Aequat. Math.

https://doi.org/10.1007/s00010-024-01070-4

Aequationes Mathematicae

An explicit example of an iteration group in the ring of formal power series

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Abstract. We give an example of some iteration group in a ring of formal power series over a field of characteristic 0. It allows us to obtain an explicit formula for some one-parameter group of (truncated) formal power series under an additional condition. Consequently, we are able to show some non-commutative groups of solutions of the third Aczél-Jabotinsky differential equation in the ring of truncated formal power series.

Mathematics Subject Classification. Primary 13F25, 34M25; Secondary 13J05, 13H05.

Keywords. Formal power series, Translation equation, Aczél-Jabotinsky differential equation.

1. Introduction

Let $\boldsymbol{k}$ be a field of characteristic 0 with the prime field $\boldsymbol{q} \subset \boldsymbol{k}$ which is isomorphic to the field $\mathbb{Q}$ of all rational numbers. Assume that $(G, +)$ is a commutative group. For $s \in \mathbb{N} \cup \{\infty\}$ by $\boldsymbol{k}[\![x]\!]_s$ we denote the set

$$\left\{ \sum_{j=0}^{s} a_j x^j : a_j \in \boldsymbol{k} \ \text{ for } \ j \in \{0\} \cup \mathbb{N} \right\}.$$

If $s < \infty$ it is the ring of all s-truncated formal power series over $\boldsymbol{k}$. Otherwise $\boldsymbol{k}[\![x]\!]_\infty$ is the ring of all formal power series over $\boldsymbol{k}$, so we have $\boldsymbol{k}[\![x]\!] = \boldsymbol{k}[\![x]\!]_\infty$. More details about $\boldsymbol{k}[\![x]\!]_s$ are presented in the next section. Let $\Gamma^s \subset \boldsymbol{k}[\![x]\!]_s$ be the set of all s-truncated formal power series which are invertible with respect to substitution $\circ$ in $\boldsymbol{k}[\![x]\!]_s$. Clearly $(\Gamma^s, \circ)$ are groups for all $s \in \mathbb{N} \cup \{\infty\}$.

A non-empty family $\mathcal{F} = (F_t)_{t \in G} \subset \Gamma^s$ satisfying

$$F_{t_1+t_2} = F_{t_1} \circ F_{t_2} \qquad \text{for } \ t_1, t_2 \in G$$

is called a one-parameter group of (s-truncated) formal power series. A characterization of one-parameter groups of formal power series can be found among

Published online: 06 May 2024

Birkhäuser

others in [2]. In the case when $\mathcal{F} \ni F_t(x) = c_1(t)x + \sum_{j=2}^{s} c_j(t)x^j$ and either the set $\boldsymbol{F}_1 = \{c_1(t) \in \boldsymbol{k}^\star : t \in G\}$ is infinite or the family $\mathcal{F} = \{F_t : t \in G\}$ is finite, one can find $S \in \Gamma^s$ such that

$$F_t(x) = S^{-1}(c_1(t)S(x)) \qquad \text{for } \ t \in G.$$

The case when $\boldsymbol{F}_1$ is finite but $\mathcal{F}$ is infinite is much more complicated and no explicit form of such a group is known. A possible and known description uses sequences of polynomials defined recursively (see [2,3]).

It was proved in [3,5,6] that each element $\mathcal{F} \ni \Phi = F_{t_0}$ for $t_0 \in G$ of a one-parameter group $(F_t)_{t\in G}$ is a solution of the *third Aczél-Jabotinsky* formal differential equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \cdot H = (H \circ \Phi), \tag{1}$$

where $H(x) = \frac{\partial F_t}{\partial t}(x)|_{t=0}$ is the so-called *infinitesimal generator* of the group $(F_t)_{t\in G}$ (assuming that $(F_t)_{t\in G}$ is formally differentiable). In [3] all groups of solutions of (1) are described in the ring $\boldsymbol{k}[\![x]\!]_s$ over an arbitrary field $\boldsymbol{k}$ of characteristic 0. Those descriptions are based on recurrent constructions of two sequences of polynomials over $\boldsymbol{q}$. Earlier results (see [5]) were proved in the ring of formal power series (only the case $s = \infty$) over $\mathbb{C}$. It is known (see [3,5]) that for $s = \infty$ all possible groups of solutions of (1) are commutative. The situation for finite s is different (cf. [3]) and then also non-commutative groups of solutions appear.

Here we will construct some *two-parameter* family of formal power series. This will allow us to give explicit forms of groups of solutions of (1) for a specific form of the generator H. In particular cases we obtain also explicit forms of non-commutative groups of solutions of (1).

2. The rings of formal power series and truncated formal power series

In the ring $\boldsymbol{k}[\![x]\!]$ of formal power series $\sum_{j=0}^{\infty} c_j x^j$ over $\boldsymbol{k}$ we define the order of a formal power series by

$$\operatorname{ord}\left(\sum_{j=0}^{\infty} c_j x^j\right) = \min\{j \in \{0\} \cup \mathbb{N} : c_j \neq 0\},$$

where $\min\emptyset := \infty$. In the ideal $\mathfrak{m} = (x) = x\boldsymbol{k}[\![x]\!]$ of formal power series f with $\operatorname{ord} f \geq 1$ we define a substitution in the following way:

$$(f \circ g)(x) = \sum_{j=1}^{\infty} c_j \left(\sum_{l=1}^{\infty} d_l x^l\right)^j$$

for $f(x) = \sum_{j=1}^{\infty} c_j x^j \in \mathfrak{m}$ and $g(x) = \sum_{j=1}^{\infty} d_j x^j \in \mathfrak{m}$. Then f is invertible with respect to substitution if and only if $\operatorname{ord} f = 1$, whence,

$$\Gamma^{\infty} = \{ f \in \boldsymbol{k}[\![x]\!] : \operatorname{ord} f = 1 \}.$$

It is a group under substitution $\circ$ with unit element $L_1(x) = x$.

