i-1, 1 \le i \le r/2, \\ \overline{A^{(i-1)m+1}} \cup U \cup V^{(i-1)m+1} & \text{if } j = 2i, r/2 \le i \le r, \\ A^{(i-2)m+1} \cup B \cup C \cup \overline{V^{(i-1)m+1}} & \text{if } j = 2i-1, r/2 < i \le r+1. \end{cases} \tag{7}$$

$$W'_j = \begin{cases} \overline{B^{im}} \cup U^{im} \cup V & \text{if } j = 2i, 0 \le i < r/2, \\ A \cup B^{(i-1)m} \cup C \cup \overline{U^{im}} & \text{if } j = 2i-1, 1 \le i \le r/2, \\ \overline{B^{(i-1)m+l}} \cup U^{(i-1)m+l} \cup V & \text{if } j = 2i, r/2 \le i \le r, \\ A \cup B^{(i-2)m+l} \cup C \cup \overline{U^{(i-1)m+l}} & \text{if } j = 2i-1, r/2 < i \le r+1. \end{cases} \tag{8}$$

Note that for $r/2 \le i \le r$ we have

$$\begin{aligned} W'_{2i} &= \overline{B^{(i-1)m+l}} \cup U^{(i-1)m+l} \cup V = W'_{2(r/2+h)} \\ &= \overline{B^{|B|-(r/2-h)m}} \cup U^{|U|-(r/2+1-h)m} \cup V \end{aligned}$$

with $0 \le h \le r/2$.

If r is odd, we consider the following sets:

$$W_j = \begin{cases} \overline{A^{im}} \cup U \cup V^{im} & \text{if } j{=}2i, 0 \le i < r/2, \\ A^{(i-1)m} \cup B \cup C \cup \overline{V^{im}} & \text{if } j{=}2i-1, 1 \le i \le \lceil r/2 \rceil, \\ \overline{A^{(i-1)m+1}} \cup U \cup V^{(i-1)m+1} & \text{if } j{=}2i, r/2 < i \le r, \\ A^{(i-2)m+1} \cup B \cup C \cup \overline{V^{(i-1)m+1}} & \text{if } r < j{=}2i-1, \lceil r/2 \rceil < i \le r{+}1. \end{cases} \tag{9}$$

$$W'_j = \begin{cases} \overline{B^{im}} \cup U^{im} \cup V & \text{if } j{=}2i, 0 \le i < r/2, \\ A \cup B^{(i-1)m} \cup C \cup \overline{U^{im}} & \text{if } j{=}2i{-}1, 1 \le i \le \lceil r/2 \rceil, \\ \overline{B^{(i-1)m+l}} \cup U^{(i-1)m+l} \cup V & \text{if } j{=}2i, r/2 < i \le r, \\ A \cup B^{(i-2)m+l} \cup C \cup \overline{U^{(i-1)m+l}} & \text{if } r < j{=}2i-1, \lceil r/2 \rceil < i \le r{+}1. \end{cases} \tag{10}$$

Let x be the midpoint of $w_0w'_0$ and $y = w_D = w'_D$. Let P and P' be geodesics joining x to y such that $P \cap V(K(2k+m,k)) = \{w_0, w_1, \ldots, w_d\}$ and $P' \cap V(K(2k+m,k)) = \{w'_0, w'_1, \ldots, w'_d\}$. Consider the geodesic triangle $T = \{[xw'_r], [yw'_r], P\}$. Let p be the midpoint of P, we have $p \in w_rw_{r+1}$.

We are going to prove that $d(w_r, P' \cap V(K(2k+m,k))) \ge r+1$ and $d(w_{r+1}, P') = r$.

Assume that r is even. We have $W_r = \overline{A^{(r/2-1)m+1}} \cup U \cup V^{(r/2-1)m+1}$ and $W_{r+1} = A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}}$.

