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On when the union of two algebraic sets is algebraic

Erhard Aichinger, Mike Behrisch, and Bernardo Rossi

Abstract. In universal algebraic geometry, an algebra is called an *equational domain* if the union of two algebraic sets is algebraic. We characterize equational domains, with respect to polynomial equations, inside congruence permutable varieties, and with respect to term equations, among all algebras of size two and all algebras of size three with a cyclic automorphism. Furthermore, for each size at least three, we prove that, modulo term equivalence, there is a continuum of equational domains of that size.

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1. Introduction

A basic fact in classical algebraic geometry is that the union of two algebraic sets is again algebraic. In universal algebraic geometry, which studies the algebraic sets over an arbitrary algebraic structure [11,19,40], this is no longer true in general. In [18], algebras with the property that the union of two algebraic sets is algebraic have been called *equational domains*; for example, a commutative ring with unity is an equational domain if and only it is an integral domain. In such algebras, the non-empty algebraic sets coincide with the non-empty closed sets of a topology, which is called *Zariski topology* [18] as in classical algebraic geometry.

In this paper, we seek to characterize equational domains. A first observation is that every equational domain is finitely subdirectly irreducible. For a more detailed study, we need to specify whether the equations defining algebraic sets involve *term functions* or *polynomial functions*; the difference lies in

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whether constants from the algebra are allowed. In [18], an algebraic set is defined as the solution set of a system of *term equations*; we follow this viewpoint and treat polynomial equations by passing from an algebra to its expansion with all constant operations. From the fundamental result [18, Theorem 2.5], we see that if an algebra is an equational domain, then so is its expansion with all constant operations; in other words, if it is an equational domain with respect to term functions, then it is an equational domain with respect to polynomial functions. The converse is not true, as witnessed, e.g., by the alternating group A_5 (cf. [18, Proposition 2.19, Claim 2.22(4), Corollary 2.31]). For algebras in congruence permutable varieties, we obtain a structural description of those algebras that are equational domains with respect to polynomial equations. Our description uses a generalization of the ideal product in rings to universal algebra, the binary commutator [23,35,44]. The equational domains inside congruence permutable varieties, with respect to polynomials, are then those algebras $\mathbf{A}$ with at least two elements that satisfy $[\alpha, \beta] > 0_A$ for all congruences $\alpha, \beta > 0_A$ (Theorem 4.8); for a finite algebra this is equivalent to saying that the algebra is subdirectly irreducible with non-Abelian monolith. In each of these finite algebras, every subset of A^n is algebraic, hence they all have the same collection of algebraic sets, in other words, they are *algebraically equivalent* (cf. [38]).

One can view being an equational domain as a property of the clone of term functions of an algebra. Following [38] we say that a clone $\mathcal{C}$ on A is *equationally additive* if $(A; \mathcal{C})$ is an equational domain. When $\mathcal{C}$ does not contain all constant operations, we do not have a complete description of equationally additive clones, even when they contain a Mal'cev operation. One difficulty in finding a structural description is explained by the fact that every finite algebra is weakly isomorphic to an algebra that is polynomially equivalent to the quotient of an equational domain modulo its monolith (Theorem 3.15). However, we obtain a complete description of two-element equational domains: A two-element algebra is an equational domain if and only if it generates a congruence distributive variety; in Sect. 6 the order filter of equationally additive clones on a two-element set is described in detail (cf. Theorem 6.5). Part of this description carries over to all *E-minimal* algebras; these are finite algebras in which every idempotent unary polynomial function is either bijective or constant, which is the case, e.g., for all finite p-groups. Again, an E-minimal algebra is an equational domain if and only if it generates a congruence distributive variety (Theorem 6.11). A similar description can be obtained for all clones on a three-element set that are contained in the maximal clone of self-dual operations (cf. [46]): such a clone $\mathcal{C}$ is equationally additive if and only if $(A; \mathcal{C})$ generates a congruence distributive variety (Theorem 7.6).

Finally, we investigate the number of equationally additive clones on a finite set. Modulo algebraic equivalence, this number is finite [38, Theorem 3], but as we will see, there can be infinitely many equationally additive clones on a finite set that all induce the same algebraic sets. We prove that on each finite set with at least three elements, there is a continuum of equationally additive clones (Theorem 8.3), and we determine for which finite Abelian groups the number of equationally additive clones above the clone of polynomial functions is infinite (Theorem 5.2).

2. Notation and preliminaries

We write $\mathbb{N}$ for the set of positive integers and, for $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. For a set A, the i-th component of $\boldsymbol{a} \in A^n$ is denoted by a_i and $\boldsymbol{a}(i)$, and $\mathcal{P}(A)$ is the power set of A. A *relation* on A is an element of $\bigcup_{n\in\mathbb{N}} \mathcal{P}(A^n)$, and an *operation* on A is an element of $\bigcup_{n\in\mathbb{N}} A^{A^n}$. For $n \in \mathbb{N}$ and $i \in [n]$, the i-th n-ary *projection* is the operation $e_i^{[n]}\colon A^n \to A$ that is given by $e_i^{[n]}(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in A$; we also abbreviate $\mathrm{id}_A := e_1^{[1]}$. For $a \in A$ the n-ary *constant* with value a is the operation $c_a^{[n]}\colon A^n \to A$ given for all $x_1, \ldots, x_n \in A$ by $c_a^{[n]}(x_1, \ldots, x_n) := a$. We sometimes write $c_a^{[1]}$ as c_a. For a set B and a function $f\colon B \to A$ we denote the image of f by $f[B]$. For a partial order $\leq$ on A, we write $a < b$ if $a \leq b$ and $a \neq b$. Moreover, we write $a \prec b$ if $a < b$ and there is no $x \in A$ with $a < x < b$. For basic notions from universal algebra and lattice theory we refer to [16,35]. In particular we will use *lattices* as defined in [35, Chapter 1, p. 16]. For a lattice $\mathbf{L}$ and $a, b \in L$ such that $a \leq b$, we define $\mathtt{I}[a, b]$ to be the set $\{l \in L \mid a \leq l \text{ and } l \leq b\}$. If $a \prec b$, we say that $\mathtt{I}[a, b]$ is a *prime quotient* of $\mathbf{L}$. A *clone* on a set A is a set of operations on A that is closed under composition and contains all the projections (cf. [41, 1.1.2, 1.1.3], [13, Section 6.1], and [35, Definition 4.1]). A clone on A is *constantive* if it includes all operations c_a where $a \in A$. For the definition of an *algebra* $\mathbf{A}$ on A we refer to [35, Definition 1.1].

We will fix some notation. For a set of relations $\mathcal{R}$ on A, $\mathsf{Pol}\,\mathcal{R}$ is the clone of *polymorphisms* of $\mathcal{R}$, and for a set of functions $\mathcal{F}$ on A, $\mathsf{Inv}\,\mathcal{F}$ is the relational clone of *invariant relations* of $\mathcal{F}$ (cf. [41, Section E2]). For an algebra $\mathbf{A}$, $\mathrm{Clo}\,\mathbf{A}$ is the clone of *term operations* of $\mathbf{A}$ (cf. [35, Definition 4.2]), while $\mathrm{Pol}\,\mathbf{A}$ is the clone of *polynomial operations* of $\mathbf{A}$ (cf. [35, Definition 4.4]). An element e of $\mathrm{Pol}_1\,\mathbf{A}$ is called *idempotent* if it satisfies $e \circ e = e$. We write $\mathrm{Con}\,\mathbf{A}$ for the congruence lattice of $\mathbf{A}$, and we denote its bottom element by 0_A and its top element by 1_A. For an algebra $\mathbf{B}$ of the same signature as $\mathbf{A}$, we denote the set of all homomorphisms from $\mathbf{B}$ to $\mathbf{A}$ by $\mathrm{Hom}(\mathbf{B}, \mathbf{A})$. We say that $\mathbf{A}$ is *finitely subdirectly irreducible* if for all $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}\backslash\{0_A\}$ we have $\alpha \cap \beta \neq 0_A$. Clearly, a finite algebra is subdirectly irreducible (cf. [35, Definition 4.39]) if and only if it is finitely subdirectly irreducible and has at least two elements. For a subset X of A^2 we denote by $\langle X \rangle_{\mathrm{Con}\,\mathbf{A}}$ the congruence generated by X in $\mathbf{A}$ (cf. [35, Definition 1.19]). For $n \in \mathbb{N}$ and for a subset X of A^n we denote by $\langle X \rangle_{\mathbf{A}^n}$

the subalgebra of $\mathbf{A}^n$ generated by X (cf. [35, Definition 1.8]). For a group $\mathbf{G}$ and $g \in G$, $\langle g \rangle$ is the subgroup of $\mathbf{G}$ generated by $\{g\}$. For $n \in \mathbb{N}$, $\boldsymbol{a}, \boldsymbol{b} \in A^n$ and $\alpha \in \operatorname{Con} \mathbf{A}$, we write $\boldsymbol{a} \equiv_\alpha \boldsymbol{b}$ if $(\boldsymbol{a}(i), \boldsymbol{b}(i)) \in \alpha$ for all $i \in [n]$; for $n = 1$ and $(a, b) \in \alpha$ we will sometimes just write $a \; \alpha \; b$. Given a clone $\mathcal{C}$ on A, the symbol $\mathcal{C}^{[n]}$ denotes the set of all n-ary functions in $\mathcal{C}$. For a congruence θ of the algebra $(A; \mathcal{C})$ and an n-ary function $f \in \mathcal{C}^{[n]}$, f_θ is the function from $(A/\theta)^n$ to A/θ defined by $f_\theta(a_1/\theta, \ldots, a_n/\theta) = f(a_1, \ldots, a_n)/\theta$ for all $a_1, \ldots, a_n \in A$ (cf. [35, Definition 1.15]). We observe that for an algebra $\mathbf{A}$ and for $\theta \in \operatorname{Con} \mathbf{A}$ we have

$$\operatorname{Pol}(\mathbf{A}/\theta) = \{f_\theta \mid f \in \operatorname{Pol} \mathbf{A}\}. \tag{2.1}$$