Let $s \in \mathbb{N}$ be a positive integer. The ring $\boldsymbol{k}[\![x]\!]_s$ of s-truncated formal power series is the quotient ring $\boldsymbol{k}[\![x]\!]/\mathfrak{m}^{s+1}$ where

$$\mathfrak{m}^{s+1} = x^{s+1}\boldsymbol{k}[\![x]\!] = \{ f \in \boldsymbol{k}[\![x]\!] : \operatorname{ord} f \geq s+1 \}.$$

To each coset $f + \mathfrak{m}^{s+1}$ with $f(x) = \sum_{j=0}^{\infty} c_j x^j \in \boldsymbol{k}[\![x]\!]$ we associate the s-truncation $f^{[s]}$ of f given by

$$f^{[s]}(x) := \sum_{j=0}^{s} c_j x^j \in \boldsymbol{k}[\![x]\!]_s \subset \boldsymbol{k}[x] \subset \boldsymbol{k}[\![x]\!].$$

In $\boldsymbol{k}[\![x]\!]_s$ we introduce operations of addition, multiplication and substitution in the following way:

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x), \\ (f_1 \cdot f_2)(x) &= (f_1 \cdot f_2)^{[s]}(x), \\ (f_1 \circ f_2)(x) &= (f_1 \circ f_2)^{[s]}(x) \end{aligned}$$

for $f_1, f_2 \in \boldsymbol{k}[\![x]\!]_s$. Then Γ^s is the set $\{ f \in \boldsymbol{k}[\![x]\!]_s : \operatorname{ord} f = 1 \}$. It is a group under substitution, with unit element L_1.

It is known that if $\pi_l^k : \Gamma^k \to \Gamma^l$ for $k \geq l$ are natural projections defined by l-truncation, then the group Γ^{∞} can be treated as the projective limit of $(\Gamma^s)_{s \in \mathbb{N}}$, that is $\Gamma^{\infty} = \lim_{\leftarrow} \Gamma^s$ with the canonical projections $\pi_l^{\infty} : \Gamma^{\infty} \to \Gamma^l$. Moreover, for $s \in \mathbb{N} \cup \{\infty\}$ we put $\Gamma_1^s := \ker \pi_1^s$.

For a fixed positive integer n by $\boldsymbol{E}_n \subset \boldsymbol{k}^{\star} := \boldsymbol{k} \setminus \{0\}$ we denote the set of all roots of order n of $1 \in \boldsymbol{k}$, that is the set of all roots of the polynomial $x^n - 1 \in \boldsymbol{k}[x]$ in $\boldsymbol{k}$. A root $c \in \boldsymbol{E}_n$ is called *primitive* of order $n \geq 2$ provided c is not a root of any polynomial $x^k - 1$ for $1 \leq k < n$. By a *semicanonical form* of order $l \in \mathbb{N}$ in Γ^s we mean any $f(x) = \sum_{j=0}^{r} c_{jl+1} x^{jl+1}$, where r is either the greatest positive integer with $rl + 1 \leq s$ for finite s, or $r = \infty$. Let $\mathcal{N}_l^s$ be the family of all semicanonical forms in Γ^s of order l and let $c \in \boldsymbol{E}_l$ be a primitive root of order l. Put $L_c(x) = cx \in \Gamma^s$. Then (see [1, Fact 2.2])

$$\mathcal{N}_l^s = \{ f \in \Gamma^s : f \circ L_c = L_c \circ f \},$$

and thus $\mathcal{N}_l^s$ is a subgroup of Γ^s. Note that $\mathcal{N}_1^s = \Gamma^s$.

3. Descriptions and properties of the substitution

We will need two descriptions of the substitution law in Γ^s. Fix $k, l \in \mathbb{Z}$ with $k \le l$. Put $|k,l| = \{n \in \mathbb{Z} : k \le n \le l\}$ and $|k,\infty| = \{n \in \mathbb{Z} : n \ge k\}$. We assume that $0^0 = 1$, $|k,l| = \emptyset$ for $k > l$, $\sum_{t\in\emptyset} a_t = 0$ and $\prod_{t\in\emptyset} a_t = 1$.

We begin with the following lemma, which is here an important tool in the construction of an iteration group given in the next section.

Lemma 1. (see [4]) *Fix $s \in \mathbb{N} \cup \{\infty\}$, $s \ge 2$. If $F_1(x) = \sum_{i=1}^{s} a_i x^i \in \boldsymbol{k}[\![x]\!]_s$, $F_2(x) = \sum_{i=1}^{s} b_i x^i \in \boldsymbol{k}[\![x]\!]_s$ and $(F_1 \circ F_2)(x) = \sum_{n=1}^{s} d_n x^n \in \boldsymbol{k}[\![x]\!]_s$, then*

$$d_n = \sum_{k=1}^{n} a_k \sum_{\overline{v}_k \in V_{k,n}} \prod_{j=1}^{k} b_{v_j} \qquad \text{for } n \in |1,s|, \tag{2}$$

for every positive integer n, where

$$V_{k,n} = \left\{ \overline{v}_k = (v_1, \ldots, v_k) \in |1,n|^k : \sum_{i=1}^{k} v_i = n \right\} \qquad \text{for } 1 \le k \le n.$$

For example, for $n = 1, 2, 3$, from (2) we get

$$d_1 = a_1 b_1, \quad d_2 = a_1 b_2 + a_2 b_1^2, \quad d_3 = a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3.$$

We prove now the characterization of substitution in the subgroup $\mathcal{N}_l^s$. For a fixed integer $l \ge 1$ we put $\mathbb{N}_l = \{j \in \mathbb{N} : j \equiv 1 \bmod l\}$.

