Aequationes Mathematicae



An explicit example of an iteration group in the ring of formal power series

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Abstract. We give an example of some iteration group in a ring of formal power series over a field of characteristic 0. It allows us to obtain an explicit formula for some one-parameter group of (truncated) formal power series under an additional condition. Consequently, we are able to show some non-commutative groups of solutions of the third Aczél-Jabotinsky differential equation in the ring of truncated formal power series.

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1. Introduction

Let k be a field of characteristic 0 with the prime field $q \subset k$ which is isomorphic to the field \mathbb{Q} of all rational numbers. Assume that (G, +) is a commutative group. For $s \in \mathbb{N} \cup \{\infty\}$ by $k[\![x]\!]_s$ we denote the set

$$\left\{\sum_{j=0}^{s} a_j x^j : a_j \in \boldsymbol{k} \text{ for } j \in \{0\} \cup \mathbb{N}\right\}.$$

If $s < \infty$ it is the ring of all *s*-truncated formal power series over \boldsymbol{k} . Otherwise $\boldsymbol{k}[\![x]\!]_{\infty}$ is the ring of all formal power series over \boldsymbol{k} , so we have $\boldsymbol{k}[\![x]\!] = \boldsymbol{k}[\![x]\!]_{\infty}$. More details about $\boldsymbol{k}[\![x]\!]_s$ are presented in the next section. Let $\Gamma^s \subset \boldsymbol{k}[\![x]\!]_s$ be the set of all *s*-truncated formal power series which are invertible with respect to substitution \circ in $\boldsymbol{k}[\![x]\!]_s$. Clearly (Γ^s, \circ) are groups for all $s \in \mathbb{N} \cup \{\infty\}$.

A non-empty family $\mathcal{F} = (F_t)_{t \in G} \subset \Gamma^s$ satisfying

 $F_{t_1+t_2} = F_{t_1} \circ F_{t_2}$ for $t_1, t_2 \in G$

is called a one-parameter group of (s-truncated) formal power series. A characterization of one-parameter groups of formal power series can be found among

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others in [2]. In the case when $\mathcal{F} \ni F_t(x) = c_1(t)x + \sum_{j=2}^s c_j(t)x^j$ and either the set $\mathbf{F}_1 = \{c_1(t) \in \mathbf{k}^* : t \in G\}$ is infinite or the family $\mathcal{F} = \{F_t : t \in G\}$ is finite, one can find $S \in \Gamma^s$ such that

$$F_t(x) = S^{-1}(c_1(t)S(x))$$
 for $t \in G$.

The case when F_1 is finite but \mathcal{F} is infinite is much more complicated and no explicit form of such a group is known. A possible and known description uses sequences of polynomials defined recursively (see [2,3]).

It was proved in [3,5,6] that each element $\mathcal{F} \ni \Phi = F_{t_0}$ for $t_0 \in G$ of a one-parameter group $(F_t)_{t \in G}$ is a solution of the *third Aczél-Jabotinsky* formal differential equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \cdot H = (H \circ \Phi),\tag{1}$$

where $H(x) = \frac{\partial F_t}{\partial t}(x)|_{t=0}$ is the so-called *infinitesimal generator* of the group $(F_t)_{t\in G}$ (assuming that $(F_t)_{t\in G}$ is formally differentiable). In [3] all groups of solutions of (1) are described in the ring $\mathbf{k}[\![x]\!]_s$ over an arbitrary field \mathbf{k} of characteristic 0. Those descriptions are based on recurrent constructions of two sequences of polynomials over \mathbf{q} . Earlier results (see [5]) were proved in the ring of formal power series (only the case $s = \infty$) over \mathbb{C} . It is known (see [3,5]) that for $s = \infty$ all possible groups of solutions of (1) are commutative. The situation for finite s is different (cf. [3]) and then also non-commutative groups of solutions appear.

Here we will construct some *two-parameter* family of formal power series. This will allow us to give explicit forms of groups of solutions of (1) for a specific form of the generator H. In particular cases we obtain also explicit forms of non-commutative groups of solutions of (1).

2. The rings of formal power series and truncated formal power series

In the ring k[x] of formal power series $\sum_{j=0}^{\infty} c_j x^j$ over k we define the order of a formal power series by

ord
$$\left(\sum_{j=0}^{\infty} c_j x^j\right) = \min\{j \in \{0\} \cup \mathbb{N} : c_j \neq 0\},\$$

where $\min \emptyset := \infty$. In the ideal $\mathfrak{m} = (x) = x \mathbf{k} \llbracket x \rrbracket$ of formal power series f with ord $f \ge 1$ we define a substitution in the following way:

$$(f \circ g)(x) = \sum_{j=1}^{\infty} c_j \left(\sum_{l=1}^{\infty} d_l x^l\right)^j$$

for $f(x) = \sum_{j=1}^{\infty} c_j x^j \in \mathfrak{m}$ and $g(x) = \sum_{j=1}^{\infty} d_j x^j \in \mathfrak{m}$. Then f is invertible with respect to substitution if and only if ord f = 1, whence,

$$\Gamma^{\infty} = \{ f \in \boldsymbol{k}[\![x]\!] : \text{ord} f = 1 \}.$$

It is a group under substitution \circ with unit element $L_1(x) = x$.