A clone $\mathcal{C}$ on a set A is called a *Mal'cev clone* if there exists $d \in \mathcal{C}^{[3]}$ such that for all $a, b \in A$ the equalities $d(a, b, b) = d(b, b, a) = a$ hold. An algebra $\mathbf{A}$ is called a *Mal'cev algebra* if $\operatorname{Clo} \mathbf{A}$ is a Mal'cev clone. Moreover, we say that $\mathbf{A}$ has a *Mal'cev polynomial* if $\operatorname{Pol} \mathbf{A}$ is a Mal'cev clone. Note that for a group $\mathbf{G}$ the term function t, defined by $t(x_1, x_2, x_3) = x_1 - x_2 + x_3$ for all $x_1, x_2, x_3 \in G$, is a Mal'cev term. We will often use the following basic facts on Mal'cev algebras.

Lemma 2.1. (cf. [35, Theorem 4.70(iii)]) *Let* $\mathbf{A}$ *be an algebra with a Mal'cev polynomial, let* $k \in \mathbb{N}$ *and let* $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$*. Then*

$$\langle\{(a_1, b_1), \ldots, (a_k, b_k)\}\rangle_{\operatorname{Con} \mathbf{A}} = \{(p(a_1, \ldots, a_k), p(b_1, \ldots, b_k)) \mid p \in \operatorname{Pol}_k \mathbf{A}\}.$$

Later we shall also use the following observation.

Lemma 2.2. (cf. [25, Lemma 5.22]) *Every reflexive subuniverse of the square of an algebra* $\mathbf{A}$ *with a Mal'cev polynomial is a congruence of* $\mathbf{A}$*.*

We will use the notions of *centralizing relation* and *commutator* as defined in [35, Section 4.13]. To aid the reader we give the definitions explicitly. Following [3], for an algebra $\mathbf{A}$, $m, n \in \mathbb{N}$ and $\alpha, \beta, \eta \in \operatorname{Con} \mathbf{A}$, we say that $C(m, n, \alpha, \beta, \eta)$ holds if for all $p \in \operatorname{Pol}_{m+n} \mathbf{A}$, for all $\boldsymbol{a}, \boldsymbol{b} \in A^m$, $\boldsymbol{u}, \boldsymbol{v} \in A^n$ with $\boldsymbol{a} \equiv_\alpha \boldsymbol{b}$, $\boldsymbol{u} \equiv_\beta \boldsymbol{v}$ and $p(\boldsymbol{a}, \boldsymbol{u}) \; \eta \; p(\boldsymbol{a}, \boldsymbol{v})$ we have $p(\boldsymbol{b}, \boldsymbol{u}) \; \eta \; p(\boldsymbol{b}, \boldsymbol{v})$. We say that α *centralizes* β *modulo* η, and write $C(\alpha, \beta; \eta)$, if $C(1, k, \alpha, \beta, \eta)$ is satisfied for all $k \in \mathbb{N}$. Note that this definition of the centralizing relation is proved to be equivalent to [35, Definition 4.148] in [3, Proposition 2.1]. Following [35, Definition 4.150], for $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ we define their *commutator*, denoted by $[\alpha, \beta]$, to be the smallest congruence η of $\mathbf{A}$ for which $C(\alpha, \beta; \eta)$. The fact that there is such a smallest congruence is a consequence of [35, Lemma 4.149]. Given an algebra $\mathbf{A}$ and $\theta \in \operatorname{Con} \mathbf{A}$, we say that θ is *Abelian* [23] if $[\theta, \theta] = 0_A$, and we say that $\mathbf{A}$ is *Abelian* if $[1_A, 1_A] = 0_A$. Groups are Abelian if and only if they are commutative.

An algebra $\mathbf{V}$ that has a group reduct is called an *expanded group*. If the group reduct is $\mathbf{G} = (V; +, -, 0)$, we will say that $\mathbf{V}$ is an *expansion* of $\mathbf{G}$. A subset I of V is an *ideal* if it is a normal subgroup of $\mathbf{G}$ and for all $n \in \mathbb{N}$, for each n-ary basic operation f of $\mathbf{V}$, for all $\boldsymbol{i} \in I^n$ and for all $\boldsymbol{v} \in V^n$ we have $f(\boldsymbol{v} + \boldsymbol{i}) - f(\boldsymbol{v}) \in I$. We denote the lattice of ideals of an expanded group $\mathbf{V}$ by $\operatorname{Id} \mathbf{V}$. We remark that the function $\psi\colon \operatorname{Id} \mathbf{V} \to \operatorname{Con} \mathbf{V}$ defined by $\psi(I) = \{(a_1, a_2) \mid a_1 - a_2 \in I\}$ for all $I \in \operatorname{Id} \mathbf{V}$, induces a lattice isomorphism between $\operatorname{Con} \mathbf{V}$ and $\operatorname{Id} \mathbf{V}$. On $\operatorname{Id} \mathbf{V}$ we define a binary operation, the *ideal commutator* (cf. [43]), as follows: For $A, B \in \operatorname{Id} \mathbf{V}$ we let $[A, B]$ be the ideal generated by

$$\{p(a, b) \mid a \in A, b \in B, p \in \operatorname{Pol}_2 \mathbf{V} \text{ and } \forall v \in V\colon p(v, 0) = p(0, v) = 0\}.$$

The lattice $\operatorname{Id} \mathbf{V}$ expanded with the ideal commutator is isomorphic, via the isomorphism ψ, to $\operatorname{Con} \mathbf{V}$ expanded with the commutator operation defined for congruences above. A proof can be found in [6, Section 2] and in [7, Section 4]. Thus, for two ideals M, N of an expanded group $\mathbf{V}$, their ideal commutator $[M, N]$ is the ideal $\psi^{-1}([\psi(M), \psi(N)])$.

In [25], Hobby and McKenzie developed a structure theory for finite algebras called *tame congruence theory* (TCT). The central notions of this theory are that of *minimal set* (cf. [25, Definition 2.5]), and that of *minimal algebra* (cf. [25, Definition 2.14]). Each minimal algebra has one of five types (cf. [25, Definition 4.10, Corollary 4.11]). To denote the five TCT-types we will use bold numbers: $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$. Tame congruence theory associates to each prime quotient of $\operatorname{Con} \mathbf{A}$ a set of minimal algebras that have the same type. The type of a prime quotient is then defined as the type of these minimal algebras (cf. [25, Definition 5.1]). We will denote the type of a prime quotient $\mathtt{I}[\alpha, \beta]$ by $\mathtt{typ}(\alpha, \beta)$.

3. Algebraic consequences of equational additivity

Let A be a set and let $\mathcal{C}$ be a clone on A. Following [37], for $n \in \mathbb{N}$ and for $X \subseteq A^n$ we say that X is *algebraic with respect to* $\mathcal{C}$, or that X is $\mathcal{C}$-algebraic, if there exist an index set I and two families $(p_i)_{i \in I}$, $(q_i)_{i \in I}$ of operations in $\mathcal{C}^{[n]}$ such that $X = \{\boldsymbol{x} \in A^n \mid \forall i \in I\colon p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}$. We define $\operatorname{Alg}_n \mathcal{C}$ to be the collection of all the subsets of A^n that are algebraic with respect to $\mathcal{C}$, and we define the *algebraic geometry* of $\mathcal{C}$ by $\operatorname{Alg} \mathcal{C} := \bigcup_{n \in \mathbb{N}} \operatorname{Alg}_n \mathcal{C}$. For an algebra $\mathbf{A}$ we set $\operatorname{Alg} \mathbf{A} := \operatorname{Alg} \operatorname{Clo}(\mathbf{A})$ (cf. [11,39]). We first provide a lemma that will be useful to assess whether a set X is algebraic with respect to a clone $\mathcal{C}$.