Corollary 1. *Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$. If $F_1(x) = \sum_{j=0}^{r} a_{jl+1} x^{jl+1} \in \mathcal{N}_l^{rl+1}$ and $F_2(x) = \sum_{j=0}^{r} b_{jl+1} x^{jl+1} \in \mathcal{N}_l^{rl+1}$, then $(F_1 \circ F_2)(x) = \sum_{j=0}^{r} d_{jl+1} x^{jl+1} x \in \mathcal{N}_l^{rl+1}$ and*

$$d_{nl+1} = \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{\nu}_{kl+1} \in \widehat{V}^l_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{\nu_j l+1} \quad \text{for } n \in |1,r|, \tag{3}$$

where

$$\widehat{V}^l_{kl+1,nl+1} = \left\{ \overline{\nu}_{kl+1} = (\nu_1, \ldots, \nu_{kl+1}) \in |0, n-k|^{kl+1} : \sum_{j=1}^{kl+1} \nu_j = n-k \right\}$$

for $1 \le k \le n$.

Proof. Since $\mathcal{N}_l^{rl+1}$ is a subgroup of Γ^{rl+1}, consequently $(F_1 \circ F_2)(x) \in \mathcal{N}_l^{rl+1}$. In order to compute d_{nl+1} for $n \le r$, define

$$\widetilde{V}^l_{kl+1,nl+1} = \left\{ \overline{v}_{kl+1} = (v_1, \ldots, v_{kl+1}) \in \mathbb{N}_l^{kl+1} : \sum_{i=1}^{kl+1} v_i = nl+1 \right\}, \ k \in |0,n|.$$

It is a subset of $V_{kl+1,nl+1}$. We put $a_k = b_k = 0$ in (2) for $k \in |2, r| \setminus \mathbb{N}_l$. Since $(F_1 \circ F_2)(x) \in \mathcal{N}_l^{rl+1}$, so

$$d_{nl+1} = \sum_{k=1}^{nl+1} a_k \sum_{\overline{v}_k \in V_{k,nl+1}} \prod_{j=1}^{k} b_{v_j} = \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{v}_{kl+1} \in \widetilde{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{v_j},$$

Furthermore, for $\overline{v}_{kl+1} = (v_1, \ldots, v_{kl+1}) \in \widetilde{V}^l_{kl+1,nl+1}$ we put $v_j = \nu_j l + 1 \in \mathbb{N}_l$ with $\nu_j \in |0, n|$. Then

$$nl + 1 = \sum_{j=1}^{kl+1} (\nu_j l + 1) = l \sum_{j=1}^{kl+1} \nu_j + kl + 1,$$

hence $\sum_{j=1}^{kl+1} \nu_j = n - k$, thus $\nu_j \in |0, n-k|$ for all $j \in |1, kl+1|$. Finally,

$$\begin{aligned} d_{nl+1} &= \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{v}_{kl+1} \in \widetilde{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{v_j} \\ &= \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{\nu}_{kl+1} \in \widehat{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{\nu_j l+1}. \end{aligned}$$

4. The construction

Now, we construct a general example. For fixed $l \geq 1$ and $k \geq 0$ we define the so called l-fold factorial

$$(kl+1)!_l := \prod_{j=0}^{k} (jl+1),$$

assuming additionally $(-l+1)!_l := 1$. For $l = 1$ it coincides with the standard notion of factorial. Moreover, we introduce the following binary operation on $\boldsymbol{k}^\star \times \boldsymbol{k}$:

$$(y_1, y_2) \diamond (z_1, z_2) = (y_1 z_1, y_1 z_2 + y_2 z_1^{l+1}) \qquad \text{for } (y_1, y_2), (z_1, z_2) \in \boldsymbol{k}^\star \times \boldsymbol{k}.$$

Then $(\boldsymbol{k}^\star \times \boldsymbol{k}, \diamond)$ is a group isomorphic to $(\widehat{\Gamma}^{l+1}, \circ)$, where

$$\widehat{\Gamma}^{l+1} := \{c_1 x + c_{l+1} x^{l+1} \in \Gamma^{l+1} : c_1 \in \boldsymbol{k}^\star, c_{l+1} \in \boldsymbol{k}\}.$$

This group is non-commutative and $(\boldsymbol{E}_l \times \boldsymbol{k}, \diamond)$ is a commutative subgroup of $(\boldsymbol{k}^\star \times \boldsymbol{k}, \diamond)$. Observe that for $l = 1$ we have $\widehat{\Gamma}^2 = \Gamma^2$ as well as the family

$$\widehat{\Gamma}_1^{l+1} := \{x + c_{l+1} x^{l+1} \in \widehat{\Gamma}^{l+1} : c_{l+1} \in \boldsymbol{k}\}$$

is a commutative group which is isomorphic to $(\{1\} \times \boldsymbol{k}, \diamond) \cong (\boldsymbol{k}, +)$.