For $0 \le i < r/2$, we have

$$|W_r \cap W'_{2i}| = \left| \left(\overline{A^{(r/2-1)m+1}} \cup U \cup V^{(r/2-1)m+1} \right) \cap \left(\overline{B^{im}} \cup U^{im} \cup V \right) \right|$$

$$= \left|V^{(r/2-1)m+1}\right| + \left|U^{im}\right| = (r/2-1)m+1+im$$
$$= (r/2-1+i)m+1.$$

For $r/2 < i \le r$:

$$|W_r \cap W'_{2i}|$$
$$= \left|\left(\overline{A^{(r/2-1)m+1}} \cup U \cup V^{(r/2-1)m+1}\right) \cap \left(\overline{B^{(i-1)m+l}} \cup U^{(i-1)m+l} \cup V\right)\right|$$
$$= \left|V^{(r/2-1)m+1}\right| + \left|U^{(i-1)m+l}\right|$$
$$= (r/2-1)m+1+(i-1)m+l = (r/2+i-2)m+1+l.$$

If $l = m$, then $|W_r \cap W'_{2i}| = (r/2+i-1)m+1$. Note that $(r-1)m+1 < rm+1 \le (r/2+i-1)m+1 \le (r+r/2-1)m+1 = rm+1+(r/2)m$. If $i = r/2+m'$ with $1 \le m' \le r/2$, then $|W_r \cap W'_{2i}| = (r+m'-1)m+1$ and so, $d(w_r, w'_{2i}) = D-2(m'-1)-1$.

If $l < k$, then

$$|W_r \cap W'_{2i}| = (r/2+i-1)m+1 \ge (r-1)m+l+1$$
$$= |B|+1 = k-rm-1+1 = k-rm.$$

If $i = r/2+m'$ with $1 \le m' \le r/2$, then $|W_r \cap W'_{2i}| = (r+m'-2)m+l+1 = (r-1)m+l+(m'-1)m+1 = k-rm-1+(m'-1)m+1 = k-(r-m'+1)m$ and so, $d(w_r, w'_{2i}) = 2(r-m'+1)$.

For $1 \le i \le r/2$ we have

$$|W_r \cap W'_{2i-1}|$$
$$= \left|\left(\overline{A^{(r/2-1)m+1}} \cup U \cup V^{(r/2-1)m+1}\right) \cap \left(A \cup B^{(i-1)m} \cup C \cup \overline{U^{im}}\right)\right|$$
$$= \left|\overline{A^{(r/2-1)m+1}}\right| + \left|\overline{U^{im}}\right| = |A| - ((r/2-1)m+1) + |U| - (im)$$
$$= k-(r/2-1+i)m-1.$$

For $j = 2i-1$ with $r/2 < i \le r+1$ (note that $r < j$), we have

$$|W_r \cap W'_{2i-1}|$$
$$= \left|\left(\overline{A^{(r/2-1)m+1}} \cup U \cup V^{(r/2-1)m+1}\right) \cap \left(A \cup B^{(i-2)m+l} \cup C \cup \overline{U^{(i-1)m+l}}\right)\right|$$
$$= \left|\overline{A^{(r/2-1)m+1}}\right| + \left|\overline{U^{(i-1)m+l}}\right|$$
$$= |A| - ((r/2-1)m+1) + |U| - ((i-1)m+l)$$
$$= k-(r/2+i-2)m-1-l.$$

Consider $i = r/2+m'$, with $1 \le m' \le r/2+1$. We have

$$|W_r \cap W'_{2i-1}|$$
$$= k-(r/2+i-2)m-1-l = k-(r/2+r/2+m'-2)m-1-l$$

$$= k - (r + m' - 2)m - 1 - l = k - (r - 1)m - 1 - (m' - 1)m - l$$
$$= |U| - (m' - 1)m - l = rm + l - (m' - 1)m - l = (r - m' + 1)m.$$

We have

$$|W_r \cap W_j'| = \begin{cases} (r/2 - 1 + i)m + 1 & \text{if } j = 2i, 0 \le i < r/2, \\ k - (r/2 - 1 + i)m - 1 & \text{if } j = 2i - 1, 1 \le i \le r/2, \\ (r - 2)m + 1 + l & \text{if } j = r, \\ (r/2 + i - 2)m + 1 + l & \text{if } j = 2i, r/2 < i \le r, \\ k - (r/2 + i - 2)m - 1 - l & \text{if } j = 2i - 1, r/2 < i \le r + 1, \end{cases} \tag{11}$$

$$d(w_r, w_j') = \begin{cases} 2(r/2 - 1 + i) + 3 & \text{if } j = 2i, 0 \le i < r/2, \\ 2(r/2 + i) & \text{if } j = 2i - 1, 1 \le i \le r/2, \\ D & \text{if } j = r \text{ and } l = m, \\ D - 1 & \text{if } j = r \text{ and } l < m, \\ D - 2m' + 1 & \text{if } j = 2i, r/2 < i \le r \\ & \quad i = r/2 + m', 1 \le m' \le r/2, \\ D & \text{if } j = r + 1, l = m, \\ D - 1 & \text{if } j = r + 1, l < m, \\ D - 2m' + 2 & \text{if } j = 2i - 1, i = r/2 + m', \\ & 1 < m' \le r/2 + 1. \end{cases} \tag{12}$$

Note that $d(w_r, w_j') \ge r + 1$ for $0 \le j \le D$, therefore $d(w_r, P') = \min\{d(w_r, \{x, y\}), d(w_r, P \cap V(K(2k + m, k)))\} = d(w_r, x) = r + 1/2$.

