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On some classes of multiplicative functions

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Abstract. An arithmetical function f is multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are coprime. We study connections between certain subclasses of multiplicative functions, such as strongly multiplicative functions, over-multiplicative functions and totients. It appears, among others, that the over-multiplicative functions are exactly same as the totients and the strongly multiplicative functions are exactly same as the so-called level totients. All these functions satisfy nice arithmetical identities which are recursive in character.

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1. Introduction

An arithmetical function is a complex valued function defined on the set of positive integers. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(mn) = f(m)f(n) \tag{1}$$

whenever $(m, n) = 1$. Multiplicative functions constitute perhaps the most important class of arithmetical functions. There are in the literature various superclasses and subclasses of multiplicative functions, see e.g. [7,10,12,14,15]

A multiplicative function f is completely multiplicative if (1) holds for all m, n. The power function $N_k(n) = n^k$ is an example of completely multiplicative functions. The function λ_k is another example of completely multiplicative functions, where $\lambda_k(n) = k^{\Omega(n)}$ and $\Omega(n)$ is the total number of prime factors of n with $\Omega(1) = 0$. See [16].

A multiplicative function f is strongly multiplicative if $f(p^a) = f(p)$ for all primes p and integers a (≥ 1), see [11,12]. The function E_k is an example of

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strongly multiplicative functions, where $E_k(n) = k^{\omega(n)}$ and $\omega(n)$ is the number of distinct prime factors of n with $\omega(1) = 0$.

A multiplicative function is over-multiplicative if there exists an arithmetical function F with $F(1) = 1$ such that

$$f(mn) = f(m)f(n)F((m,n)) \tag{2}$$

for all positive integers m, n, see [12]. Euler's totient function $\phi(n)$ is defined as the number of integers $x \pmod n$ with $(x, n) = 1$. Euler's totient function ϕ possesses the property

$$\phi(mn)\phi((m,n)) = \phi(m)\phi(n)(m,n) \tag{3}$$

for all positive integers m, n, see [2]. Dedekind's totient $\psi(n)$ is defined as

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where the product is over the distinct primes p dividing n. Dedekind's totient ψ satisfies the arithmetical equation

$$\psi(mn)\psi((m,n)) = \psi(m)\psi(n)(m,n) \tag{4}$$

for all positive integers m, n, see [8]. Therefore the functions ϕ and ψ are over-multiplicative with $F(n) = n/\phi(n)$ and $F(n) = n/\psi(n)$.

Equation (2) is closely related to

$$f(mn)f((m,n)) = f(m)f(n)g((m,n)), \tag{5}$$

see [3,8]. We consider this equation at the end of this paper.

The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1) \neq 0$. The Dirichlet inverse of a completely multiplicative function f is of the form $f^{-1} = \mu f$, where μ is the Möbius function.

A multiplicative function f is said to be a totient if there exist completely multiplicative functions f_t and f_v such that $f = f_t \star f_v^{-1}$. See [6,10,14,16]. Totients can be characterized with various arithmetical equations, see [6]. For example, an arithmetical function f is a totient if and only if there is a completely multiplicative function h such that

$$f(mn) = f(m) \sum_{\substack{d|n \\ \gamma(d)|m}} f(n/d)h(d) \tag{6}$$

for all positive integers m and n, where γ is the strongly multiplicative function with $\gamma(p) = p$ for all primes p. In this case $f_v = h$.

It is well known that Euler's totient function ϕ can be written as

$$\phi = N * \mu = N * \zeta^{-1},$$

where $N(n) = n$ and $\zeta(n) = 1$ for all positive integers n. Thus ϕ is a totient in the sense of the above definition with $\phi_t = N$ and $\phi_v = \zeta$.

Dedekind's totient ψ can be written as $\psi = N*|\mu|$. It is another example of a totient, since $|\mu| = \lambda^{-1}$, where $|\mu|(n) = |\mu(n)|$ and λ is Liouville's function, which is a completely multiplicative function such that $\lambda(p) = -1$ for all primes p. Note that $\lambda = \lambda_{-1}$.

A totient f is said to be a level totient if $f_t = \zeta$. See [6,16]. The functions E_k are examples of level totients. In fact, it can be verified that $E_k = E_1 \star \lambda_{1-k}^{-1} = \zeta \star \lambda_{1-k}^{-1}$. See [16].

Totients belong to the class of rational arithmetical functions. In fact, totients are rational arithmetical functions of order $(1, 1)$. See [9,16].