Lemma 3.1. *Let A be a set, let $\mathcal{C}$ be a clone on A, let $n \in \mathbb{N}$ and let $X \subseteq A^n$. Then $X \in \operatorname{Alg}_n \mathcal{C}$ if and only if for all $\boldsymbol{a} \in A^n \backslash X$ there exist $f_{\boldsymbol{a}}, g_{\boldsymbol{a}} \in \mathcal{C}^{[n]}$ such that $f_{\boldsymbol{a}}(\boldsymbol{a}) \neq g_{\boldsymbol{a}}(\boldsymbol{a})$ and $f_{\boldsymbol{a}}(\boldsymbol{x}) = g_{\boldsymbol{a}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$.*

Proof. Let us assume that $X \in \operatorname{Alg}_n \mathcal{C}$. Then there exist an index set I and $\{p_i \mid i \in I\}, \{q_i \mid i \in I\} \subseteq \mathcal{C}^{[n]}$ so that $X = \{\boldsymbol{a} \in A^n \mid \forall i \in I\colon p_i(\boldsymbol{a}) = q_i(\boldsymbol{a})\}$. Let $\boldsymbol{a} \in A^n \backslash X$. Clearly, there exists some $i \in I$ such that $p_i(\boldsymbol{a}) \neq q_i(\boldsymbol{a})$. Thus, it suffices to set $f_{\boldsymbol{a}} = p_i$ and $g_{\boldsymbol{a}} = q_i$.

Let us assume that for all $\boldsymbol{a} \in A^n \backslash X$ there exist $f_{\boldsymbol{a}}, g_{\boldsymbol{a}} \in \mathcal{C}^{[n]}$ such that $f_{\boldsymbol{a}}(\boldsymbol{a}) \neq g_{\boldsymbol{a}}(\boldsymbol{a})$ and $f_{\boldsymbol{a}}(\boldsymbol{x}) = g_{\boldsymbol{a}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$. Then we can obtain X in the form $X = \{\boldsymbol{x} \in A^n \mid \forall \boldsymbol{a} \in A^n \backslash X \colon f_{\boldsymbol{a}}(\boldsymbol{x}) = g_{\boldsymbol{a}}(\boldsymbol{x})\}$. □

We report the definition of equationally additive clone as given in [38].

Definition 3.2. (*Equationally additive*) A clone $\mathcal{C}$ on a set A is called *equationally additive* if for all $n \in \mathbb{N}$ and for all $A, B \in \operatorname{Alg}_n \mathcal{C}$ we have $A \cup B \in \operatorname{Alg}_n \mathcal{C}$. An algebra $\mathbf{A}$ is an *equational domain* [18, Definition 1] if $\operatorname{Clo} \mathbf{A}$ is equationally additive.

To each set A we associate the following quaternary relation

$$\Delta_A^{(4)} := \{(x_1, x_2, x_3, x_4) \in A^4 \mid x_1 = x_2 \text{ or } x_3 = x_4\}.$$

We observe that $\Delta_A^{(4)} = \pi_4(A)$ as defined in [41, Lemma 1.3.1]. Next, we shall state a theorem by Daniyarova, Myasnikov and Remeslennikov that characterizes equationally additive clones in terms of their quaternary algebraic sets.

Lemma 3.3. (cf. [18, proof of Theorem 2.5]) *Let $\mathcal{C}$ be a clone on a set A and $n \in \mathbb{N}$. Suppose that $\Delta_A^{(4)}$ and $B, C \subseteq A^n$ are algebraic over $\mathcal{C}$, expressed as*

$$\begin{aligned} \Delta_A^{(4)} &= \{\boldsymbol{a} \in A^4 \mid \forall i \in I \colon p_i(\boldsymbol{a}) = q_i(\boldsymbol{a})\} \\ B &= \{\boldsymbol{a} \in A^n \mid \forall j \in J \colon f_j(\boldsymbol{a}) = g_j(\boldsymbol{a})\} \\ C &= \{\boldsymbol{a} \in A^n \mid \forall k \in K \colon h_k(\boldsymbol{a}) = t_k(\boldsymbol{a})\} \end{aligned}$$

for some index sets I, J, K and operations $\{p_i \mid i \in I\}, \{q_i \mid i \in I\} \subseteq \mathcal{C}^{[4]}$ $\{f_j \mid j \in J\}, \{g_j \mid j \in J\}, \{h_k \mid k \in K\}, \{t_k \mid k \in K\} \subseteq \mathcal{C}^{[n]}$. Then we have

$$\begin{aligned} B \cup C = \{\boldsymbol{a} \in A^n \mid{}& \forall (i, j, k) \in I \times J \times K \colon \\ & p_i(f_j(\boldsymbol{a}), g_j(\boldsymbol{a}), h_k(\boldsymbol{a}), t_k(\boldsymbol{a})) = q_i(f_j(\boldsymbol{a}), g_j(\boldsymbol{a}), h_k(\boldsymbol{a}), t_k(\boldsymbol{a}))\}. \end{aligned}$$

Theorem 3.4. [18, Theorem 2.5] *A clone $\mathcal{C}$ on a set A is equationally additive if and only if $\Delta_A^{(4)} \in \operatorname{Alg}_4 \mathcal{C}$.*

Proof. If $\Delta_A^{(4)} \in \operatorname{Alg}_4 \mathcal{C}$, then Lemma 3.3 yields that the union of any two $\mathcal{C}$-algebraic sets is always a $\mathcal{C}$-algebraic set. If $\mathcal{C}$ is equationally additive, then $\Delta_A^{(4)} \in \operatorname{Alg}_4 \mathcal{C}$ since it is the union of two algebraic sets, namely

$$\Delta_A^{(4)} = \{\boldsymbol{a} \in A^4 \mid a_1 = a_2\} \cup \{\boldsymbol{a} \in A^4 \mid a_3 = a_4\}.$$

□

Corollary 3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be clones on a set A such that $\mathcal{C} \subseteq \mathcal{D}$. If $\mathcal{C}$ is equationally additive, then so is $\mathcal{D}$.*

Hence if $\mathbf{A}$ is an equational domain, then not only is $\operatorname{Clo}\mathbf{A}$ equationally additive, but also its extension $\operatorname{Pol}\mathbf{A}$.

An algebra $\mathbf{A}$ is called *essentially at most unary* if the clone $\operatorname{Clo}\mathbf{A}$ is generated by its unary part. We shall now prove that non-trivial equational domains must contain a function with at least two essential arguments.

Theorem 3.6. *Let $\mathbf{A}$ be an essentially at most unary algebra with at least two elements. Then $\operatorname{Pol}\mathbf{A}$ is not equationally additive.*

Proof. The algebra $\mathbf{A}$ being essentially at most unary means that its clone $\operatorname{Clo}\mathbf{A}$ is generated by its unary part; hence $\operatorname{Pol}\mathbf{A}$ is generated by $\mathcal{F}$ defined as the union of $\operatorname{Clo}_1\mathbf{A}$ and all unary constants. That is, for every $n \in \mathbb{N}$ and $g \in \operatorname{Pol}_n\mathbf{A}$ there is some $i \in [n]$ and $f \in \mathcal{F}$ such that $g(x_1, \dots, x_n) = f(x_i)$ for all $x_1, \dots, x_n \in A$. In order to obtain a contradiction, let us assume that $\operatorname{Pol}\mathbf{A}$ is equationally additive, which means that $\Delta_A^{(4)}$ is the solution set of some system of equations over $\operatorname{Pol}_4\mathbf{A}$ (cf. Theorem 3.4). Each of the equations is of the form $f(x_i) = g(x_j)$ for some $f, g \in \mathcal{F}$ and $i, j \in [4]$, and it must be satisfied by all tuples in $\Delta_A^{(4)}$. Let us now consider any particular such equation.

As a first case we assume that in this equation $i \neq j$. For any $a, b \in A$ we can find a tuple $\boldsymbol{x} \in \Delta_A^{(4)}$ such that $x_i = a$ and $x_j = b$. For instance, if $(i, j) = (1, 4)$ we may choose (a, a, a, b), if $(i, j) = (1, 2)$ we may choose (a, b, b, b), etc. Since $f(x_i) = g(x_j)$ is satisfied by the constructed $\boldsymbol{x} \in \Delta_A^{(4)}$, we obtain $f(a) = g(b)$ for all $a, b \in A$. This implies that f and g are constant with the same value; but then the equation $f(x_i) = g(x_j)$ is satisfied by all $\boldsymbol{x} \in A^4$.

Let us now investigate the case where $i = j$, that is, the considered equation is of the form $f(x_i) = g(x_i)$ with $i \in [4]$. Again, for any $a \in A$ we can choose $\boldsymbol{x} = (a, a, a, a) \in \Delta_A^{(4)}$ to show that $f(a) = g(a)$ holds for all $a \in A$. Thus $f = g$ and the equation $f(x_i) = g(x_i)$ is again satisfied by all tuples in A^4.