Proposition 1. *Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$. The family* $\left(F^{(l)}_{(z_1,z_2)}(x)\right)_{(z_1,z_2)\in \boldsymbol{k}^\star \times \boldsymbol{k}}$,

$$F^{(l)}_{(z_1,z_2)}(x) = \sum_{n=0}^{r} \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{z_2^n}{z_1^{n-1}} \right) x^{nl+1} \quad \text{for } (z_1,z_2) \in \boldsymbol{k}^\star \times \boldsymbol{k}, \tag{4}$$

is a non-commutative two-parameter iteration group in $\mathcal{N}_l^{rl+1}$ if and only if

$$\frac{((n-1)l+1)!_l}{(n-k)!((k-1)l+1)!_l} = \sum_{\overline{\nu}_{kl+1} \in \widehat{V}^l_{kl+1,nl+1}} \prod_{j=1}^{kl+1} \frac{((\nu_j-1)l+1)!_l}{\nu_j!} \tag{5}$$

holds true for all $n \in \mathbb{N}$ and $k \in |0,n|$.

Proof. Fix a positive integer l. We have to show that

$$F^{(l)}_{(y_1,y_2)\diamond(z_1,z_2)} = F^{(l)}_{(y_1,y_2)} \circ F^{(l)}_{(z_1,z_2)} \qquad \text{for } (y_1,y_2),(z_1,z_2) \in \boldsymbol{k}^\star \times \boldsymbol{k} \tag{6}$$

holds if and only if (5) is satisfied for $n \in \mathbb{N}$ and $k \in |0,n|$. Put

$$c_{nl+1}(z_1,z_2) = \frac{((n-1)l+1)!_l}{n!} \cdot \frac{z_2^n}{z_1^{n-1}} \qquad \text{for } (z_1,z_2) \in \boldsymbol{k}^\star \times \boldsymbol{k},\ n \in \{0\} \cup \mathbb{N}.$$

On account of Corollary 1 condition (6) is equivalent to

$$\begin{aligned}
&\sum_{n=0}^{r} c_{nl+1}(y_1z_1, y_1z_2 + y_2z_1^{l+1}) x^{nl+1} \\
&= \sum_{k=0}^{r} c_{kl+1}(y_1,y_2) \left(\sum_{j=0}^{r} c_{jl+1}(z_1,z_2) x^{jl+1} \right)^{kl+1} \\
&= \sum_{n=0}^{r} \left(\sum_{k=0}^{n} c_{kl+1}(y_1,y_2) \sum_{\overline{\nu}_{kl+1} \in \widehat{V}^l_{kl+1,nl+1}} \prod_{j=0}^{kl+1} c_{\nu_{jl+1}}(z_1,z_2) \right) x^{nl+1} \mod x^{rl+2}.
\end{aligned}$$

We have

$$\begin{aligned}
c_{nl+1}(y_1z_1, y_1z_2 + y_2z_1^{l+1}) &= \frac{((n-1)l+1)!_l}{n!} \frac{(y_1z_2 + y_2z_1^{l+1})^n}{(y_1z_1)^{n-1}} \\
&= \frac{((n-1)l+1)!_l}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{\left(y_2z_1^{l+1}\right)^k \cdot (y_1z_2)^{n-k}}{(y_1z_1)^{n-1}} \\
&= \sum_{k=0}^{n} \frac{((n-1)l+1)!_l}{k!(n-k)!} \frac{y_2^k}{y_1^{k-1}} \cdot \frac{z_2^{n-k}}{z_1^{n-(l+1)k-1}}.
\end{aligned}$$

Moreover, $\sum_{j=0}^{kl+1} \nu_j = n-k$ for $\overline{\nu}_{kl+1} = (\nu_1, \ldots, \nu_{kl+1}) \in \widehat{V}^l_{kl+1,nl+1}$, hence

$$\sum_{k=0}^{n} c_{kl+1}(y_1,y_2) \sum_{\overline{\nu}_{kl+1}\in \widehat{V}^l_{nl+1,kl+1}} \prod_{j=1}^{kl+1} c_{\nu_{jl+1}}(z_1,z_2)$$

$$= \sum_{k=0}^{n} \frac{((k-1)l+1)!_l}{k!} \frac{y_2^k}{y_1^{k-1}} \sum_{\overline{\nu}_{kl+1}\in \widehat{V}^l_{kl+1,nl+1}} \prod_{j=1}^{kl+1} \frac{((\nu_j-1)l+1)!_l}{\nu_j!} \frac{z_2^{\nu_j}}{z_1^{\nu_j-1}}$$

$$= \sum_{k=0}^{n} \left(\frac{((k-1)l+1)!_l}{k!} \cdot \sum_{\overline{\nu}_{kl+1}\in \widehat{V}^l_{kl+1,nl+1}} \prod_{j=1}^{kl+1} \frac{((\nu_j-1)l+1)!_l}{\nu_j!} \right) \frac{y_2^k}{y_1^{k-1}} \frac{z_2^{n-k}}{z_1^{n-(l+1)k-1}}.$$

Thus (6) is equivalent to the system (5) for every $n \in \mathbb{N}$ and $k \in |0,n|$. □

Remark 1. Note, that if $l=1$, (5) holds true for every $n \in \mathbb{N}$ and $k \in |0,n|$. It is a consequence of the equality

$$\sum_{\overline{\nu}_{k+1}\in \widehat{V}^1_{k+1,n+1}} 1 = \binom{n}{k} \qquad \text{for } n\in\mathbb{N},\, k\in|0,n|$$

(the number of all compositions of the number $n-k$ into $k+1$ non-negative integers, or, which is the same, the number of all compositions of the number $n+1$ onto $k+1$ positive integers).