We denote by $\mathcal{C}, \mathcal{S}, \mathcal{O}, \mathcal{T}$, and $\mathcal{L}$, respectively, the class of completely multiplicative functions, the class of strongly multiplicative functions, the class of over-multiplicative functions, the class of totients, and the class of level totients. The symbol $\mathcal{CL}$ refers to the class of usual products of completely multiplicative functions and level totients. For a class $\mathcal{A}$ of arithmetical functions let $\mathcal{A}^\bullet$ denote the class of those arithmetical functions $f \in \mathcal{A}$ for which $f(n) \neq 0$ for all n. In this paper we show that $\mathcal{S} = \mathcal{L}$, $\mathcal{O} = \mathcal{T}$, $\mathcal{L} \subsetneq \mathcal{CL} \subsetneq \mathcal{T}$ and $\mathcal{L}^\bullet \subsetneq (\mathcal{CL})^\bullet = \mathcal{T}^\bullet$.

2. Results

Theorem 2.1. *$\mathcal{S} = \mathcal{L}$.*

Proof. Suppose that $f \in \mathcal{S}$. Then $f(p^a) = f(p)$ for all primes p and integers a (≥ 1). Let f_v be a completely multiplicative function such that $f_v(p) = 1 - f(p)$ for all primes p. Then $(\zeta \star f_v^{-1})(p^a) = (\zeta \star (\mu f_v))(p^a) = 1 - f_v(p) = f(p) = f(p^a)$ for all primes p and integers a (≥ 1). Thus $f = \zeta \star f_v^{-1}$, which means that $f \in \mathcal{L}$.

Suppose that $f \in \mathcal{L}$. Then for all primes p and all integers $a \geq 1$, $f(p^a) = (\zeta \star f_v^{-1})(p^a) = (\zeta \star (\mu f_v))(p^a) = 1 - f_v(p)$, which does not depend on a. Thus $f(p^a) = f(p)$, that is, $f \in \mathcal{S}$. □

Proposition 2.1. (See [6]) *A multiplicative function f is a totient if and only if for each prime p there exists a complex number $z(p)$ such that*

$$f(p^a) = f(p)\,[z(p)]^{a-1} \tag{7}$$

for all $a \geq 1$. In this case $z(p) = f_t(p)$.

Theorem 2.2. $\mathcal{O} = \mathcal{T}$.

Proof. Suppose that $f \in \mathcal{O}$. Then there exists an arithmetical function F such that $f(mn) = f(m)f(n)F((m,n))$ for all m, n. Let $m = p^{a-1}$ and $n = p$, where p is a prime and a is an integer (≥ 2). Thus $f(p^a) = f(p^{a-1})f(p)F(p)$. Applying this recursion we obtain $f(p^a) = f(p)\,[f(p)F(p)]^{a-1}$ for all primes p and integers a (≥ 1). Thus, according to Proposition 2.1, $f \in \mathcal{T}$.

Suppose that $f \in \mathcal{T}$. Then, according to Proposition 2.1,

$$f(p^a) = f(p)\,[f_t(p)]^{a-1} \tag{8}$$

for all primes p and integers a (≥ 1). Let F be a multiplicative function such that

$$F(p^a) = \begin{cases} \frac{f_t(p)}{f(p)} & \text{if } f(p) \neq 0, \\ 0 & \text{if } f(p) = 0 \end{cases} \tag{9}$$

for all primes p and integers a (≥ 1). We show that (2) holds. Since f and F are multiplicative, it suffices to show that

$$f(p^{a+b}) = f(p^a)f(p^b)F(p^{\min\{a,b\}}) \tag{10}$$

for all primes p and integers a, b (≥ 0). If $a = 0$ or $b = 0$, then (10) holds. Suppose that $a \neq 0$ and $b \neq 0$. We distinguish two cases: $f(p) = 0$, $f(p) \neq 0$.

If $f(p) = 0$, then, according to (8), $f(p^{a+b}) = f(p^a) = f(p^b) = 0$, and thus (10) holds. If $f(p) \neq 0$, then, according to (8) and (9),

$$\begin{aligned} f(p^{a+b}) &= f(p)\,[f_t(p)]^{a+b-1} = f(p)\,[f_t(p)]^{a-1}\,f(p)\,[f_t(p)]^{b-1}\,\frac{f_t(p)}{f(p)} \\ &= f(p^a)f(p^b)F(p^{\min\{a,b\}}), \end{aligned}$$

and thus (10) holds.

Now, we have proved that (10) holds. Thus (2) holds, that is, $f \in \mathcal{O}$. □

Remark. It is easy to see that Equations (1)–(6) are recursive in character. For example, for a recursive character of Equation (2), see the proof of Theorem 2.2. The function values are totally determined by certain "initial values". It is easy to see and well known that a multiplicative function is totally determined by its values at prime powers, and a completely multiplicative function is totally determined by its values at primes. A strongly multiplicative function is likewise totally determined by its values at primes. According to Proposition 2.1, a totient f is totally determined by the values of f and f_t at primes. It can be shown that a totient f is also totally determined by the values of f and f_v at primes or by the values of f_t and f_v at primes. A level totient f is totally determined by the values of f (or f_v) at primes. From the proof of Theorem 2.2 we see that an over-multiplicative function f is totally determined by the values of f and F at primes.