As a consequence, all the equations that were assumed to define $\Delta_A^{(4)}$ are actually satisfied by any quadruple in A^4. This, however, means that their solution set is A^4, which properly contains $\Delta_A^{(4)}$, due to $|A| \geq 2$. This contradiction shows that $\operatorname{Pol}\mathbf{A}$ cannot be equationally additive. $\square$

Lemma 3.7. *Let $\mathcal{C}$ be a clone on A, let $\{p_i \mid i \in I\}, \{q_i \mid i \in I\} \subseteq \mathcal{C}^{[4]}$ such that $\Delta_A^{(4)} = \{\boldsymbol{a} \in A^4 \mid \forall i \in I\colon p_i(\boldsymbol{a}) = q_i(\boldsymbol{a})\}$, and let $\mathbf{A} = (A; \mathcal{C})$. Then for all $\alpha \in \operatorname{Con}\mathbf{A}$, for all $(a_1, a_2) \in \alpha$, for all $x, y \in A$ and for all $i \in I$, we have $p_i(a_1, a_2, x, y) \mathrel{\alpha} q_i(a_1, a_2, x, y)$ and $p_i(x, y, a_1, a_2) \mathrel{\alpha} q_i(x, y, a_1, a_2)$.*

Proof. Let $\alpha \in \operatorname{Con}\mathbf{A}$, let $(a_1, a_2) \in \alpha$, let $x, y \in A$ and let $i \in I$. We have

$$p_i(a_1, a_2, x, y) \mathrel{\alpha} p_i(a_1, a_1, x, y) = q_i(a_1, a_1, x, y) \mathrel{\alpha} q_i(a_1, a_2, x, y)$$

and

$$p_i(x, y, a_1, a_2) \mathrel{\alpha} p_i(x, y, a_1, a_1) = q_i(x, y, a_1, a_1) \mathrel{\alpha} q_i(x, y, a_1, a_2).$$

$\square$

The next result tells that every equational domain is finitely subdirectly irreducible.

Proposition 3.8. *For any set A and every equationally additive clone $\mathcal{C}$ on A the algebra $\mathbf{A} = (A; \mathcal{C})$ is finitely subdirectly irreducible.*

Proof. If $\mathcal{C}$ is equationally additive, then there exist an index set I and functions $p_i, q_i \in \mathcal{C}^{[4]}$ for $i \in I$, such that $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid \forall i \in I \colon p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}$. Let $\alpha, \beta \in \operatorname{Con} \mathbf{A} \backslash \{0_A\}$. We show that $(\alpha \cap \beta) \backslash 0_A \neq \emptyset$. Since $\alpha \neq 0_A$ and $\beta \neq 0_A$, there exist $(a_1, a_2) \in \alpha \backslash 0_A$ and $(b_1, b_2) \in \beta \backslash 0_A$. Let $i \in I$ be such that $p_i(a_1, a_2, b_1, b_2) \neq q_i(a_1, a_2, b_1, b_2)$. Since $(a_1, a_2) \in \alpha$, Lemma 3.7 yields that $p_i(a_1, a_2, b_1, b_2) \; \alpha \; q_i(a_1, a_2, b_1, b_2)$; likewise $(b_1, b_2) \in \beta$ implies $p_i(a_1, a_2, b_1, b_2) \; \beta \; q_i(a_1, a_2, b_1, b_2)$. Thus, $(p_i(a_1, a_2, b_1, b_2), q_i(a_1, a_2, b_1, b_2))$ belongs to $(\alpha \cap \beta) \backslash 0_A$. $\square$

We say that an algebra $\mathbf{A}$ has a *weak difference term* if there exists some $d \in \operatorname{Clo}_3 \mathbf{A}$ such that for all $\theta \in \operatorname{Con} \mathbf{A}$ and all $(a, b) \in \theta$ we have the condition $d(a, b, b) \; [\theta, \theta] \; a \; [\theta, \theta] \; d(b, b, a)$. A *weak difference polynomial* is defined analogously using $\operatorname{Pol} \mathbf{A}$. Note that a Mal'cev polynomial is also a weak difference polynomial.

Proposition 3.9. *Let $\mathbf{A}$ be an algebra with a weak difference polynomial. If $\operatorname{Pol} \mathbf{A}$ is equationally additive, then for all congruences $\alpha \in \operatorname{Con} \mathbf{A} \backslash \{0_A\}$ it follows that $[\alpha, \alpha] > 0_A$.*

Proof. If $\operatorname{Pol} \mathbf{A}$ is equationally additive, then there exist an index set I and functions $p_i, q_i \in \operatorname{Pol}_4 \mathbf{A}$ for $i \in I$, such that

$$\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid \forall i \in I \colon p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}.$$

Let $\alpha \in \operatorname{Con} \mathbf{A} \backslash \{0_A\}$ and let $(a, b) \in \alpha \backslash 0_A$. As $(a, b, a, b) \notin \Delta_A^{(4)}$, there exists $i \in I$ such that $p_i(a, b, a, b) \neq q_i(a, b, a, b)$. Let us define the polynomial operation f for all $x_1, x_2 \in A$ by

$$f(x_1, x_2) := d(p_i(a, b, a, b), p_i(a, x_1, x_2, b), q_i(a, x_1, x_2, b)).$$

By using the definition of weak difference polynomial and noting that, due to Lemma 3.7, we have $p_i(a, b, a, b) \; \alpha \; q_i(a, b, a, b)$, we can verify that

$$\begin{aligned}
f(a, a) &= d(p_i(a, b, a, b), p_i(a, a, a, b), q_i(a, a, a, b)) \quad &[\alpha, \alpha] \quad &p_i(a, b, a, b),\\
f(a, b) &= d(p_i(a, b, a, b), p_i(a, a, b, b), q_i(a, a, b, b)) \quad &[\alpha, \alpha] \quad &p_i(a, b, a, b),\\
f(b, a) &= d(p_i(a, b, a, b), p_i(a, b, a, b), q_i(a, b, a, b)) \quad &[\alpha, \alpha] \quad &q_i(a, b, a, b),\\
f(b, b) &= d(p_i(a, b, a, b), p_i(a, b, b, b), q_i(a, b, b, b)) \quad &[\alpha, \alpha] \quad &p_i(a, b, a, b).
\end{aligned}$$

Therefore, we have that $f(a, a) \; [\alpha, \alpha] \; f(a, b)$. Thus, applying the definition of commutator to f yields that

$$q_i(a, b, a, b) \; [\alpha, \alpha] \; f(b, a) \; [\alpha, \alpha] \; f(b, b) \; [\alpha, \alpha] \; p_i(a, b, a, b).$$

Since $q_i(a, b, a, b) \neq p_i(a, b, a, b)$ we deduce that $[\alpha, \alpha] > 0_A$. $\square$

We will use the notion of *Taylor operation* on a set A as defined, e.g., in [13, Definition 6.6.1]. We say that $\mathbf{A}$ has a *Taylor term* (cf. [13, Definition 6.6.2]) if $\operatorname{Clo}\mathbf{A}$ contains a Taylor operation, and that $\mathbf{A}$ has a *Taylor polynomial* if $\operatorname{Pol}\mathbf{A}$ contains a Taylor operation.

Corollary 3.10. *Let $\mathbf{A}$ be a finite, at least two-element algebra with an idempotent Taylor polynomial. If $\operatorname{Pol}\mathbf{A}$ is equationally additive, then $\mathbf{A}$ is subdirectly irreducible and its monolith is non-Abelian.*

Proof. Let $\mathbf{A}' = (A; \operatorname{Pol}\mathbf{A})$. Since A is finite and has at least two elements and since $\operatorname{Con}\mathbf{A} = \operatorname{Con}\mathbf{A}'$, Proposition 3.8 yields that $\mathbf{A}$ and $\mathbf{A}'$ are subdirectly irreducible. Let μ be the monolithic congruence of $\mathbf{A}$ and $\mathbf{A}'$. Since $\mathbf{A}'$ has an idempotent Taylor operation, it generates a variety satisfying a non-trivial idempotent Mal'cev condition. Hence that variety satisfies condition (2) of [25, Theorem 9.6], and thus, by the latter theorem, the variety omits type **1**. Therefore, by [25, Theorem 7.12], $\mathbf{A}'$ has a weak difference term. Consequently, $\mathbf{A}$ has a weak difference polynomial; and therefore Proposition 3.9 yields that μ is non-Abelian. $\square$

Hence, using Corollary 3.5, it follows that all finite non-trivial equational domains having a Taylor polynomial are subdirectly irreducible with a non-Abelian monolith.

Since a Mal'cev operation is a Taylor operation, we obtain the following.

Corollary 3.11. *Let $\mathbf{A}$ be a finite algebra with at least two elements and a Mal'cev polynomial. If $\operatorname{Pol}\mathbf{A}$ is equationally additive, then $\mathbf{A}$ is subdirectly irreducible and its monolith is non-Abelian.*

We now focus on those clones on a finite set A with the property that $\Delta_A^{(4)}$ is the solution set of a single equation of the form $f \approx a$ with $a \in A$. An example is given by the clone of polynomial functions of a ring with no zero divisors, where $f(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_3 - x_4)$ and $a = 0$.

Lemma 3.12. *Let A be a finite set with $|A| \geq 2$, let $0 \in A$, let $f\colon A^4 \to A$ be such that $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = 0\}$, and let $\mathbf{A} = (A; f)$. Then there exists $p \in \operatorname{Pol}_1\mathbf{A}$ and there exists $i \in f[A^4]\backslash\{0\}$ such that $p(0) = 0$ and $p(x) = i$ for all $x \in A\backslash\{0\}$.*

Proof. We proceed by induction on $|A| \geq 2$.

Base step. $|A| = 2$, $A = \{0, i\}$ with $i \neq 0$: The unary polynomial p, defined by $x \mapsto f(0, x, 0, x)$ for all $x \in A$, satisfies all the desired properties. In fact $(0, 0, 0, 0) \in \Delta_A^{(4)}$, hence $p(0) = f(0, 0, 0, 0) = 0$, and $(0, i, 0, i) \notin \Delta_A^{(4)}$, hence $p(i) = i$.