Corollary 2. *Fix $r \in \mathbb{N}\cup\{\infty\}$, $l\in\mathbb{N}$. If the equalities (5) hold for $n\in\mathbb{N}$ and $k\in|0,n|$, then the iteration group $\left(F^{(l)}_{(z_1,z_2)}(x)\right)_{(z_1,z_2)\in \boldsymbol{k}^*\times \boldsymbol{k}}$ defined by (4) is isomorphic to $(\widehat{\Gamma}^{l+1},\circ)$.*

Proof. Observe that (see the proof of Proposition 1) the coefficient functions c_{nl+1} of the iteration group $\mathcal{F} = \left(F^{(l)}_{(z_1,z_2)}(x)\right)_{(z_1,z_2)\in \boldsymbol{k}^*\times \boldsymbol{k}}$ depend on two variables $(z_1,z_2)\in \boldsymbol{k}^*\times \boldsymbol{k}$. Moreover,

$$\pi^{rl+1}_{l+1}\left(F^{(l)}_{(z_1,z_2)}\right)(x) = z_1x + z_2x^{l+1} \in \widehat{\Gamma}^{l+1} \qquad \text{for } (z_1,z_2)\in \boldsymbol{k}^*\times \boldsymbol{k}.$$

This implies that the projection $\pi^{rl+1}_{l+1}|_{\mathcal{F}}$ is injective. Whence $\pi^{rl+1}_{l+1} : \mathcal{F} \to \widehat{\Gamma}^{l+1}$ is an isomorphism. □

Since $(\{1\}\times \boldsymbol{k},\diamond)$ is a subgroup of the group $(\boldsymbol{k}^*\times \boldsymbol{k},\diamond)$ and $(\{1\}\times \boldsymbol{k},\diamond)$ is isomorphic to $(\boldsymbol{k},+)$, from Proposition 1 and Corollary 2 one can derive the following result.

Corollary 3. *Fix $r\in\mathbb{N}\cup\{\infty\}$ and $l\in\mathbb{N}$. The family $\left(G^l_t\right)_{t\in \boldsymbol{k}}$,*

$$G^{(l)}_t(x) = F^{(l)}_{(1,t)}(x) = \sum_{n=0}^{r} \left(\frac{((n-1)l+1)!_l}{n!} \cdot t^n \right) x^{nl+1} \quad \text{for } t\in \boldsymbol{k}, \tag{7}$$

is a commutative one-parameter iteration group in $\mathcal{N}^{rl+1}_l$ if and only if (5) holds for $n\in\mathbb{N}$ and $k\in|0,n|$. It is isomorphic to $(\boldsymbol{k},+)$.

Remark 2. Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$ and assume that condition (5) holds for $n \in \mathbb{N}$ and $k \in |0, n|$. For the group $\left(G_t^l\right)_{t \in \boldsymbol{k}}$ we have

$$\frac{\partial G_t^{(l)}}{\partial t}(x) = \sum_{n=1}^{r} \left(\frac{((n-1)l+1)!_l}{(n-1)!} t^{n-1} \right) x^{nl+1} \quad \text{for } t \in \boldsymbol{k},$$

hence $H(x) = \frac{\partial G_t^{(l)}}{\partial t}(x)|_{t=0} = x^{l+1}$ is the infinitesimal generator of $\left(G_t^{(l)}\right)_{t \in \boldsymbol{k}}$.

It is known, that (5) is valid for $l = 1$ (see Remark 1). We show now that (5) also holds true for an arbitrary positive integer $l \geq 2$ and some values $k \in |0, n|$.

Lemma 2. *Condition* (5) *is trivially satisfied for* $k \in \{0, n\}$*. Moreover, it is valid for all* $n \in \mathbb{N}$ *and* $k \in |0, n|$*, for which* $n - k \leq 4$.

The proof of the above lemma is very technical and seems to be natural, but we present it for the convenience of the reader.

Proof of Lemma 2. For $k = n$ we have $\widehat{V}^l_{nl+1,nl+1} = \{(0, \dots, 0)\}$, whereas for $k = 0$ we have $\widehat{V}^l_{1,nl+1} = \{(n)\}$. Thus (5) is valid for $k \in \{0, n\}$.

For $k = n - 1$ and $\overline{\nu}_{(n-1)l+1} = (\nu_1, \dots, \nu_{(n-1)l+1}) \in \widehat{V}^l_{(n-1)l+1,nl+1}$ we have $\sum_{j=1}^{(n-1)l+1} \nu_j = n - (n-1) = 1$. There are $(n-1)l + 1$ sequences with one element equal to 1 and all remaining ones equal to 0. Hence

$$\begin{aligned} &((n-2)l+1)!_l \sum_{\overline{\nu}_{(n-1)l+1} \in \widehat{V}^l_{(n-1)l+1,nl+1}} \prod_{j=1}^{(n-1)l+1} \frac{((\nu_j - 1)l + 1)!_l}{(\nu_j)!} \\ &\quad = ((n-2)l+1)!_l \cdot ((n-1)l+1) \cdot 1 = ((n-1)l+1)!_l. \end{aligned}$$

Now, for $k = n - 2$ and $\overline{\nu}_{(n-2)l+1} = (\nu_1, \dots, \nu_{(n-2)l+1}) \in \widehat{V}^l_{(n-2)l+1,nl+1}$ exactly one of the following two possibilities holds:

(a) either one element of the sequence $\overline{\nu}_{(n-2)l+1}$ is equal to 2 and the remaining ones are equal to 0; there are $(n-2)l + 1$ such sequences,

(b) two elements of the sequence $\overline{\nu}_{(n-2)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-2)l+1}{2}$ such sequences.