Let $s \in \mathbb{N}$ be a positive integer. The ring $k[\![x]\!]_s$ of s-truncated formal power series is the quotient ring $k[\![x]\!]/\mathfrak{m}^{s+1}$ where

$$\mathfrak{m}^{s+1} = x^{s+1} \boldsymbol{k}[\![x]\!] = \{f \in \boldsymbol{k}[\![x]\!] : \text{ord} f \ge s+1\}.$$

To each coset $f + \mathfrak{m}^{s+1}$ with $f(x) = \sum_{j=0}^{\infty} c_j x^j \in \mathbf{k}[\![x]\!]$ we associate the s-truncation $f^{[s]}$ of f given by

$$f^{[s]}(x) := \sum_{j=0}^{s} c_j x^j \in \mathbf{k}[\![x]\!]_s \subset \mathbf{k}[\![x]\!] \subset \mathbf{k}[\![x]\!].$$

In $k[x]_s$ we introduce operations of addition, multiplication and substitution in the following way:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), (f_1 \cdot f_2)(x) = (f_1 \cdot f_2)^{[s]}(x), (f_1 \circ f_2)(x) = (f_1 \circ f_2)^{[s]}(x)$$

for $f_1, f_2 \in \mathbf{k}[\![x]\!]_s$. Then Γ^s is the set $\{f \in \mathbf{k}[\![x]\!]_s : \text{ord } f = 1\}$. It is a group under substitution, with unit element L_1 .

It is known that if $\pi_l^k : \Gamma^k \to \Gamma^l$ for $k \ge l$ are natural projections defined by *l*-truncation, then the group Γ^{∞} can be treated as the projective limit of $(\Gamma^s)_{s\in\mathbb{N}}$, that is $\Gamma^{\infty} = \lim_{\leftarrow} \Gamma^s$ with the canonical projections $\pi_l^{\infty} : \Gamma^{\infty} \to \Gamma^l$. Moreover, for $s \in \mathbb{N} \cup \{\infty\}$ we put $\Gamma_1^s := \ker \pi_1^s$.

For a fixed positive integer n by $\mathbf{E}_n \subset \mathbf{k}^* := \mathbf{k} \setminus \{0\}$ we denote the set of all roots of order n of $1 \in \mathbf{k}$, that is the set of all roots of the polynomial $x^n - 1 \in \mathbf{k}[x]$ in \mathbf{k} . A root $c \in \mathbf{E}_n$ is called *primitive* of order $n \ge 2$ provided c is not a root of any polynomial $x^k - 1$ for $1 \le k < n$. By a *semicanonical* form of order $l \in \mathbb{N}$ in Γ^s we mean any $f(x) = \sum_{j=0}^r c_{jl+1} x^{jl+1}$, where r is either the greatest positive integer with $rl + 1 \le s$ for finite s, or $r = \infty$. Let \mathcal{N}_l^s be the family of all semicanonical forms in Γ^s of order l and let $c \in \mathbf{E}_l$ be a primitive root of order l. Put $L_c(x) = cx \in \Gamma^s$. Then (see [1, Fact 2.2])

$$\mathcal{N}_l^s = \left\{ f \in \Gamma^s : f \circ L_c = L_c \circ f \right\},\$$

and thus \mathcal{N}_l^s is a subgroup of Γ^s . Note that $\mathcal{N}_1^s = \Gamma^s$.

3. Descriptions and properties of the substitution

We will need two descriptions of the substitution law in Γ^s . Fix $k, l \in \mathbb{Z}$ with $k \leq l$. Put $|k, l| = \{n \in \mathbb{Z} : k \leq n \leq l\}$ and $|k, \infty| = \{n \in \mathbb{Z} : n \geq k\}$. We assume that $0^0 = 1$, $|k, l| = \emptyset$ for k > l, $\sum_{t \in \emptyset} a_t = 0$ and $\prod_{t \in \emptyset} a_t = 1$.

We begin with the following lemma, which is here an important tool in the construction of an iteration group given in the next section.

Lemma 1. (see [4]) Fix $s \in \mathbb{N} \cup \{\infty\}$, $s \ge 2$. If $F_1(x) = \sum_{i=1}^s a_i x^i \in \mathbf{k}[\![x]\!]_s$, $F_2(x) = \sum_{i=1}^s b_i x^i \in \mathbf{k}[\![x]\!]_s$ and $(F_1 \circ F_2)(x) = \sum_{n=1}^s d_n x^n \in \mathbf{k}[\![x]\!]_s$, then

$$d_n = \sum_{k=1}^n a_k \sum_{\overline{v}_k \in V_{k,n}} \prod_{j=1}^k b_{v_j} \qquad \text{for } n \in |1, s|,$$

$$\tag{2}$$

for every positive integer n, where

$$V_{k,n} = \left\{ \overline{v}_k = (v_1, \dots, v_k) \in |1, n|^k : \sum_{i=1}^k v_i = n \right\} \quad \text{for } 1 \le k \le n.$$

For example, for n = 1, 2, 3, from (2) we get

$$d_1 = a_1b_1, \quad d_2 = a_1b_2 + a_2b_1^2, \quad d_3 = a_1b_3 + 2a_2b_1b_2 + a_3b_1^3$$

We prove now the characterization of substitution in the subgroup \mathcal{N}_l^s . For a fixed integer $l \ge 1$ we put $\mathbb{N}_l = \{j \in \mathbb{N} : j \equiv 1 \mod l\}$.