Induction step. For each element $a \in A$, let us define a unary polynomial $p_a \in \operatorname{Pol}_1\mathbf{A}$ by $p_a(x) = f(0, x, 0, a)$ for all $x \in A$. Note that, if $a \in A\backslash\{0\}$, then

p_a preserves $A\backslash\{0\}$ as a subuniverse since $(0,x,0,a) \notin \Delta_A^{(4)}$ for all $x \in A\backslash\{0\}$. Moreover, we have $p_a(0)=0$. We split the induction step into two cases.

Case 1. *For all $a \in A\backslash\{0\}$ the function p_a induces a permutation on A.* Set $m := |A|!$ and consider any $a \in A\backslash\{0\}$. The order of p_a in the full symmetric group on A divides m, hence $p_a^m(x) = x$ for all $x \in A$. Since $(0,x,0,0) \in \Delta_A^{(4)}$ the m-th iterated power of p_0 is still the constant zero function of arity one. Therefore, for all $x \in A$, given $a \neq 0$, we have $p_a^m(x) = x$, while $p_a^m(x) = 0$ if $a = 0$. We now pick an arbitrary element $i \in f[A^4]\backslash\{0\}$ (this is possible since there is some $a \in A\backslash\{0\}$, for which p_a is a permutation) and define $p(x) := p_x^m(i)$ for all $x \in A$. Clearly, if $x \neq 0$, then $p(x) = p_x^m(i) = i$, and $p(0) = p_0^m(i) = 0$. Moreover, $p \in \operatorname{Pol}_1 \mathbf{A}$ because it is constructed as an iterated substitution of f within itself wherein some positions have been filled by constant values.

Case 2. *There is $a \in A\backslash\{0\}$ where the function p_a is not a permutation of A.* Let $m \in \mathbb{N}$ be such that $e := p_a^m \in \operatorname{Pol}_1 \mathbf{A}$ is idempotent, i.e., $e \circ e = e$. Let $B := e[A]$ be its image, which contains 0 since $p_a(0) = 0$. Since p_a preserves $\{0\}$ and $A\backslash\{0\}$, so does e, and hence we have

$$\forall x \in A\colon \quad e(x) = 0 \iff x = 0. \tag{3.1}$$

Moreover, we have $B = e[A] \subseteq p_a[A] \subsetneq A$ since p_a is not surjective; hence the algebra $\mathbf{B} = (B; (e \circ f)|_B)$ is defined on a set with smaller cardinality than A. Given $n \in \mathbb{N}$, a straightforward induction on the polynomial terms describing $\operatorname{Pol}_n \mathbf{A}$ shows that

$$\forall n \in \mathbb{N}\, \forall g \in \operatorname{Pol}_n \mathbf{B}\, \exists \hat{g} \in \operatorname{Pol}_n \mathbf{A}\, \forall \boldsymbol{b} \in B^n\colon \quad g(\boldsymbol{b}) = \hat{g}(\boldsymbol{b}). \tag{3.2}$$

Because of (3.1), for all $\boldsymbol{x} \in A^4$ we have

$$(e \circ f)(\boldsymbol{x}) = 0 \iff f(\boldsymbol{x}) = 0 \iff \boldsymbol{x} \in \Delta_A^{(4)},$$

which implies that

$$\Delta_B^{(4)} = B^4 \cap \Delta_A^{(4)} = \left\{ \boldsymbol{b} \in B^4 \;\middle|\; \boldsymbol{b} \in \Delta_A^{(4)} \right\} = \left\{ \boldsymbol{b} \in B^4 \mid (e \circ f)|_B(\boldsymbol{b}) = 0 \right\}.$$

This means that the induction hypothesis can be applied to $\mathbf{B}$, as $|B| < |A|$. Thus, there are $q \in \operatorname{Pol}_1 \mathbf{B}$ and $i \in B\backslash\{0\} \subseteq p_a[A]\backslash\{0\} \subseteq f[A^4]\backslash\{0\}$ such that $q(0) = 0$ and $q(b) = i$ for all $b \in B\backslash\{0\}$. Moreover, (3.2) yields that there exists $\hat{q} \in \operatorname{Pol}_1 \mathbf{A}$ such that $q(b) = \hat{q}(b)$ for all $b \in B$. Let us define $p := \hat{q} \circ e \in \operatorname{Pol}_1 \mathbf{A}$. Then (3.1) yields $p(0) = q(e(0)) = q(0) = 0$. Moreover, for all $a \in A\backslash\{0\}$ we have by (3.1) that $e(a) \in B\backslash\{0\}$, and therefore $p(a) = \hat{q}(e(a)) = q(e(a)) = i$. This concludes the proof. □

Proposition 3.13. *Let A be a finite set with at least two elements, let $\mathcal{C}$ be a clone on A, let $f \in \mathcal{C}^{[4]}$, let $0 \in A$ be such that $\Delta_A^{(4)} = \{\boldsymbol{a} \in A^4 \mid f(\boldsymbol{a}) = 0\}$, and let $\mathbf{A} = (A; \mathcal{C})$. Then $\mathbf{A}$ is subdirectly irreducible, there is $i \in f[A^4]\backslash\{0\}$ such that $\mu = \langle\{(0,i)\}\rangle_{\operatorname{Con} \mathbf{A}}$ is the monolith of $\mathbf{A}$, and $\mathtt{typ}(0_A, \mu) = \mathbf{3}$.*

Proof. Lemma 3.12 yields that there exists $i \in f[A^4]\backslash\{0\}$ and there exists $p \in \mathrm{Pol}_1 \mathbf{A}$ such that $p(0) = 0$ and $p(a) = i$ for all $a \in A\backslash\{0\}$. Take any $\theta \in \mathrm{Con}\,\mathbf{A}\backslash\{0_A\}$ and $(a, b) \in \theta\backslash\{0_A\}$. We show that $(0, i) \in \theta$. Let $h\colon A^4 \to A$ be defined by $h(\boldsymbol{x}) = p(f(\boldsymbol{x}))$ for all $\boldsymbol{x} \in A^4$. Clearly, $h \in \mathrm{Pol}\,\mathbf{A}$. Moreover, $(0, i, a, b) \equiv_\theta (0, i, a, a)$. Thus, we have $0 = h(0, i, a, a)\ \theta\ h(0, i, a, b) = i$, and therefore $(0, i) \in \theta$. Hence $\mathbf{A}$ is subdirectly irreducible and the monolith is $\mu = \langle\{(0, i)\}\rangle_{\mathrm{Con}\,\mathbf{A}}$. Since p is idempotent and has image $\{0, i\}$, the set $\{0, i\}$ is $(0_A, \mu)$-minimal in the sense of tame congruence theory (cf. [25, Definition 2.5]).

Next, we define $c\colon A \to A$ by letting $c(x) = p(f(x, i, x, i))$ for all $x \in A$, and we introduce $m\colon A^2 \to A$ by $m(x_1, x_2) = p(f(p(f(x_1, i, x_1, i)), i, x_2, 0))$ for all $x_1, x_2 \in A$. Clearly, $c \in \mathrm{Pol}_1 \mathbf{A}$ and $m \in \mathrm{Pol}_2 \mathbf{A}$. Moreover, we have

$$(0, i, 0, i), (0, i, i, 0) \notin \Delta_A^{(4)},$$
$$(i, i, i, i), (i, i, i, 0), (0, i, 0, 0), (i, i, 0, 0) \in \Delta_A^{(4)}.$$

Therefore, we know that

$$\begin{aligned} &f(i, i, i, i) = f(i, i, i, 0) = f(0, i, 0, 0) = f(i, i, 0, 0) = 0,\\ &f(0, i, 0, i) \neq 0,\\ &f(0, i, i, 0) \neq 0. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &p(f(i, i, i, i)) = p(f(i, i, i, 0)) = p(f(0, i, 0, 0)) = p(f(i, i, 0, 0)) = 0,\\ &p(f(0, i, 0, i)) = p(f(0, i, i, 0)) = i. \end{aligned}$$

Hence $c(0) = p(f(0, i, 0, i)) = i$ and $c(i) = p(f(i, i, i, i)) = 0$. Consequently, $c \in \mathrm{Pol}(\mathbf{A}|_{\{0,i\}})$, and c acts as a complement on $\{0, i\}$. Moreover, m satisfies

$$\begin{aligned} m(0, 0) &= p(f(p(f(0, i, 0, i)), i, 0, 0)) = p(f(i, i, 0, 0)) = 0,\\ m(i, 0) &= p(f(p(f(i, i, i, i)), i, 0, 0)) = p(f(0, i, 0, 0)) = 0,\\ m(0, i) &= p(f(p(f(0, i, 0, i)), i, i, 0)) = p(f(i, i, i, 0)) = 0,\\ m(i, i) &= p(f(p(f(i, i, i, i)), i, i, 0)) = p(f(0, i, i, 0)) = i. \end{aligned}$$

Consequently, $m \in \mathrm{Pol}(\mathbf{A}|_{\{0,i\}})$ and it acts as a meet on $\{0, i\}$. Hence $\mathbf{A}|_{\{0,i\}}$ is polynomially equivalent to a two-element Boolean algebra, and therefore $\mathtt{typ}(0_A, \mu) = \mathbf{3}$. □

For the subsequent three results, the following notation to extend an algebra $\mathbf{A} = (A; \mathcal{F})$ by a single operation $f\colon A^k \to A$, $k \in \mathbb{N}$, comes handy. We define $\mathbf{A} + f$ as an abbreviation of the algebra $(A; \mathcal{F} \cup \{f\})$.