Thus

$$\begin{aligned} &((n-3)l+1)!_l \sum_{\overline{\nu}_{(n-2)l+1} \in \widehat{V}^l_{(n-2)l+1,nl+1}} \prod_{j=1}^{(n-2)l+1} \frac{((\nu_j - 1)l + 1)!_l}{(\nu_j)!} \\ &= ((n-3)l+1)!_l \cdot \left(((n-2)l+1) \cdot \frac{l+1}{2} + \binom{(n-2)l+1}{2} \cdot 1 \right) \\ &= \frac{((n-1)l+1)!_l}{2!}. \end{aligned}$$

For $k = n-3$ and $\overline{\nu}_{(n-3)l+1} = (\nu_1, \ldots, \nu_{(n-3)l+1}) \in \widehat{V}^l_{(n-3)l+1,nl+1}$ exactly one of the following possibilities holds:

(a) one element of the sequence $\overline{\nu}_{(n-3)l+1}$ is equal to 3 and the remaining ones are equal to 0; there are $(n-3)l+1$ such sequences,
(b) one element of the sequence $\overline{\nu}_{(n-3)l+1}$ is equal to 2, another one is equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-3)l+1}{2} \cdot 2$ such sequences,
(c) three elements of the sequence $\overline{\nu}_{(n-3)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-3)l+1}{3}$ such sequences.

Then

$$((n-4)l+1)!_l \sum_{\overline{\nu}_{(n-3)l+1} \in \widehat{V}^l_{(n-3)l+1,nl+1}} \prod_{j=1}^{(n-3)l+1} \frac{((\nu_j - 1)l+1)!_l}{(\nu_j)!}$$

$$= ((n-4)l+1)!_l \cdot \Bigg(((n-3)l+1) \cdot \frac{(l+1)(2l+1)}{3!}$$

$$+ 2 \cdot \binom{(n-3)l+1}{2} \cdot \frac{l+1}{2} + \binom{(n-3)l+1}{3} \cdot 1 \Bigg) = \frac{((n-1)l+1)!_l}{3!}.$$

Finally, for $k = n-4$, $\overline{\nu}_{(n-4)l+1} = (\nu_1, \ldots, \nu_{(n-4)l+1}) \in \widehat{V}^l_{(n-4)l+1,nl+1}$ exactly one of the following possibilities holds:

(a) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 4 and the remaining ones are equal to 0; there are $(n-4)l+1$ such sequences,
(b) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 3, another one is equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{2} \cdot 2$ such sequences,
(c) two elements of the sequence $\overline{\nu}_{(n-4)l+1}$ are equal to 2 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{2}$ such sequences,
(d) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 2, two others are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{3} \cdot 3$ such sequences,
(e) four elements of the sequence $\overline{\nu}_{(n-4)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{4}$ such sequences.

Then

$$((n-5)l+1)!_l \sum_{\overline{\nu}_{(n-4)l+1}\in \widehat{V}^l_{(n-4)l+1,nl+1}} \prod_{j=1}^{(n-4)l+1} \frac{((\nu_j-1)l+1)!_l}{(\nu_j)!}$$

$$= ((n-5)l+1)!_l \cdot \Bigg(((n-4)l+1)\cdot \frac{(l+1)(2l+1)(3l+1)}{4!} + 2\cdot \binom{(n-4)l+1}{2}\cdot \frac{(l+1)(2l+1)}{3!} + \binom{(n-4)l+1}{2}\cdot \left(\frac{l+1}{2}\right)^2 + 3\cdot \binom{(n-4)l+1}{3}\cdot \frac{l+1}{2} + \binom{(n-4)l+1}{4}\cdot 1\Bigg) = \frac{((n-1)l+1)!_l}{4!}.$$

This completes the proof. □

Remark 3. Observe that on account of Lemma 2 condition (5) holds true for $n \le 5$ and $k \in |0,n|$.

Since (5) is always satisfied with $l=1$, we obtain what follows.

Corollary 4. *Fix $s \in \mathbb{N}\cup\{\infty\}$ with $s \ge 2$. Then*

$$F^{(1)}_{(z_1,z_2)}(x) = z_1x + z_2x^2 + \sum_{n=2}^{s} \frac{z_2^n}{z_1^{n-1}} x^{n+1} \qquad \textit{for } (z_1,z_2)\in \boldsymbol{k}^\star \times \boldsymbol{k},$$

is a non-commutative two-parameter iteration group of invertible formal power series. It is an injective embedding of the group Γ^2 into Γ^s. In particular,

$$G^{(1)}_t(x) = F^{(1)}_{(1,t)}(x) = x + tx^2 + \sum_{n=2}^{s} t^n x^{n+1} \qquad \textit{for } t\in \boldsymbol{k},$$

is a commutative one-parameter iteration group of formal power series over $\boldsymbol{k}$ with infinitesimal generator $H(x) = \frac{\partial G^{(1)}_t}{\partial t}(x)|_{t=0} = x^2$.