Corollary 1. Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$. If $F_1(x) = \sum_{j=0}^r a_{jl+1} x^{jl+1} \in \mathcal{N}_l^{rl+1}$ and $F_2(x) = \sum_{j=0}^r b_{jl+1} x^{jl+1} \in \mathcal{N}_l^{rl+1}$, then $(F_1 \circ F_2)(x) = \sum_{j=0}^r d_{jl+1} x^{jl+1} x \in \mathcal{N}_l^{rl+1}$ and

$$d_{nl+1} = \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{\nu}_{kl+1} \in \widehat{V}_{kl+1,nl+1}^{l}} \prod_{j=1}^{kl+1} b_{\nu_j l+1} \text{ for } n \in |1,r|,$$
(3)

where

$$\widehat{V}_{kl+1,nl+1}^{l} = \left\{ \overline{\nu}_{kl+1} = (\nu_1, \dots, \nu_{kl+1}) \in |0, n-k|^{kl+1} : \sum_{j=1}^{kl+1} \nu_j = n-k \right\}$$

for $1 \leq k \leq n$.

Proof. Since \mathcal{N}_l^{rl+1} is a subgroup of Γ^{rl+1} , consequently $(F_1 \circ F_2)(x) \in \mathcal{N}_l^{rl+1}$. In order to compute d_{nl+1} for $n \leq r$, define

$$\widetilde{V}_{kl+1,nl+1}^{l} = \left\{ \overline{v}_{kl+1} = (v_1, \dots, v_{kl+1}) \in \mathbb{N}_l^{kl+1} : \sum_{i=1}^{kl+1} v_i = nl+1 \right\}, \ k \in [0, n].$$

It is a subset of $V_{kl+1,nl+1}$. We put $a_k = b_k = 0$ in (2) for $k \in [2, r] \setminus \mathbb{N}_l$. Since $(F_1 \circ F_2)(x) \in \mathcal{N}_l^{rl+1}$, so

$$d_{nl+1} = \sum_{k=1}^{nl+1} a_k \sum_{\overline{v}_k \in V_{k,nl+1}} \prod_{j=1}^k b_{v_j} = \sum_{k=0}^n a_{kl+1} \sum_{\overline{v}_{kl+1} \in \widetilde{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{v_j},$$

Furthermore, for $\overline{v}_{kl+1} = (v_1, \dots, v_{kl+1}) \in \widetilde{V}_{kl+1, nl+1}^l$ we put $v_j = \nu_j l + 1 \in \mathbb{N}_l$ with $\nu_j \in [0, n]$. Then

$$nl + 1 = \sum_{j=1}^{kl+1} (\nu_j l + 1) = l \sum_{j=1}^{kl+1} \nu_j + kl + 1,$$

hence $\sum_{j=1}^{kl+1} \nu_j = n-k$, thus $\nu_j \in [0, n-k]$ for all $j \in [1, kl+1]$. Finally,

$$d_{nl+1} = \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{v}_{kl+1} \in \widetilde{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{v_j}$$
$$= \sum_{k=0}^{n} a_{kl+1} \sum_{\overline{v}_{kl+1} \in \widehat{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} b_{\nu_j l+1}.$$

4. The construction

Now, we construct a general example. For fixed $l \ge 1$ and $k \ge 0$ we define the so called *l*-fold factorial

$$(kl+1)!_l := \prod_{j=0}^k (jl+1),$$

assuming additionally $(-l+1)!_l := 1$. For l = 1 it coincides with the standard notion of factorial. Moreover, we introduce the following binary operation on $\mathbf{k}^* \times \mathbf{k}$:

$$(y_1, y_2) \diamond (z_1, z_2) = (y_1 z_1, y_1 z_2 + y_2 z_1^{l+1})$$
 for $(y_1, y_2), (z_1, z_2) \in \mathbf{k}^* \times \mathbf{k}$.

Then $(\mathbf{k}^{\star} \times \mathbf{k}, \diamond)$ is a group isomorphic to $(\widehat{\Gamma}^{l+1}, \circ)$, where

$$\widehat{\Gamma}^{l+1} := \{ c_1 x + c_{l+1} x^{l+1} \in \Gamma^{l+1} : c_1 \in \mathbf{k}^*, c_{l+1} \in \mathbf{k} \}.$$

This group is non-commutative and $(\boldsymbol{E}_l \times \boldsymbol{k}, \diamond)$ is a commutative subgroup of $(\boldsymbol{k}^* \times \boldsymbol{k}, \diamond)$. Observe that for l = 1 we have $\widehat{\Gamma}^2 = \Gamma^2$ as well as the family

$$\widehat{\Gamma}_{1}^{l+1} := \{ x + c_{l+1} x^{l+1} \in \widehat{\Gamma}^{l+1} : c_{l+1} \in \mathbf{k} \}$$

is a commutative group which is isomorphic to $(\{1\} \times \mathbf{k}, \diamond) \cong (\mathbf{k}, +)$.