Similarly, we have $W_{r+1} = W_{2(r/2+1)-1} = A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}}$. Let us consider their intersections and the distances from w_j' to $w_{r+1} = w_{2(r/2+1)-1}$.

For $0 \le i < r/2$:

$$\begin{aligned} |W_{r+1} \cap W_{2i}'| &= \left| \left(A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}} \right) \cap \left(\overline{B^{im}} \cup U^{im} \cup V \right) \right| \\ &= \left| \overline{B^{im}} \right| + \left| \overline{V^{(r/2)m+1}} \right| = |B| - (im) + |V| - ((r/2)m + 1) \\ &= k - (r/2 + i)m - 1. \end{aligned}$$

Then $d(w_{r+1}, w_{2i}') = 2(r/2+i+1)$, for $0 \le i < r/2$. Note that $d(w_{r+1}, w_{2i}') \ge r + 2$.

For $r/2 \le i \le r$:

$$\begin{aligned} &|W_{r+1} \cap W_{2i}'| \\ &\quad = \left| \left(A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}} \right) \cap \left(\overline{B^{(i-1)m+l}} \cup U^{(i-1)m+l} \cup V \right) \right| \\ &\quad = \left| \overline{B^{(i-1)m+l}} \right| + \left| \overline{V^{(r/2)m+1}} \right| = |B| - ((i - 1)m + l) + |V| - ((r/2)m + 1) \\ &\quad = (r - 1)m + l - (i - 1)m - l + rm + 1 - (r/2)m - 1 = \left(\frac{3}{2} r - i \right) m. \end{aligned}$$

For $r/2+1 \le i \le r$ we have $\left(\frac{3}{2}r-i\right)m \le (r-1)m$ and so, Proposition 4.7 gives $d(w_{r+1}, w'_{2i}) = 2(\frac{3}{2}r-i)+1$. If $i = r/2$ and $l = m$, then $d(w_{r+1}, w'_{2i}) = D$; if $i = r/2$ and $l < m$, then $d(w_{r+1}, w'_{2i}) = D-1$.

For $1 \le i \le r/2$:

$$\begin{aligned}|W_{r+1} \cap W'_{2i-1}| \\ &= \left|\left(A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}}\right) \cap \left(A \cup B^{(i-1)m} \cup C \cup \overline{U^{im}}\right)\right| \\ &= \left|A^{(r/2-1)m+1}\right| + \left|B^{(i-1)m}\right| + |C| = (r/2-1)m+1+(i-1)m+m \\ &= (r/2-1+i)m+1.\end{aligned}$$

Since $i \le r/2$, we have $|W_{r+1} \cap W'_{2i-1}| = (r/2-1+i)m+1 \le (r-1)m+1$ and Proposition 4.7 gives $d(w_{r+1}, w'_{2i}) = 2(r/2-1+i)+3$. In particular, if $i = r/2$, then $d(w_{r+1}, w'_{2i}) = D$; if $i = 1$, then $d(w_{r+1}, w'_{2i-1}) = r+3$.

For $r/2 < i \le r+1$:

$$\begin{aligned}|W_{r+1} \cap W'_{2i-1}| \\ &= \left|\left(A^{(r/2-1)m+1} \cup B \cup C \cup \overline{V^{(r/2)m+1}}\right) \cap \left(A \cup B^{(i-2)m+l} \cup C \cup \overline{U^{(i-1)m+l}}\right)\right| \\ &= \left|A^{(r/2-1)m+1}\right| + \left|B^{(i-2)m+l}\right| + |C| = (r/2-1)m+1+(i-2)m+l+m \\ &= (r/2+i-2)m+1+l.\end{aligned}$$