Lemma 3.14. *Let $\mathbf{A}$ be a finite algebra, let $a, b \in A$ such that $a \neq b$, and let $\alpha = \langle\{(a, b)\}\rangle_{\mathrm{Con}\,\mathbf{A}}$. Then there exists $f\colon A^4 \to A$ such that $(\mathbf{A} + f) + c_a$ is a* *subdirectly irreducible equational domain with monolith* α, $\mathtt{typ}(0_A,\alpha) = \mathbf{3}$ *in* $(\mathbf{A}+f)+c_a$, *and* $((\mathbf{A}+f)+c_a)/\alpha = (\mathbf{A}/\alpha + c^{[4]}_{a/\alpha}) + c_{a/\alpha}$.

Proof. Let $f\colon A^4 \to A$ be defined for all $\boldsymbol{x} \in A^4$ by $f(\boldsymbol{x}) = a$ if $\boldsymbol{x} \in \Delta_A^{(4)}$ and $f(\boldsymbol{x}) = b$ otherwise. Proposition 3.13 yields that $\mathrm{Clo}((\mathbf{A}+f)+c_a)$ is equationally additive, $(\mathbf{A}+f)+c_a$ is subdirectly irreducible with monolith $\nu := \langle\{(a,b)\}\rangle_{\mathrm{Con}\,(\mathbf{A}+f)+c_a}$, and $\mathtt{typ}(0_A,\nu) = \mathbf{3}$. Since the image of f is a subset of a/α, the equivalence relation α is preserved by f and c_a; thus we have $\alpha \in \mathrm{Con}((\mathbf{A}+f)+c_a)$. As $\nu \in \mathrm{Con}((\mathbf{A}+f)+c_a) \subseteq \mathrm{Con}\,\mathbf{A}$ and $(a,b) \in \nu$, we have $\alpha \subseteq \nu$, and since ν is the monolithic congruence of $(\mathbf{A}+f)+c_a$, we infer that $\nu = \alpha$. The final equality of the lemma follows from $f_\alpha = c^{[4]}_{a/\alpha}$. □

We say that an algebra $\mathbf{A}$ is *weakly isomorphic* to an algebra $\mathbf{C}$ if there exists an algebra $\mathbf{B}$ with the same universe as $\mathbf{A}$ such that $\mathrm{Clo}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{B}$ and $\mathbf{B} \cong \mathbf{C}$.

Theorem 3.15. *Let* $\mathbf{A}$ *be a finite algebra with at least two elements. Then there exists a subdirectly irreducible finite equational domain* $\mathbf{B}$ *with monolith* $\mu_{\mathbf{B}}$ *and an algebra* $\mathbf{C}$ *such that* $\mathtt{typ}(0_B,\mu_{\mathbf{B}}) = \mathbf{3}$, $\mathbf{A}$ *is weakly isomorphic to* $\mathbf{C}$, *and* $\mathbf{C}$ *is polynomially equivalent to* $\mathbf{B}/\mu_{\mathbf{B}}$.

Proof. Let $\mathbf{D}$ be an algebra on the same universe as $\mathbf{A}$ with at least one at least binary functional symbol in its type such that $\mathrm{Clo}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{D}$. For example, we may take $\mathbf{D} = \mathbf{A} + e^{[2]}_1$, adding the binary projection onto the first argument to $\mathbf{A}$. By [32, Theorem 3.1] there exists a finite subdirectly irreducible algebra $\mathbf{E}$ with monolith $\mu_{\mathbf{E}}$ such that $\mathbf{D} \cong \mathbf{E}/\mu_{\mathbf{E}} =: \mathbf{C}$, i.e., $\mathbf{A}$ is weakly isomorphic to $\mathbf{C}$.

Let $a,b \in E$ such that $\mu_{\mathbf{E}} = \langle\{(a,b)\}\rangle_{\mathrm{Con}\,\mathbf{E}}$. Then Lemma 3.14 states that there exists $f\colon E^4 \to E$ such that $\mathbf{B} := (\mathbf{E}+f)+c_a$ is a finite subdirectly irreducible equational domain with monolith $\mu_{\mathbf{B}} = \mu_{\mathbf{E}}$, $\mathtt{typ}(0_E,\mu_{\mathbf{B}}) = \mathbf{3}$ in $\mathbf{B}$, and $\mathbf{B}/\mu_{\mathbf{B}} = ((\mathbf{E}+f)+c_a)/\mu_{\mathbf{E}} = (\mathbf{E}/\mu_{\mathbf{E}} + c^{[4]}_{a/\mu_{\mathbf{E}}}) + c_{a/\mu_{\mathbf{E}}}$.

Then $\mathbf{A}$ is weakly isomorphic to $\mathbf{C}$, and $\mathbf{C}$ is polynomially equivalent to the quotient $\mathbf{B}/\mu_{\mathbf{B}}$. □

Theorem 3.15 can be improved if we assume that $\mathbf{A}$ generates a congruence modular variety. For the basic properties of modular lattices and congruence modular varieties we refer the reader to [35, Section 2.3].

Theorem 3.16. *Let* $\mathbf{A}$ *be a finite at least two-element algebra in a congruence modular variety and let* $a \in A$. *Then there exist an algebra* $\mathbf{B}$ *with universe* B *in the variety generated by* $\mathbf{A}$, $b \in B$, *and* $f\colon B^4 \to B$ *such that* $(\mathbf{B}+f)+c_b$ *is a subdirectly irreducible equational domain with monolith* α, $\mathtt{typ}(0_B,\alpha) = \mathbf{3}$ *in* $(\mathbf{B}+f)+c_b$, *and* $((\mathbf{B}+f)+c_b)/\alpha \cong (\mathbf{A}+c^{[4]}_a)+c_a$.

Proof. Let $\mathbf{B} := \mathbf{A} \times \mathbf{S}$ where $\mathbf{S}$ is a simple quotient of $\mathbf{A}$ with at least two elements. Then $\mathbf{B}$ belongs to the variety generated by $\mathbf{A}$, and thus $\operatorname{Con}\mathbf{B}$ is a modular lattice. Since $\mathbf{S}$ is a simple quotient of $\mathbf{A}$ and $|S| \geq 2$, there are $s_1, s_2 \in S$ such that $s_1 \neq s_2$; we define $\boldsymbol{a}_1 = (a, s_1)$ and $\boldsymbol{a}_2 = (a, s_2)$, and we set $\alpha = \langle\{(\boldsymbol{a}_1, \boldsymbol{a}_2)\}\rangle_{\operatorname{Con}\mathbf{B}}$. Let Π_1 be the canonical homomorphism from $\mathbf{B}$ onto $\mathbf{A}$, and let Π_2 be the canonical homomorphism of $\mathbf{B}$ onto $\mathbf{S}$. Since $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \ker \Pi_1$, we have $\ker \Pi_1 \supseteq \alpha$. Moreover, since $\mathbf{B}/\ker \Pi_2 \cong \mathbf{S}$ and $\mathbf{S}$ is simple with more than one element, $\ker \Pi_2$ is a co-atom in $\operatorname{Con}\mathbf{B}$. Since $\mathtt{I}[0_B, \ker \Pi_1]$ and $\mathtt{I}[\ker \Pi_2, 1_B]$ are transposes, and thus projective, and $\operatorname{Con}\mathbf{B}$ is modular, we infer that these intervals are isomorphic [35, Corollary 2.28]. Therefore, $\ker \Pi_1$ is an atom of $\operatorname{Con}\mathbf{B}$, whence we conclude that $\alpha = \ker \Pi_1$; accordingly, we have $\mathbf{A} \cong \mathbf{B}/\ker \Pi_1 = \mathbf{B}/\alpha$.

Next, Lemma 3.14 yields that there exists $f\colon (A \times S)^4 \to A \times S$ such that $(\mathbf{B} + f) + c_{\boldsymbol{a}_1}$ is a subdirectly irreducible equational domain with monolith α, $\mathtt{typ}(0_B, \alpha) = \mathbf{3}$ in $(\mathbf{B} + f) + c_{\boldsymbol{a}_1}$, and

$$((\mathbf{B} + f) + c_{\boldsymbol{a}_1})/\alpha = (\mathbf{B}/\alpha + c^{[4]}_{\boldsymbol{a}_1/\alpha}) + c_{\boldsymbol{a}_1/\alpha} \cong (\mathbf{A} + c^{[4]}_a) + c_a.$$

□

4. Characterization of equationally additive constantive Mal'cev clones

In this section we provide a characterization of equationally additive constantive Mal'cev clones in terms of properties of the term condition commutator (cf. Theorem 4.8). We start by stating a few well-known properties of the commutator for algebras with a Mal'cev polynomial.