Theorem 2.3. $\mathcal{L} \subsetneq \mathcal{CL} \subsetneq \mathcal{T}$.

Proof. Since $\zeta \in \mathcal{C}$, it follows that $\mathcal{L} \subseteq \mathcal{CL}$. It is clear that $\phi \notin \mathcal{S} = \mathcal{L}$, by Theorem 2.1. However, $\phi = N \star \mu = N(\zeta \star \mu \frac{1}{N})$, where $N, \frac{1}{N} \in \mathcal{C}$. Thus $\phi \in \mathcal{CL}$ and therefore $\mathcal{L}$ is a proper subclass of $\mathcal{CL}$.

Assume that $f \in \mathcal{CL}$. Then $f = g(\zeta \star \mu h)$, where $g, h \in \mathcal{C}$. Thus $f = g \star (\mu g h)$, where $g, gh \in \mathcal{C}$, and therefore $f \in \mathcal{T}$. This proves that $\mathcal{CL} \subseteq \mathcal{T}$. Next we show that $\mu \in (\mathcal{T} \setminus \mathcal{CL})$. Since $\mu = \delta \star \mu \zeta$, where $\delta, \zeta \in \mathcal{C}$, we have $\mu \in \mathcal{T}$. Assume that $\mu \in \mathcal{CL}$, that is, $\mu \in \mathcal{CS}$, by Theorem 2.1. Then $\mu = gh$, where $g \in \mathcal{C}, h \in \mathcal{S}$, and thus for each prime p, $g(p)h(p) = -1$ and $g(p^2)h(p^2) = g(p)^2 h(p) = 0$, which is impossible. Therefore $\mu \notin \mathcal{CL}$. So we have proved that $\mu \in (\mathcal{T} \setminus \mathcal{CL})$ and further that $\mathcal{CL}$ is a proper subclass of $\mathcal{T}$. $\square$

Theorem 2.4. $\mathcal{L}^\bullet \subsetneq (\mathcal{CL})^\bullet = \mathcal{T}^\bullet$.

Proof. From Theorem 2.3 we can conclude that $\mathcal{L}^\bullet \subseteq (\mathcal{CL})^\bullet$. Since $\phi \in (\mathcal{CL})^\bullet \setminus \mathcal{L}^\bullet$, we see that $\mathcal{L}^\bullet$ is a proper subclass of $(\mathcal{CL})^\bullet$.

From Theorem 2.3 we also can conclude that $(\mathcal{CL})^\bullet \subseteq \mathcal{T}^\bullet$. We prove that $\mathcal{T}^\bullet \subseteq (\mathcal{CL})^\bullet$. Assume that $f \in \mathcal{T}^\bullet$, that is, $f \in \mathcal{T}$ and $f(n) \neq 0$ for all n. Since $f(p^a) = f_t(p^a) - f_t(p^{a-1}) f_v(p) = f_t(p)^{a-1}(f_t(p) - f_v(p))$, we see that $f_t(p) \neq 0$ for all primes p. It is thus possible to define a completely multiplicative function g as $g(p) = f_v(p)/f_t(p)$ for all primes p. Then $[f_t(\zeta \star \mu g)](p^a) = f_t(p)^a[1 - f_v(p)/f_t(p)] = f_t(p^a) - f_t(p^{a-1}) f_v(p) = f(p^a)$ for all primes p and integers a (≥ 1). Thus $f = f_t(\zeta \star \mu g) \in (\mathcal{CL})^\bullet$. This proves that $\mathcal{T}^\bullet \subseteq (\mathcal{CL})^\bullet$ and further that $(\mathcal{CL})^\bullet = \mathcal{T}^\bullet$. $\square$

Remark. A problem related to Equation (2) is to characterize the arithmetical functions f with $f(1) = 1$ satisfying Equation (5) for all positive integers m, n, where g is a completely multiplicative function. In fact, Apostol and Zuckerman [3] have shown that an arithmetical function f with $f(1) = 1$ satisfies (5) if and only if f is multiplicative and

$$f(p^{a+b}) f(p^b) = f(p^a) f(p^b) g(p^b) \tag{11}$$

for all primes p and integers $a \geq b \geq 1$. Apostol and Zuckerman [3] assume that g is a completely multiplicative function. Their result holds even more generally, namely if g is a multiplicative function, see [13].

We obtain a more illustrative result if we assume that f possesses Property O which is defined as follows: an arithmetical function f satisfies *Property O* if for each prime p, $f(p) = 0$ implies $f(p^a) = 0$ for all $a > 1$. Under this condition, (5) is a characterization of totients if g is a completely multiplicative function. See [8]. If f is always nonzero, then (5) reduces to (2) with $F = g/f$ and again, (5) is a characterization of totients.

Equation (5) has been studied in [1,2,5,6]. For further material relating to this type of equations we refer to [4,13].

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Declarations

Conflict of interest The authors declare no competing interests.

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