If $i = r/2+i'$ with $1 \le i' \le r/2+1$, then

$$\begin{aligned}(r/2+i-2)m+1+l &= (r/2+r/2+i'-2)m+1+l = (r+i'-2)m+1+l \\ &= (r-1)m+l+1+(i'-1)m = |B|+1+(i'-1)m \\ &= k-rm-1+1+(i'-1)m = k-(r+1-i')m.\end{aligned}$$

So, Proposition 4.7 gives $d(w_{r+1}, w'_{2i-1}) = 2(r+1-i')$ with $1 \le i' \le r/2+1$. Note that if $i' = r/2+1$ then $d(w_{r+1}, w'_{2(r/2+1)-1}) = d(w_{r+1}, w'_D) = r$. Hence, $d(w_{r+1}, P') = d(w_{r+1}, w'_D) = r$.

Thus,

$$\begin{aligned}2d(p, [xw'_r] \cup [xw'_r]) &= d(w_r, P') + d(p, w_r) + d(p, w_{r+1}) + d(w_{r+1}, P') \\ &= r + 1/2 + d(w_r, p) + d(p, w_{r+1}) + r = D + 1/2,\end{aligned}$$

and therefore, $\delta\left(K\left(2k+m, k\right)\right) \ge \delta(T) \ge \frac{D+1/2}{2}$.

The case with odd r is similar. □

4.4. Even diameter

In this subsection we compute the precise value of the hyperbolicity constant of Kneser graphs $K(2k+m, k)$ with even diameter D and $1 < m < (k-1)/2$.

Unless otherwise specified we assume $D = 2r$. Note that it suffices to consider the case $1 < m < (k-1)/2$, since the case $r = 1$ was studied previously.

Proposition 4.7 gives the following result.

Proposition 4.15. *Let $K(2k+m,k)$ be a Kneser graph with $1 < m < (k-1)/2$ and $D = 2r$, and let $w_i, w_j \in V(K(2k+m,k))$ with $s = |W_i \cap W_j|$. The following statements hold:*

(1) If $(r-1)m + 1 \le s \le k - (r-1)m - 1$, then $d(w_i, w_j) = D$.
(2) If $(j-1)m + 1 \le s \le jm$ for $1 \le j \le r-1$, then $d(w_i, w_j) = 2j + 1$.
(3) If $k - jm \le s \le k - (j-1)m - 1$ for $1 \le j \le r-1$, then $d(w_i, w_j) = 2j$.

Theorem 4.16. *If $K(2k+m,k)$ is a Kneser graph with $1 < m < (k-1)/2$ and D is even, then*

$$\delta\left(K\left(2k+m,k\right)\right) = \frac{D+1}{2}.$$

Proof. Lemma 2.2 gives $\delta\left(K\left(2k+m,k\right)\right) \le \frac{D+1}{2}$. Let us prove the converse inequality.

Let $D = 2r$. If $w_i, w_j \in V\left(K\left(2k+m,k\right)\right)$ with $d(w_i, w_j) = D$, then Propositions 4.7 and 4.15 give $(r-1)m + 1 \le |W_i \cap W_j| \le k - (r-1)m + 1$.

Let A, B, C, U, V be pairwise disjoint sets such that $|A| = |B| = (r-1)m + 1 = s_1$, $|C| = m$, $|U| = |V| = k - (r-1)m - 1 = s_2$.

Consider the following sets:

$$W_j = \begin{cases} \overline{A^{im}} \cup B^{im} \cup U & \text{if } j = 2i, 0 \le i \le \lfloor r/2 \rfloor, \\ A^{(i-1)m} \cup \overline{B^{im}} \cup C \cup V & \text{if } j = 2i-1, 1 \le i \le \lceil r/2 \rceil, \\ \overline{A^{(i-1)m+1}} \cup B^{(i-1)m+1} \cup U & \text{if } j = 2i, \lfloor r/2 \rfloor < i \le r, \\ A^{(i-2)m+1} \cup \overline{B^{(i-1)m+1}} \cup C \cup V & \text{if } j = 2i-1, \lceil r/2 \rceil < i \le r. \end{cases} \tag{13}$$

$$W'_j = \begin{cases} \overline{A^{s_1-im}} \cup B^{s_1-im} \cup V & \text{if } j = 2i, 0 \le i < \lceil r/2 \rceil, \\ A^{s_1-im} \cup \overline{B^{s_1-(i-1)m}} \cup C \cup U & \text{if } j = 2i-1, 1 \le i \le \lfloor r/2 \rfloor, \\ \overline{A^{s_1-(i-1)m-l}} \cup B^{s_1-(i-1)m-l} \cup V & \text{if } j = 2i, \lceil r/2 \rceil \le i \le r, \\ A^{s_1-(i-1)m-l} \cup \overline{B^{s_1-(i-2)m-l}} \cup C \cup U & \text{if } j = 2i-1, \lfloor r/2 \rfloor < i \le r. \end{cases} \tag{14}$$