Lemma 4.1. (cf. [3, Propositions 2.3 and 2.4]) *Let $\mathbf{A}$ be an algebra with a Mal'cev polynomial and let $\alpha, \beta, \alpha', \beta' \in \operatorname{Con}\mathbf{A}$ satisfy $\alpha \leq \alpha'$ and $\beta \leq \beta'$. Then we have*

(a) $C(1, 1, \alpha, \beta, \eta) \iff C(\alpha, \beta; \eta) \iff [\alpha, \beta] \leq \eta$, *in particular, the commutator is completely determined by the binary polynomials of $\mathbf{A}$;*
(b) $[\alpha, \beta] \leq \alpha \wedge \beta$;
(c) $[\alpha, \beta] \leq [\alpha', \beta] \leq [\alpha', \beta']$.

Proof. The first equivalence of statement (a) is shown in [3, Proposition 2.3], the second one in [3, Proposition 2.4]. Statements (b) and (c) are obvious consequences of the definition of $[\alpha, \beta]$ that hold for every algebra $\mathbf{A}$. □

Lemma 4.2. (cf. [3, Proposition 2.6]) *Let $k \in \mathbb{N}$, let $\mathbf{A}$ be an algebra with a Mal'cev polynomial d, let $\alpha, \beta \in \operatorname{Con}\mathbf{A}$, and let $p \in \operatorname{Pol}_k \mathbf{A}$. For all k-tuples $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in A^k$ such that $\boldsymbol{u} \equiv_\alpha \boldsymbol{v} \equiv_\beta \boldsymbol{w}$, we have*

$$d(p(\boldsymbol{u}), p(\boldsymbol{v}), p(\boldsymbol{w})) \equiv_{[\alpha,\beta]} p(d(u_1, v_1, w_1), d(u_2, v_2, w_2), \ldots, d(u_k, v_k, w_k)).$$

Following [8], for $p \in \operatorname{Pol}_2 \mathbf{A}$ and $u_1, u_2 \in A$, we say that p is *absorbing at* (u_1, u_2) if for all $x_1, x_2 \in A$ we have $p(x_1, u_2) = p(u_1, x_2) = p(u_1, u_2)$.

The following lemma is a direct consequence of [8, Lemma 6.13], which was shown using the theory of higher commutators. We here present a different proof, which offers the advantage that it is solely based on the more elementary binary commutator.

Lemma 4.3. (cf. [8, Lemma 6.13]) *Let* $\mathbf{A}$ *be an algebra with a Mal'cev polynomial* d, *let* $\alpha = \langle\{(u_1, v_1)\}\rangle_{\operatorname{Con}\mathbf{A}}$ *and let* $\beta = \langle\{(u_2, v_2)\}\rangle_{\operatorname{Con}\mathbf{A}}$. *Then*

$$[\alpha, \beta] = \{(z(v_1, v_2), z(u_1, u_2)) \mid z \in \operatorname{Pol}_2 \mathbf{A} \text{ is absorbing at } (u_1, u_2)\}. \tag{4.1}$$

Proof. Let η denote the right-hand side of (4.1); we first prove that this set is a congruence. Since constant functions are absorbing at (u_1, u_2), the relation η is reflexive. Let $n \in \mathbb{N}$ and let f be an n-ary basic operation of $\mathbf{A}$. If $z_1, \ldots, z_n$ are binary polynomials absorbing at (u_1, u_2), then $f(z_1, \ldots, z_n)$ is a binary polynomial absorbing at (u_1, u_2). Thus η is a subalgebra of $\mathbf{A} \times \mathbf{A}$. Hence Lemma 2.2 yields that η is a congruence of $\mathbf{A}$.

Next, we prove that $C(\alpha, \beta; \eta)$. For this, according to Lemma 4.1(a), let us take an arbitrary $q \in \operatorname{Pol}_2 \mathbf{A}$ and any $a, b, u, v \in A$ with $a \mathrel{\alpha} b$ and $u \mathrel{\beta} v$. We assume that $q(a, u) \mathrel{\eta} q(a, v)$ and want to show $q(b, u) \mathrel{\eta} q(b, v)$. Since α and β are generated by a single pair, Lemma 2.1 yields that there are unary polynomials $p_1, p_2 \in \operatorname{Pol}_1 \mathbf{A}$ such that $a = p_1(u_1)$, $b = p_1(v_1)$, $u = p_2(u_2)$, $v = p_2(v_2)$. Setting $p(x, y) := q(p_1(x), p_2(y))$ for $x, y \in A$, we are able to infer $p(u_1, u_2) \mathrel{\eta} p(u_1, v_2)$. Let us define $f\colon A^2 \to A$ by

$$\begin{aligned} &f(x_1, x_2) := \\ &d(d(p(x_1, x_2), p(x_1, u_2), p(v_1, u_2)), d(p(u_1, x_2), p(u_1, u_2), p(v_1, u_2)), p(v_1, u_2)) \end{aligned}$$

for all $x_1, x_2 \in A$. Clearly, $f \in \operatorname{Pol}_2 \mathbf{A}$, for $p \in \operatorname{Pol}_2 \mathbf{A}$. For arbitrary $a_1, a_2 \in A$ we have

$$\begin{aligned} &f(u_1, a_2) \\ &\quad = d(d(p(u_1, a_2), p(u_1, u_2), p(v_1, u_2)), d(p(u_1, a_2), p(u_1, u_2), p(v_1, u_2)), p(v_1, u_2)) \\ &\quad = p(v_1, u_2) \end{aligned}$$

and

$$\begin{aligned} &f(a_1, u_2) \\ &\quad = d(d(p(a_1, u_2), p(a_1, u_2), p(v_1, u_2)), d(p(u_1, u_2), p(u_1, u_2), p(v_1, u_2)), p(v_1, u_2)) \\ &\quad = d(p(v_1, u_2), p(v_1, u_2), p(v_1, u_2)) = p(v_1, u_2). \end{aligned}$$

Hence f is absorbing at (u_1, u_2) with value $p(v_1, u_2)$. Therefore, we have

$$f(v_1, v_2) \mathrel{\eta} f(u_1, u_2) = p(v_1, u_2). \tag{4.2}$$

Moreover, since $p(u_1, u_2) \mathrel{\eta} p(u_1, v_2)$, we may derive that

$$f(v_1, v_2) = d(p(v_1, v_2), d(p(u_1, v_2), p(u_1, u_2), p(v_1, u_2)), p(v_1, u_2))$$

$$\begin{aligned}&\eta\ d(p(v_1,v_2),d(p(u_1,v_2),p(u_1,v_2),p(v_1,u_2)),p(v_1,u_2))\\&=d(p(v_1,v_2),p(v_1,u_2),p(v_1,u_2))=p(v_1,v_2).\end{aligned}\tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$q(b,u)=p(v_1,u_2)=f(u_1,u_2)\ \eta\ f(v_1,v_2)\ \eta\ p(v_1,v_2)=q(b,v).$$

Thus, we have that $C(\alpha,\beta;\eta)$ and hence $[\alpha,\beta]\subseteq\eta$.

For the converse inclusion let $\gamma:=[\alpha,\beta]$ and $(a,b)\in\eta$. Thus, there is some $c\in\mathrm{Pol}_2\,\mathbf{A}$ that is absorbing at (u_1,u_2) such that $a=c(v_1,v_2)$ and $b=c(u_1,u_2)$. We have $u_1\ \alpha\ v_1$ and $u_2\ \beta\ v_2$; moreover, $c(u_1,u_2)=c(u_1,v_2)$ by the absorption property at (u_1,u_2), hence $c(u_1,u_2)\ \gamma\ c(u_1,v_2)$. Since, by the definition of the commutator, α centralizes β modulo γ and c absorbs at (u_1,u_2), it follows that $b=c(u_1,u_2)=c(v_1,u_2)\ \gamma\ c(v_1,v_2)=a$, i.e., $(b,a)\in\gamma$ and hence $(a,b)\in\gamma$. This concludes the proof that $\eta\subseteq\gamma=[\alpha,\beta]$. ☐

Proposition 4.4. *Let* $\mathbf{A}$ *be a subdirectly irreducible algebra with a non-Abelian monolith* $\mu\in\mathrm{Con}\,\mathbf{A}$*, let* d *be a Mal'cev polynomial, let* $o\in A$*, let* $U=o/\mu$*, let* $k\in\mathbb{N}$*, let* $D\subseteq A^k$ *and let* $l\colon D\to U$*. Then, for all* $T\subseteq D$ *finite, there exists a polynomial* $p_T\in\mathrm{Pol}_k\,\mathbf{A}$ *such that* $p_T(\boldsymbol{t})=l(\boldsymbol{t})$ *for all* $\boldsymbol{t}\in T$*, and* $p_T(\boldsymbol{x})\in U$ *for all* $\boldsymbol{x}\in A^k$*.*

Proof. Let $T\subseteq D$ be finite. We prove that there exists $p_T\in\mathrm{Pol}_k\,\mathbf{A}$ such that $p_T(\boldsymbol{t})=l(\boldsymbol{t})$ for all $\boldsymbol{t}\in T$, and $p_T(\boldsymbol{x})\in U$ for all $\boldsymbol{x}\in A^k$. We proceed by induction on the cardinality of $T=\{\boldsymbol{t}_1,\dots,\boldsymbol{t}_n\}$.