Proposition 1. Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$. The family $\left(F_{(z_1, z_2)}^{(l)}(x)\right)_{(z_1, z_2) \in \mathbf{k}^* \times \mathbf{k}}$,

$$F_{(z_1,z_2)}^{(l)}(x) = \sum_{n=0}^r \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{z_2^n}{z_1^{n-1}} \right) x^{nl+1} \text{ for } (z_1,z_2) \in \mathbf{k}^* \times \mathbf{k}, \quad (4)$$

is a non-commutative two-parameter iteration group in \mathcal{N}_l^{rl+1} if and only if

$$\frac{((n-1)l+1)!_l}{(n-k)!((k-1)l+1)!_l} = \sum_{\overline{\nu}_{kl+1} \in \widehat{V}_{kl+1,nl+1}} \prod_{j=1}^{kl+1} \frac{((\nu_j - 1)l+1)!_l}{\nu_j!}$$
(5)

holds true for all $n \in \mathbb{N}$ and $k \in [0, n]$.

Proof. Fix a positive integer l. We have to show that

$$F_{(y_1,y_2)\diamond(z_1,z_2)}^{(l)} = F_{(y_1,y_2)}^{(l)} \circ F_{(z_1,z_2)}^{(l)} \quad \text{for } (y_1,y_2), (z_1,z_2) \in \boldsymbol{k}^* \times \boldsymbol{k} \quad (6)$$

holds if and only if (5) is satisfied for $n \in \mathbb{N}$ and $k \in [0, n]$. Put

$$c_{nl+1}(z_1, z_2) = \frac{((n-1)l+1)!_l}{n!} \cdot \frac{z_2^n}{z_1^{n-1}} \quad \text{for } (z_1, z_2) \in \boldsymbol{k}^* \times \boldsymbol{k}, n \in \{0\} \cup \mathbb{N}.$$

On account of Corollary 1 condition (6) is equivalent to

$$\sum_{n=0}^{r} c_{nl+1}(y_1 z_1, y_1 z_2 + y_2 z_1^{l+1}) x^{nl+1}$$

$$= \sum_{k=0}^{r} c_{kl+1}(y_1, y_2) \left(\sum_{j=0}^{r} c_{jl+1}(z_1, z_2) x^{jl+1} \right)^{kl+1}$$

$$= \sum_{n=0}^{r} \left(\sum_{k=0}^{n} c_{kl+1}(y_1, y_2) \sum_{\overline{\nu}_{kl+1} \in \widehat{V}_{kl+1,nl+1}^{l}} \prod_{j=0}^{kl+1} c_{\nu_{jl+1}}(z_1, z_2) \right) x^{nl+1} \mod x^{rl+2}.$$

We have

$$c_{nl+1}(y_1z_1, y_1z_2 + y_2z_1^{l+1}) = \frac{((n-1)l+1)!_l}{n!} \frac{(y_1z_2 + y_2z_1^{l+1})^n}{(y_1z_1)^{n-1}}$$

= $\frac{((n-1)l+1)!_l}{n!} \sum_{k=0}^n \binom{n}{k} \frac{(y_2z_1^{l+1})^k \cdot (y_1z_2)^{n-k}}{(y_1z_1)^{n-1}}$
= $\sum_{k=0}^n \frac{((n-1)l+1)!_l}{k!(n-k)!} \frac{y_2^k}{y_1^{k-1}} \cdot \frac{z_2^{n-k}}{z_1^{n-(l+1)k-1}}.$

Moreover, $\sum_{j=0}^{kl+1} \nu_j = n-k$ for $\overline{\nu}_{kl+1} = (\nu_1, \dots, \nu_{kl+1}) \in \widehat{V}_{kl+1, nl+1}^l$, hence

$$\sum_{k=0}^{n} c_{kl+1}(y_1, y_2) \sum_{\overline{\nu}_{kl+1} \in \hat{V}_{nl+1,kl+1}^l} \prod_{j=1}^{m+1} c_{\nu_{jl+1}}(z_1, z_2)$$

$$= \sum_{k=0}^{n} \frac{((k-1)l+1)!_l}{k!} \frac{y_2^k}{y_1^{k-1}} \sum_{\overline{\nu}_{kl+1} \in \hat{V}_{kl+1,nl+1}^l} \prod_{j=1}^{kl+1} \frac{((\nu_j - 1)l+1)!_l}{\nu_j!} \frac{z_2^{\nu_j}}{z_1^{\nu_j-1}}$$

$$= \sum_{k=0}^{n} \left(\frac{((k-1)l+1)!_l}{k!} \cdot \sum_{\overline{\nu}_{kl+1} \in \hat{V}_{kl+1,nl+1}^l} \prod_{j=1}^{kl+1} \frac{((\nu_j - 1)l+1)!_l}{\nu_j!} \right) \frac{y_2^k}{y_1^{k-1}} \frac{z_2^{n-k}}{z_1^{n-(l+1)k-1}}$$

Thus (6) is equivalent to the system (5) for every $n \in \mathbb{N}$ and $k \in [0, n]$. *Remark 1.* Note, that if l = 1, (5) holds true for every $n \in \mathbb{N}$ and $k \in [0, n]$. It is a consequence of the equality

$$\sum_{\overline{\nu}_{k+1} \in \widehat{V}^1_{k+1,n+1}} 1 = \binom{n}{k} \quad \text{for } n \in \mathbb{N}, \, k \in [0,n]$$

(the number of all compositions of the number n - k into k + 1 non-negative integers, or, which is the same, the number of all compositions of the number n + 1 onto k + 1 positive integers).

Corollary 2. Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$. If the equalities (5) hold for $n \in \mathbb{N}$ and $k \in [0, n]$, then the iteration group $\left(F_{(z_1, z_2)}^{(l)}(x)\right)_{(z_1, z_2) \in \mathbf{k}^* \times \mathbf{k}}$ defined by (4) is isomorphic to $(\widehat{\Gamma}^{l+1}, \circ)$.