Let x, y be the midpoints of $w_0w'_0$ and $w_Dw'_D$ respectively. Let P and P' be geodesics joining x and y such that $P \cap V\left(K\left(2k+m,k\right)\right) = \{w_0, w_1, \ldots, w_d\}$ and $P' \cap V\left(K\left(2k+m,k\right)\right) = \{w'_0, w'_1, \ldots, w'_d\}$. Consider the geodesic triangle $T = \{[xw'_r], [yw'_r], P\}$. Let p be the midpoint of P, i.e., $p = w_r$.

We are going to prove that $d\left(w_r, P' \cap V\left(K\left(2k+m,k\right)\right)\right) = r + 1$.

Assume that r is even. Then $W_r = W_{2(r/2)} = \overline{A^{(r/2)m}} \cup B^{(r/2)m} \cup U$, and for $0 \le i < \lceil r/2 \rceil$,

$$(r/2-1)m + 1 < (r/2)m + 1 \le s_1 - im = (r-1-i)m + 1 \le (r-1)m + 1,$$
$$|W_r \cap W'_{2i}| = \left|\left(\overline{A^{(r/2)m}} \cup B^{(r/2)m} \cup U\right) \cap \left(\overline{A^{s_1-im}} \cup B^{s_1-im} \cup V\right)\right|$$

$$
\begin{aligned}
&= \left|\overline{A^{(r/2)m}} \cap \overline{A^{s_1-im}}\right| + \left|B^{(r/2)m} \cap B^{s_1-im}\right| \\
&= \left|\overline{A^{(r/2)m}} \cap \overline{A^{(r-1-i)m+1}}\right| + \left|B^{(r/2)m} \cap B^{(r-1-i)m+1}\right| \\
&= \left|\overline{A^{(r-1-i)m+1}}\right| + \left|B^{(r/2)m}\right| = |A| - ((r-1-i)m+1) + (r/2)m \\
&= (r-1)m+1-(r-1)m-1+im+(r/2)m = (r/2+i)m < s_1.
\end{aligned}
$$

Thus, we have $d(w_r, w_{2i}) = 2(r/2+i)+1 \geq r+1$ and $d(w_r, w_0) = r+1$.

If $\lceil r/2 \rceil \leq i \leq r$, then

$$
\begin{aligned}
|W_r \cap W'_{2i}| &= \left|\left(\overline{A^{(r/2)m}} \cup B^{(r/2)m} \cup U\right) \cap \left(\overline{A^{s_1-(i-1)m-l}} \cup B^{s_1-(i-1)m-l} \cup V\right)\right| \\
&= \left|\overline{A^{(r/2)m}} \cap \overline{A^{s_1-(i-1)m-l}}\right| + \left|B^{(r/2)m} \cap B^{s_1-(i-1)m-l}\right| \\
&= \left|\overline{A^{(r/2)m}} \cap \overline{A^{(r-i)m}}\right| + \left|B^{(r/2)m} \cap B^{(r-i)m}\right|.
\end{aligned}
$$

If $i = r/2$, then $|W_r \cap W'_{2i}| = |A| - (r/2)m + (r/2)m = s_1$ and so, $d(w_r, w_{2i}) = D$. If $i > r/2$, then $(r-i)m < (r/2)m$ and so,

$$
\begin{aligned}
|W_r \cap W'_{2i}| &= \left|\overline{A^{(r/2)m}}\right| + \left|B^{(r-i)m}\right| = |A| - (r/2)m + (r-i)m \\
&= (r-1)m+1-(r/2)m+(r-i)m = (3r/2-1-i)m+1.
\end{aligned}
$$

We can write $i = r/2 + i'$ with $1 \leq i' \leq r/2$ and so, $|W_r \cap W'_{2i}| = (r-1-i')m+1$ and $(r/2-1)m+1 \leq |W_r \cap W'_{2i}| \leq (r-2)m+1 < s$. Therefore, $d(w_r, w_{2i}) = 2(r-1-i')+3 \geq r+1$.