The group $\left(F^{(1)}_{(z_1,z_2)}\right)_{(z_1,z_2)\in \boldsymbol{k}^*\times \boldsymbol{k}}$ is isomorphic to Γ^2, whereas $\left(G^{(1)}_t\right)_{t\in\boldsymbol{k}}$ is isomorphic to $(\boldsymbol{k},+)$.

In order to describe solutions of some special case of the Aczél-Jabotinsky differential equation we need the following groups (see [3]). For $l,s\in\mathbb{N}$ with and $2l+1\le s$ let us consider the product $\boldsymbol{E}_l \times \boldsymbol{k}^{l+1}$ with an operation $\overline{\diamond} : (\boldsymbol{E}_l\times \boldsymbol{k}^{l+1}) \times (\boldsymbol{E}_l\times \boldsymbol{k}^{l+1}) \to \boldsymbol{E}_{l+1}\times \boldsymbol{k}^{l+1}$,

$$(c_1,(c_j)_{j\in\{l\}\cup|s-l+1,s|})\overline{\diamond}(d_1,(d_j)_{j\in\{l\}\cup|s-l+1,s|}) = (c_1d_1,(c_1d_j + d_1^jc_j)_{j\in\{l\}\cup|s-l+1,s|})$$

for $(c_1,(c_j)_{j\in\{l\}\cup|s-l+1,s|}),(d_1,(d_j)_{j\in\{l\}\cup|s-l+1,s|}) \in \boldsymbol{E}_l \times \boldsymbol{k}^{l+1}$. Similarly, if $l,s \in \mathbb{N}$ with $l+1 \le s \le 2l$, we use the product $\boldsymbol{E}_l \times \boldsymbol{k}^l$ with an operation $\widehat{\diamond} : (\boldsymbol{E}_l \times \boldsymbol{k}^l) \times (\boldsymbol{E}_l \times \boldsymbol{k}^l) \to \boldsymbol{E}_l \times \boldsymbol{k}^l$ defined by

$$(c_1,(c_j)_{j\in|s-l+1,s|})\widehat{\diamond}(d_1,(d_j)_{j\in|s-l+1,s|}) = (c_1d_1,(c_1d_j + d_1^j c_j)_{j\in|s-l+1,s|})$$

for $(c_1,(c_j)_{j\in|s-l+1,s|}),(d_1,(d_j)_{j\in|s-l+1,s|}) \in \boldsymbol{E}_l\times\boldsymbol{k}^l$. Observe that for $l \ge 2$ the groups $(\boldsymbol{E}_l\times\boldsymbol{k}^{l+1},\overline{\diamond})$ and $(\boldsymbol{E}_l\times\boldsymbol{k}^l,\widehat{\diamond})$ are not commutative provided $\{1\} \subsetneq \boldsymbol{E}_l$.

From [3] one can derive the following result.

Lemma 3. [3, Corollaries 5 and 6] *Fix $r \in \mathbb{N}\cup\{\infty\}$ and a positive integer l. Assume that $(G_t)_{t\in\boldsymbol{k}}$, $G_t(x) = x + tx^{l+1} + \sum_{j=2}^{r} c_{jl+1}(t)x^{jl+1} \in \mathcal{N}_l^{rl+1}$ with some $c_{jl+1} : \boldsymbol{k}\to\boldsymbol{k}$ for $j \in |2,r|$, is a one-parameter group of solutions of the differential equation*

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x}\cdot x^{l+1} = (\Phi(x))^{l+1} \tag{8}$$

in the ring $\boldsymbol{k}[\![x]\!]_s$, where either $rl+1 \le s < (r+1)l+1$ for finite r or $s=\infty$ otherwise.

(i) *For $s=\infty$ the family $(\widetilde{G}_{d,t})_{(d,t)\in\boldsymbol{E}_l\times\boldsymbol{k}}$,*

$$\widetilde{G}_{d,t} = dx + tx^{l+1} + \sum_{j=2}^{\infty} dc_{jl+1}(d^{-1}t)x^{jl+1}$$

is the group of all solutions of (8). It is isomorphic to $(\boldsymbol{E}_l\times\boldsymbol{k},\diamond)$.

(ii) *For $s \in |2l+1,\infty|$ the family $\left(\widetilde{G}_{d_1,t,d_{s-l+1},\ldots,d_s}\right)_{(d_1,t,d_{s-l+1},\ldots,d_s)\in\boldsymbol{E}_l\times\boldsymbol{k}^{l+1}}$ defined by*

$$\widetilde{G}_{d_1,t,d_{s-l+1},\ldots,d_s}(x) = d_1x + tx^{l+1} + \sum_{j=2}^{r-1} d_1c_{jl+1}(d_1^{-1}t)x^{jl+1} + \sum_{j=s-l+1}^{rl} d_jx^j + (d_1c_{rl+1}(d_1^{-1}t)+d_{rl+1})x^{rl+1} + \sum_{j=rl+2}^{s} d_jx^j,$$

is the group of all solutions of (8). It is isomorphic to $(\boldsymbol{E}_l\times\boldsymbol{k}^{l+1},\overline{\diamond})$.

(iii) *For $2 \le l+1 \le s \le 2l$ the family $\left(\widetilde{G}_{d_1,d_{s-l+1},\ldots,d_s}\right)_{(d_1,d_{s-l+1},\ldots,d_s)\in\boldsymbol{E}_l\times\boldsymbol{k}^l}$ defined by*

$$\widetilde{G}_{d_1,d_{s-l+1},\ldots,d_s}(x) = d_1x + \sum_{j=s-l+1}^{s} d_jx^j$$

is the group of all solutions of (8). It is isomorphic to $(\boldsymbol{E}_l\times\boldsymbol{k}^l,\widehat{\diamond})$.