Proof. Observe that (see the proof of Proposition 1) the coefficient functions c_{nl+1} of the iteration group $\mathcal{F} = \left(F_{(z_1,z_2)}^{(l)}(x)\right)_{(z_1,z_2)\in k^*\times k}$ depend on two variables $(z_1, z_2) \in k^* \times k$. Moreover,

$$\pi_{l+1}^{rl+1}\left(F_{(z_1,z_2)}^{(l)}\right)(x) = z_1 x + z_2 x^{l+1} \in \widehat{\Gamma}^{l+1} \qquad \text{for } (z_1,z_2) \in \boldsymbol{k}^* \times \boldsymbol{k}.$$

This implies that the projection $\pi_{l+1}^{rl+1}|_{\mathcal{F}}$ is injective. Whence $\pi_{l+1}^{rl+1}: \mathcal{F} \to \widehat{\Gamma}^{l+1}$ is an isomorphism.

Since $(\{1\} \times \mathbf{k}, \diamond)$ is a subgroup of the group $(\mathbf{k}^* \times \mathbf{k}, \diamond)$ and $(\{1\} \times \mathbf{k}, \diamond)$ is isomorphic to $(\mathbf{k}, +)$, from Proposition 1 and Corollary 2 one can derive the following result.

Corollary 3. Fix $r \in \mathbb{N} \cup \{\infty\}$ and $l \in \mathbb{N}$. The family $(G_t^l)_{t \in \mathbf{k}}$,

$$G_t^{(l)}(x) = F_{(1,t)}^{(l)}(x) = \sum_{n=0}^r \left(\frac{((n-1)l+1)!_l}{n!} \cdot t^n\right) x^{nl+1} \text{ for } t \in \mathbf{k}, \quad (7)$$

is a commutative one-parameter iteration group in \mathcal{N}_l^{rl+1} if and only if (5) holds for $n \in \mathbb{N}$ and $k \in [0, n]$. It is isomorphic to $(\mathbf{k}, +)$.

Remark 2. Fix $r \in \mathbb{N} \cup \{\infty\}$, $l \in \mathbb{N}$ and assume that condition (5) holds for $n \in \mathbb{N}$ and $k \in [0, n]$. For the group $(G_t^l)_{t \in k}$ we have

$$\frac{\partial G_t^{(l)}}{\partial t}(x) = \sum_{n=1}^r \left(\frac{((n-1)l+1)!_l}{(n-1)!} t^{n-1} \right) x^{nl+1} \text{ for } t \in \mathbf{k},$$

hence $H(x) = \frac{\partial G_t^{(l)}}{\partial t}(x)|_{t=0} = x^{l+1}$ is the infinitesimal generator of $\left(G_t^{(l)}\right)_{t\in k}$.

It is known, that (5) is valid for l = 1 (see Remark 1). We show now that (5) also holds true for an arbitrary positive integer $l \ge 2$ and some values $k \in [0, n]$.

Lemma 2. Condition (5) is trivially satisfied for $k \in \{0, n\}$. Moreover, it is valid for all $n \in \mathbb{N}$ and $k \in [0, n]$, for which $n - k \leq 4$.

The proof of the above lemma is very technical and seems to be natural, but we present it for the convenience of the reader.

Proof of Lemma 2. For k = n we have $\widehat{V}_{nl+1,nl+1}^{l} = \{(0,\ldots,0)\}$, whereas for k = 0 we have $\widehat{V}_{1,nl+1}^{l} = \{(n)\}$. Thus (5) is valid for $k \in \{0,n\}$.

For k = n - 1 and $\overline{\nu}_{(n-1)l+1} = (\nu_1, \dots, \nu_{(n-1)l+1}) \in \widehat{V}_{(n-1)l+1,nl+1}^l$ we have $\sum_{j=1}^{(n-1)l+1} \nu_j = n - (n-1) = 1$. There are (n-1)l + 1 sequences with one element equal to 1 and all remaining ones equal to 0. Hence

$$((n-2)l+1)!_l \sum_{\overline{\nu}_{(n-1)l+1} \in \widehat{V}_{(n-1)l+1,nl+1}^l} \prod_{j=1}^{(n-1)l+1} \frac{((\nu_j-1)l+1)!_l}{(\nu_j)!}$$

= $((n-2)l+1)!_l \cdot ((n-1)l+1) \cdot 1 = ((n-1)l+1)!_l.$

Now, for k = n - 2 and $\overline{\nu}_{(n-2)l+1} = (\nu_1, \dots, \nu_{(n-2)l+1}) \in \widehat{V}_{(n-2)l+1,nl+1}^l$ exactly one of the following two possibilities holds:

- (a) either one element of the sequence $\overline{\nu}_{(n-2)l+1}$ is equal to 2 and the remaining ones are equal to 0; there are (n-2)l+1 such sequences,
- (b) two elements of the sequence $\overline{\nu}_{(n-2)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-2)l+1}{2}$ such sequences.