If $1 \leq i \leq \lfloor r/2 \rfloor$, then

$$
\begin{aligned}
|W_r \cap W'_{2i-1}| &= \left|\left(\overline{A^{(r/2)m}} \cup B^{(r/2)m} \cup U\right) \cap \left(A^{s_1-im} \cup \overline{B^{s_1-(i-1)m}} \cup C \cup U\right)\right| \\
&= \left|\overline{A^{(r/2)m}} \cap A^{s_1-im}\right| + \left|B^{(r/2)m} \cap \overline{B^{s_1-(i-1)m}}\right| + |U| \\
&= \left|\overline{A^{(r/2)m}} \cap A^{(r-1-i)m+1}\right| + \left|B^{(r/2)m} \cap \overline{B^{(r-i)m+1}}\right| + |U|.
\end{aligned}
$$

Since $1 \leq i \leq \dfrac{r}{2}$, we have $(r-i)m+1 > \left(\dfrac{r}{2}\right)m$ and $\left|B^{(\frac{r}{2})m} \cap \overline{B^{(r-i)m+1}}\right| = 0$. If $i = r/2$, then $\left|\overline{A^{(r/2)m}} \cap A^{(r/2-1)m+1}\right| = 0$ and $|W_r \cap W'_{2i-1}| = |U| = s_2$, and so, $d(w_r, w_{2i-1}) = D$. If $i < r/2$, then $\left|\overline{A^{(r/2)m}} \cap A^{(r/2-1)m+1}\right| = (r/2-1-i)m+1$ and $|W_r \cap W'_{2i-1}| = |U| + (r/2-1-i)m+1 = |U|+1+(r/2-1-i)m = k-(r/2+i)m$. Therefore, for $1 \leq i < r/2$, we have $d(w_r, w_{2i-1}) = 2(r/2+i) = r+2i \geq r+2$.

If $\lfloor r/2 \rfloor < i \leq r$, then

$$
\begin{aligned}
&|W_r \cap W'_{2i-1}| \\
&\quad = \left|\left(\overline{A^{(r/2)m}} \cup B^{(r/2)m} \cup U\right) \cap \left(A^{s_1-(i-1)m-l} \cup \overline{B^{s_1-(i-2)m-l}} \cup C \cup U\right)\right|
\end{aligned}
$$

$$= \left|\overline{A^{(r/2)m}} \cap A^{s_1-(i-1)m-l}\right| + \left|B^{(r/2)m} \cap \overline{B^{s_1-(i-2)m-l}}\right| + |U|.$$

Since $r/2 < i$, we have $s_1 - (i-1)m - 1 = (r-i)m < (r/2)m$ so $\left|\overline{A^{(r/2)m}} \cap A^{s_1-(i-1)m-l}\right| = 0$ and $\left|B^{(r/2)m} \cap \overline{B^{s_1-(i-2)m-l}}\right| = (i - r/2 - 1)m$ and $|W_r \cap W'_{2i-1}| = k - ((3/2)r - i)\, m - 1$. Therefore, for $r/2 < i \leq r$, we have $d(w_r, w_{2i-1}) = 2\,((3/2)r - i) + 2 \geq r + 2$.

Consequently, $d\,(w_r, P' \cap V\,(K\,(2k+m, k))) = r + 1$ and so,

$$\begin{aligned}\delta\,(K\,(2k+m,k)) \geq \delta(T) &\geq d(w_r, [xw'_r] \cup [yw'_r]) \\ &= \min\{d(w_r, \{x,y\}), d(w_r, P' \cap V\,(K\,(2k+m,k))\} \\ &= \min\{r + 1/2, r+1\} = \frac{D+1}{2}.\end{aligned}$$

The argument with odd r is similar. □

Acknowledgements

We would like to thank the referee for him/her careful reading of the manuscript and several useful comments which have helped us to improve the presentation of the paper.

Author contributions All authors contributed equally in the creation and writing of this paper.

Funding For APC: Universidad Carlos III de Madrid (Agreement CRUE-Madroño 2024). Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. * Supported in part by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00/AEI/10.13039/501100011033), Spain.

• Supported in part by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

Declarations

Conflict of interest The authors declare no conflict of interest. The founding sponsors had no role in the design of the study; in the collection, analysis, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

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Received: September 9, 2022
Revised: September 10, 2023
Accepted: April 20, 2024