Applying Corollary 4 we give an explicit form of the group of all solutions of the third Aczél-Jabotinsky formal differential equation (AJ) in the case $H(x) = x^2$, that is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \cdot x^2 = (\Phi(x))^2 . \tag{9}$$

Putting $l = 1$, thus $d = 1$, we obtain:

Corollary 5. (i) *The family* $(G_t^{(1)})_{t\in \boldsymbol{k}}$*,*

$$G_t^{(1)}(x) = x + tx^2 + \sum_{n=2}^{\infty} t^n x^{n+1} \qquad \textit{for } t \in \boldsymbol{k},$$

is the group of all solutions of (9) *for* $s = \infty$*. It is isomorphic to* $(\boldsymbol{k}, +)$ *and so commutative.*

(ii) *The family* $(\widehat{G}_{(t,c)}^{(1)})_{(t,c)\in \boldsymbol{k}^2}$*,*

$$\widehat{G}_{(t,c)}^{(1)}(x) = x + tx^2 + \sum_{n=2}^{s-2} t^n x^{n+1} + (c + t^{s-1})x^s \qquad \textit{for } (t, c) \in \boldsymbol{k}^2,$$

is the group of all solutions of (9) *for* $s \in \mathbb{N}$, $s \geq 3$*. It is isomorphic to* $(\boldsymbol{k}^2, +)$ *and so commutative.*

(iii) *The family* $\widehat{\Gamma}_1^2 = \{x + tx^2 : t \in \boldsymbol{k}\}$ *is the group of all solutions of* (9) *for* $s = 2$*. It is isomorphic to* $(\boldsymbol{k}, +)$ *and so commutative.*

According to Corollary 3, similar results for solutions of the formal differential equation (8) can be proved under the assumption that (5) holds true.

Corollary 6. *Fix an integer* $l \geq 2$ *and assume that* (5) *holds for* $n \in \mathbb{N} \cup \{0\}$ *and* $k \in |0, n|$*.*

(i) *The family* $(G_{(d,t)}^{(l)})_{(d,t)\in \boldsymbol{E}_l \times \boldsymbol{k}}$

$$\begin{aligned} G_{(d,t)}^{(l)}(x) &= \sum_{n=0}^{\infty} \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{t^n}{d^{n-1}} \right) x^{nl+1} \\ &= dx + tx^{l+1} + \sum_{n=2}^{\infty} \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{t^n}{d^{n-1}} \right) x^{nl+1} \ \textit{for } (d, t) \in \boldsymbol{E}_l \times \boldsymbol{k}, \end{aligned}$$

is the group of all solutions of (8) *for* $r = \infty$*. It is isomorphic to* $(\boldsymbol{E}_l \times \boldsymbol{k}, \diamond)$ *and so commutative.*

(ii) *The family* $(\widehat{G}_{(d_1,t,d_{s-l+1},\ldots,d_s)}^{(l)})_{(d_1,t,d_{s-l+1},\ldots,d_s)\in \boldsymbol{E}_l \times \boldsymbol{k}^{l+1}}$*,*

$$\widehat{G}^{(l)}_{(d_1,t,d_{s-l+1},\ldots,d_s)}(x) = d_1x + tx^{l+1} + \sum_{n=2}^{r-1} \frac{t^n}{d_1^{n-1}} x^{nl+1} + \sum_{j=s-l+1}^{rl} d_j x^j +$$

$$\left(d_{rl+1} + \frac{t^{rl+1}}{d_1^{rl}}\right) x^{rl+1} + \sum_{j=rl+2}^{s} d_j x^j \text{ for } (d_1, t, d_{s-l+1}, \ldots, d_s) \in \boldsymbol{E}_l \times \boldsymbol{k}^{l+1},$$

is the group of all solutions of (8) *for a finite integer* $s \geq 2l+1$, *where* $r \in \mathbb{N}$ *is such that* $rl+1 \leq s < (r+1)l+1$. *It is isomorphic to* $(\boldsymbol{E}_l \times \boldsymbol{k}^{l+1}, \overline{\diamond})$ *and so non-commutative provided* $\{1\} \subsetneq \boldsymbol{E}_l$.

(iii) *The family* $\{cx + c_{s-l+1}x^{s-l+1} + \ldots + c_s x^s : c \in \boldsymbol{E}_l, c_{s-l+1}, \ldots, c_s \in \boldsymbol{k}\}$ *is the group of all solutions of* (8) *for* $s \in |l+1, 2l|$, *which is isomorphic to* $(\boldsymbol{E}_l \times \boldsymbol{k}^l, \widehat{\diamond})$ *and so non-commutative provided* $\{1\} \subsetneq \boldsymbol{E}_l$.

Remark 4. We know that (5) holds true for all $n \in \mathbb{N}$ and $k \in |0, n|$. Since the proof of this fact uses a completely new approach, it will be proved in a separate paper.

Acknowledgements

We would like to express our gratitude to the anonymous referee whose remarks contributed to improve the present note substantially.

Author contributions The whole manuscript is prepared by myself.

Data availability Not applicable.

Declarations

Conflict of interest The author declares that he has no financial interests.

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Received: April 20, 2023
Revised: April 8, 2024
Accepted: April 9, 2024