Thus

$$\begin{split} &((n-3)l+1)!_l \sum_{\overline{\nu}_{(n-2)l+1} \in \widehat{V}_{(n-2)l+1,nl+1}^l} \prod_{j=1}^{(n-2)l+1} \frac{((\nu_j-1)l+1)!_l}{(\nu_j)!} \\ &= ((n-3)l+1)!_l \cdot \left(((n-2)l+1) \cdot \frac{l+1}{2} + \binom{(n-2)l+1}{2} \cdot 1 \right) \\ &= \frac{((n-1)l+1)!_l}{2!}. \end{split}$$

For k = n - 3 and $\overline{\nu}_{(n-3)l+1} = (\nu_1, \dots, \nu_{(n-3)l+1}) \in \widehat{V}_{(n-3)l+1,nl+1}^l$ exactly one of the following possibilities holds:

- (a) one element of the sequence $\overline{\nu}_{(n-3)l+1}$ is equal to 3 and the remaining ones are equal to 0; there are (n-3)l+1 such sequences,
- (b) one element of the sequence $\overline{\nu}_{(n-3)l+1}$ is equal to 2, another one is equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-3)l+1}{2} \cdot 2$ such sequences,
- (c) three elements of the sequence $\overline{\nu}_{(n-3)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-3)l+1}{3}$ such sequences.

Then

$$\begin{aligned} &((n-4)l+1)!_l \sum_{\overline{\nu}_{(n-3)l+1} \in \widehat{V}_{(n-3)l+1,nl+1}} \prod_{j=1}^{(n-3)l+1} \frac{((\nu_j-1)l+1)!_l}{(\nu_j)!} \\ &= ((n-4)l+1)!_l \cdot \left(((n-3)l+1) \cdot \frac{(l+1)(2l+1)}{3!} + 2 \cdot \binom{(n-3)l+1}{2} \cdot \frac{l+1}{2} + \binom{(n-3)l+1}{3} \cdot 1 \right) = \frac{((n-1)l+1)!_l}{3!}. \end{aligned}$$

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Finally, for k = n - 4, $\overline{\nu}_{(n-4)l+1} = (\nu_1, \dots, \nu_{(n-4)l+1}) \in \widehat{V}_{(n-4)l+1,nl+1}^l$ exactly one of the following possibilities holds:

- (a) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 4 and the remaining ones are equal to 0; there are (n-4)l+1 such sequences,
- (b) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 3, another one is equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{2} \cdot 2$ such sequences,
- (c) two elements of the sequence $\overline{\nu}_{(n-4)l+1}$ are equal to 2 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{2}$ such sequences,
- (d) one element of the sequence $\overline{\nu}_{(n-4)l+1}$ is equal to 2, two others are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{3} \cdot 3$ such sequences,
- (e) four elements of the sequence $\overline{\nu}_{(n-4)l+1}$ are equal to 1 and the remaining ones are equal to 0; there are $\binom{(n-4)l+1}{4}$ such sequences.

Then

$$((n-5)l+1)!_l \sum_{\overline{\nu}_{(n-4)l+1} \in \widehat{V}_{(n-4)l+1,nl+1}^l} \prod_{j=1}^{(n-4)l+1} \frac{((\nu_j-1)l+1)!_l}{(\nu_j)!}$$

$$= ((n-5)l+1)!_{l} \cdot \left(((n-4)l+1) \cdot \frac{(l+1)(2l+1)(3l+1)}{4!} + 2 \cdot \binom{(n-4)l+1}{2} \cdot \frac{(l+1)(2l+1)}{3!} + \binom{(n-4)l+1}{2} \cdot \left(\frac{l+1}{2}\right)^{2} + 3 \cdot \binom{(n-4)l+1}{3} \cdot \frac{l+1}{2} + \binom{(n-4)l+1}{4} \cdot 1 = \frac{((n-1)l+1)!_{l}}{4!}.$$

This completes the proof.

Remark 3. Observe that on account of Lemma 2 condition (5) holds true for $n \leq 5$ and $k \in [0, n]$.

Since (5) is always satisfied with l = 1, we obtain what follows.

Corollary 4. Fix $s \in \mathbb{N} \cup \{\infty\}$ with $s \ge 2$. Then

$$F_{(z_1,z_2)}^{(1)}(x) = z_1 x + z_2 x^2 + \sum_{n=2}^{s} \frac{z_2^n}{z_1^{n-1}} x^{n+1} \qquad for \ (z_1,z_2) \in \mathbf{k}^* \times \mathbf{k},$$

is a non-commutative two-parameter iteration group of invertible formal power series. It is an injective embedding of the group Γ^2 into Γ^s . In particular,

$$G_t^{(1)}(x) = F_{(1,t)}^{(1)}(x) = x + tx^2 + \sum_{n=2}^s t^n x^{n+1}$$
 for $t \in \mathbf{k}$,

is a commutative one-parameter iteration group of formal power series over \mathbf{k} with infinitesimal generator $H(x) = \frac{\partial G_t^{(1)}}{\partial t}(x)|_{t=0} = x^2$.

The group $\left(F_{(z_1,z_2)}^{(1)}\right)_{(z_1,z_2)\in \mathbf{k}^*\times\mathbf{k}}$ is isomorphic to Γ^2 , whereas $\left(G_t^{(1)}\right)_{t\in\mathbf{k}}$ is isomorphic to $(\mathbf{k},+)$.

In order to describe solutions of some special case of the Aczél-Jabotinsky differential equation we need the following groups (see [3]). For $l, s \in \mathbb{N}$ with and $2l + 1 \leq s$ let us consider the product $\mathbf{E}_l \times \mathbf{k}^{l+1}$ with an operation $\overline{\diamond} : (\mathbf{E}_l \times \mathbf{k}^{l+1}) \times (\mathbf{E}_l \times \mathbf{k}^{l+1}) \rightarrow \mathbf{E}_{l+1} \times \mathbf{k}^{l+1}$,

$$(c_1, (c_j)_{j \in \{l\} \cup |s-l+1,s|}) \overline{\diamond} (d_1, (d_j)_{j \in \{l\} \cup |s-l+1,s|})$$
$$= (c_1 d_1, (c_1 d_j + d_1^j c_j)_{j \in \{l\} \cup |s-l+1,s|})$$

differential equation

for $(c_1, (c_j)_{j \in \{l\} \cup |s-l+1,s|}), (d_1, (d_j)_{j \in \{l\} \cup |s-l+1,s|}) \in \mathbf{E}_l \times \mathbf{k}^{l+1}$. Similarly, if $l, s \in \mathbb{N}$ with $l+1 \leq s \leq 2l$, we use the product $\mathbf{E}_l \times \mathbf{k}^l$ with an operation $\widehat{\diamond} : (\mathbf{E}_l \times \mathbf{k}^l) \times (\mathbf{E}_l \times \mathbf{k}^l) \to \mathbf{E}_l \times \mathbf{k}^l$ defined by

$$(c_1, (c_j)_{j \in |s-l+1,s|}) \widehat{\diamond} (d_1, (d_j)_{j \in |s-l+1,s|}) = (c_1 d_1, (c_1 d_j + d_1^j c_j)_{j \in |s-l+1,s|})$$

for $(c_1, (c_j)_{j \in |s-l+1,s|}), (d_1, (d_j)_{j \in |s-l+1,s|}) \in \mathbf{E}_l \times \mathbf{k}^l$. Observe that for $l \ge 2$ the groups $(\mathbf{E}_l \times \mathbf{k}^{l+1}, \bar{\diamond})$ and $(\mathbf{E}_l \times \mathbf{k}^l, \hat{\diamond})$ are not commutative provided $\{1\} \subsetneq \mathbf{E}_l$. From [3] one can derive the following result.

Lemma 3. [3, Corollaries 5 and 6] Fix $r \in \mathbb{N} \cup \{\infty\}$ and a positive integer l. Assume that $(G_t)_{t \in k}$, $G_t(x) = x + tx^{l+1} + \sum_{j=2}^r c_{jl+1}(t)x^{jl+1} \in \mathcal{N}_l^{rl+1}$ with some $c_{jl+1} : \mathbf{k} \to \mathbf{k}$ for $j \in [2, r]$, is a one-parameter group of solutions of the

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \cdot x^{l+1} = \left(\Phi(x)\right)^{l+1} \tag{8}$$

in the ring $\mathbf{k}[\![x]\!]_s$, where either $rl + 1 \leq s < (r+1)l + 1$ for finite r or $s = \infty$ otherwise.

(i) For $s = \infty$ the family $(\widetilde{G}_{d,t})_{(d,t)\in E_l\times k}$,

$$\widetilde{G}_{d,t} = dx + tx^{l+1} + \sum_{j=2}^{\infty} dc_{jl+1} (d^{-1}t) x^{jl+1}$$

is the group of all solutions of (8). It is isomorphic to $(E_l \times k, \diamond)$.

(ii) For $s \in |2l+1, \infty|$ the family $\left(\widetilde{G}_{d_1,t,d_{s-l+1},\ldots,d_s}\right)_{(d_1,t,d_{s-l+1},\ldots,d_s)\in E_l\times k^{l+1}}$ defined by

$$\widetilde{G}_{d_1,t,d_{s-l+1},\dots,d_s}(x) = d_1 x + t x^{l+1} + \sum_{j=2}^{r-1} d_1 c_{jl+1} (d_1^{-1}t) x^{jl+1} + \sum_{\substack{j=s-l+1\\s}}^{rl} d_j x^j + (d_1 c_{rl+1} (d_1^{-1}t) + d_{rl+1}) x^{rl+1} + \sum_{\substack{j=rl+2\\s}}^{rl} d_j x^j,$$

is the group of all solutions of (8). It is isomorphic to $(E_l \times k^{l+1}, \bar{\diamond})$.

(iii) For $2 \leq l+1 \leq s \leq 2l$ the family $\left(\widetilde{G}_{d_1,d_{s-l+1},\ldots,d_s}\right)_{(d_1,d_{s-l+1},\ldots,d_s)\in \mathbf{E}_l\times\mathbf{k}^l}$ defined by

$$\widetilde{G}_{d_1,d_{s-l+1},...,d_s}(x) = d_1 x + \sum_{j=s-l+1}^s d_j x^j$$

is the group of all solutions of (8). It is isomorphic to $(\boldsymbol{E}_l \times \boldsymbol{k}^l, \widehat{\diamond})$.

Applying Corollary 4 we give an explicit form of the group of all solutions of the third Aczél-Jabotinsky formal differential equation (AJ) in the case $H(x) = x^2$, that is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} \cdot x^2 = \left(\Phi(x)\right)^2. \tag{9}$$

Putting l = 1, thus d = 1, we obtain:

Corollary 5. (i) The family $(G_t^{(1)})_{t \in k}$,

$$G_t^{(1)}(x) = x + tx^2 + \sum_{n=2}^{\infty} t^n x^{n+1}$$
 for $t \in \mathbf{k}$,

is the group of all solutions of (9) for $s = \infty$. It is isomorphic to $(\mathbf{k}, +)$ and so commutative.

(ii) The family $(\widehat{G}_{(t,c)}^{(1)})_{(t,c)\in k^2}$,

$$\widehat{G}_{(t,c)}^{(1)}(x) = x + tx^2 + \sum_{n=2}^{s-2} t^n x^{n+1} + (c+t^{s-1})x^s \qquad for \ (t,c) \in \mathbf{k}^2.$$

is the group of all solutions of (9) for $s \in \mathbb{N}$, $s \geq 3$. It is isomorphic to $(\mathbf{k}^2, +)$ and so commutative.

(iii) The family $\widehat{\Gamma}_1^2 = \{x + tx^2 : t \in \mathbf{k}\}$ is the group of all solutions of (9) for s = 2. It is isomorphic to $(\mathbf{k}, +)$ and so commutative.

According to Corollary 3, similar results for solutions of the formal differential equation (8) can be proved under the assumption that (5) holds true.

Corollary 6. Fix an integer $l \ge 2$ and assume that (5) holds for $n \in \mathbb{N} \cup \{0\}$ and $k \in [0, n]$.

(i) The family
$$(G_{(d,t)}^{(l)})_{(d,t)\in E_l\times k}$$

(1)

$$\begin{aligned} G_{(d,t)}^{(l)}(x) &= \sum_{n=0}^{\infty} \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{t^n}{d^{n-1}} \right) x^{nl+1} \\ &= dx + tx^{l+1} + \sum_{n=2}^{\infty} \left(\frac{((n-1)l+1)!_l}{n!} \cdot \frac{t^n}{d^{n-1}} \right) x^{nl+1} \text{ for } (d,t) \in \mathbf{E}_l \times \mathbf{k}, \end{aligned}$$

is the group of all solutions of (8) for $r = \infty$. It is isomorphic to $(\mathbf{E}_l \times \mathbf{k}, \diamond)$ and so commutative.

(ii) The family $(\widehat{G}_{(d_1,t,d_{s-l+1},\ldots,d_s)}^{(l)})_{(d_1,t,d_{s-l+1},\ldots,d_s)\in E_l\times k^{l+1}}$,

$$\widehat{G}_{(d_1,t,d_{s-l+1},\ldots,d_s)}^{(l)}(x) = d_1 x + t x^{l+1} + \sum_{n=2}^{r-1} \frac{t^n}{d_1^{n-1}} x^{nl+1} + \sum_{j=s-l+1}^{rl} d_j x^j + \left(d_{rl+1} + \frac{t^{rl+1}}{d_1^{rl}}\right) x^{rl+1} + \sum_{j=rl+2}^s d_j x^j \text{ for } (d_1,t,d_{s-l+1},\ldots,d_s) \in \mathbf{E}_l \times \mathbf{k}^{l+1},$$

is the group of all solutions of (8) for a finite integer $s \ge 2l + 1$, where $r \in \mathbb{N}$ is such that $rl + 1 \le s < (r+1)l + 1$. It is isomorphic to $(\mathbf{E}_l \times \mathbf{k}^{l+1}, \overline{\diamond})$ and so non-commutative provided $\{1\} \subsetneq \mathbf{E}_l$.

(iii) The family $\{cx + c_{s-l+1}x^{s-l+1} + \ldots + c_sx^s : c \in \mathbf{E}_l, c_{s-l+1}, \ldots, c_s \in \mathbf{k}\}$ is the group of all solutions of (8) for $s \in |l+1, 2l|$, which is isomorphic to $(\mathbf{E}_l \times \mathbf{k}^l, \widehat{\diamond})$ and so non-commutative provided $\{1\} \subsetneq \mathbf{E}_l$.

Remark 4. We know that (5) holds true for all $n \in \mathbb{N}$ and $k \in [0, n]$. Since the proof of this fact uses a completely new approach, it will be proved in a separate paper.

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References

- Jabłoński, W.: A survey on iterations in rings of formal power series in one indeterminate. In: Balcerzak, M., Jabłońska, E., Strobin, F. (eds.)Inspirations in Real Analysis: Będlewo, April 3–8, 2022, vol. 125, Banach Center Publications, pp. 45–70 (2023)
- [2] Jabłoński, W., Reich, L.: A new approach to the description of one-parameter groups of formal power series in one indeterminate. Acqu. Math. 87, 247–284 (2014)
- [3] Jabłoński, W., Reich, L.: An algebraic structure of groups of solutions of the third Aczél-Jabotinsky formal differential equation. arXiv:2304.06791
- [4] Moszner, Z.: Sur un sous-groupe à un paramètre du groupe $L^1_s.$ Opusc. Math. 6, 149–155(1990)
- [5] Reich, L.: On a differential equation arrising in iteration theory in rings of formal power series in one variable. In: Lecture Notes in Mathematics, vol. 1163. Springer, Berlin, Iteration Theory and its Functional Equations (1985)
- [6] Reich, L.: On families of commuting formal power series, selected topics in functional equations. In: Proceedings of a seminar held in Graz, May 22–23 (1986). Grazer Math. Ber. 285-296, Ber. Nr. 294 (1988)

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