Case $|T|\leq 1$. If $|T|=1$, the constant polynomial p_T with value $l(\boldsymbol{t}_1)$ interpolates l at $\boldsymbol{t}_1$. If $|T|=0$, any constant polynomial p_T with value in U, e.g., $o\in U$, will satisfy the required conditions.

Case $|T|=2$. If $l(\boldsymbol{t}_1)=l(\boldsymbol{t}_2)$, a constant polynomial with value $l(\boldsymbol{t}_1)$ interpolates l on T. Let us now assume that $l(\boldsymbol{t}_1)\neq l(\boldsymbol{t}_2)$; this implies that $|U|\geq 2$. Let $l(\boldsymbol{t}_1)=f$ and $l(\boldsymbol{t}_2)=g$. Since μ is not Abelian, Lemma 4.1(a) implies that there exist $a,b,u,v\in A$ and $t\in\mathrm{Pol}_2\,\mathbf{A}$ such that $a\,\mu\,b$, $u\,\mu\,v$, $t(a,u)=t(a,v)$ and $t(b,u)\neq t(b,v)$. Moreover, since $\boldsymbol{t}_1\neq\boldsymbol{t}_2$, there is $j\in[k]$ such that $\boldsymbol{t}_1(j)\neq\boldsymbol{t}_2(j)$, whence

$$(u,v)\in\mu\subseteq\langle\{(\boldsymbol{t}_1(1),\boldsymbol{t}_2(1)),\dots,(\boldsymbol{t}_1(k),\boldsymbol{t}_2(k))\}\rangle_{\mathrm{Con}\,\mathbf{A}},$$

for μ is the monolith of $\mathbf{A}$. Thus, by Lemma 2.1, there is $h\in\mathrm{Pol}_k\,\mathbf{A}$ such that $h(\boldsymbol{t}_1)=u$ and $h(\boldsymbol{t}_2)=v$. Since $(f,g)\in U^2\subseteq\mu\subseteq\langle\{(t(b,u),t(b,v))\}\rangle_{\mathrm{Con}\,\mathbf{A}}$, Lemma 2.1 yields a $p\in\mathrm{Pol}_1\,\mathbf{A}$ such that $p(t(b,u))=f$ and $p(t(b,v))=g$. Let us define the k-ary polynomial $p_T\colon A^k\to A$ by

$$p_T(\boldsymbol{z}):=p(d(t(b,h(\boldsymbol{z})),t(a,h(\boldsymbol{z})),t(a,u)))$$

for all $\boldsymbol{z}\in A^k$. Then for any $\boldsymbol{x}\in A^k$ such that $h(\boldsymbol{x})=u$, we have

$$p_T(\boldsymbol{x})=p(d(t(b,u),t(a,u),t(a,u)))=p(t(b,u))=f=l(\boldsymbol{t}_1).\tag{4.4}$$

In particular, this holds for $\boldsymbol{x} = \boldsymbol{t}_1$. Moreover, since $t(a,u) = t(a,v)$, we have

$$p_T(\boldsymbol{t}_2) = p(d(t(b,v), t(a,v), t(a,u))) = p(t(b,v)) = g = l(\boldsymbol{t}_2).$$

Furthermore, we obtain from $a \mathrel{\mu} b$ that for all $\boldsymbol{x} \in A^k$ the relations

$$\begin{aligned} p_T(\boldsymbol{x}) &= p(d(t(b,h(\boldsymbol{x})), t(a,h(\boldsymbol{x})), t(a,u))) \\ &\equiv_\mu p(d(t(b,h(\boldsymbol{x})), t(b,h(\boldsymbol{x})), t(b,u))) = p(t(b,u)) = f \in U \end{aligned}$$

hold. Since $U = o/\mu$, we conclude that $p_T(\boldsymbol{x}) \in U$ for every $\boldsymbol{x} \in A^k$.
Induction step. Let $|T| = n \geq 3$ and let us assume that l can be interpolated at any $n-1$ points of T by a polynomial whose image is a subset of U. We prove that l can be interpolated on T by a polynomial with image inside U. To this end, let us consider the following three sets

$$\begin{aligned} \beta &= \langle\{(\boldsymbol{t}_1(1), \boldsymbol{t}_2(1)), \ldots, (\boldsymbol{t}_1(k), \boldsymbol{t}_2(k))\}\rangle_{\operatorname{Con}\mathbf{A}}; \\ \eta &= \left\{ (p(\boldsymbol{t}_1), q(\boldsymbol{t}_1)) \;\middle|\; p, q \in \operatorname{Pol}_k \mathbf{A}, \begin{array}{ll} \forall \boldsymbol{x} \in A^k\colon & p(\boldsymbol{x}) \mathrel{\mu} q(\boldsymbol{x}), \\ \forall i \in \{2, \ldots, n\}\colon & p(\boldsymbol{t}_i) = q(\boldsymbol{t}_i) \end{array} \right\}; \\ \alpha &= \left\{ (p(\boldsymbol{t}_1), q(\boldsymbol{t}_1)) \;\middle|\; p, q \in \operatorname{Pol}_k \mathbf{A}, \begin{array}{ll} \forall \boldsymbol{x} \in A^k\colon & p(\boldsymbol{x}) \mathrel{\mu} q(\boldsymbol{x}), \\ \forall i \in \{3, \ldots, n\}\colon & p(\boldsymbol{t}_i) = q(\boldsymbol{t}_i) \end{array} \right\}. \end{aligned}$$

It is easy to see that η and α are reflexive and symmetric subuniverses of $\mathbf{A} \times \mathbf{A}$ that are contained in μ. Now, by Lemma 2.2, we have $\alpha, \beta, \eta \in \operatorname{Con}\mathbf{A}$, and $\alpha, \eta \in \{0_A, \mu\}$ since $\alpha, \eta \leq \mu$.

Our next goal is to prove that $\alpha = \eta$. The definition of α and η yields $\eta \leq \alpha$. If $\alpha = 0_A \leq \eta$, we have the desired equality; hence we assume that $0_A < \alpha \leq \mu$, i.e., $\alpha = \mu$.

We shall first prove that $C(1,1,\alpha,\beta,\eta)$. To this end let $(u,v) \in \beta$, $(a,b) \in \alpha$ and consider $p \in \operatorname{Pol}_2 \mathbf{A}$ such that $p(a,u) \mathrel{\eta} p(a,v)$. We have to show that $p(b,u) \mathrel{\eta} p(b,v)$. Since $(a,b) \in \alpha$, there exist $p_a, p_b \in \operatorname{Pol}_k \mathbf{A}$ such that

(1) $\forall \boldsymbol{x} \in A^k\colon p_a(\boldsymbol{x}) \mathrel{\mu} p_b(\boldsymbol{x})$;
(2) $\forall j \in \{3, \ldots n\}\colon p_a(\boldsymbol{t}_j) = p_b(\boldsymbol{t}_j)$;
(3) $p_a(\boldsymbol{t}_1) = a$ and $p_b(\boldsymbol{t}_1) = b$.

Since $(u,v) \in \beta$, Lemma 2.1 yields that there exist $q, q' \in \operatorname{Pol}_k \mathbf{A}$ such that $q(\boldsymbol{t}_1) = u$, $q(\boldsymbol{t}_2) = v$, $q'(\boldsymbol{t}_1) = v$ and $q'(\boldsymbol{t}_2) = u$. We define $p_u, p_v\colon A^k \to A$ by letting

$$p_u(\boldsymbol{x}) = d(q(\boldsymbol{x}), q'(\boldsymbol{x}), v) \text{ and } p_v(\boldsymbol{x}) = d(q(\boldsymbol{x}), u, v)$$

for all $\boldsymbol{x} \in A^k$. We observe that $p_u, p_v \in \operatorname{Pol}_k \mathbf{A}$, and moreover we can see that $p_u(\boldsymbol{t}_1) = u$, $w := p_u(\boldsymbol{t}_2) = d(v,u,v) = p_v(\boldsymbol{t}_2)$ and $p_v(\boldsymbol{t}_1) = v$. We further define $h, \hbar \in \operatorname{Pol}_k \mathbf{A}$ on each $\boldsymbol{x} \in A^k$ by

$$\begin{aligned} \hbar(\boldsymbol{x}) &= p(p_b(\boldsymbol{x}), p_u(\boldsymbol{x})), \\ h(\boldsymbol{x}) &= d(\hbar(\boldsymbol{x}), d(p(p_a(\boldsymbol{x}), p_v(\boldsymbol{x})), p(p_a(\boldsymbol{x}), p_u(\boldsymbol{x})), \hbar(\boldsymbol{x})), p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x}))). \end{aligned}$$

For each $j \in \{3, \dots, n\}$ we have $p(p_a(\boldsymbol{t}_j), p_u(\boldsymbol{t}_j)) = \hbar(\boldsymbol{t}_j)$, and hence

$$\begin{aligned} h(\boldsymbol{t}_j) &= d\big(\hbar(\boldsymbol{t}_j), d\big(p(p_a(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j)), p(p_a(\boldsymbol{t}_j), p_u(\boldsymbol{t}_j)), \hbar(\boldsymbol{t}_j)\big), p(p_b(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j))\big) \\ &= d\big(\hbar(\boldsymbol{t}_j), d\big(p(p_a(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j)), \hbar(\boldsymbol{t}_j), \hbar(\boldsymbol{t}_j)\big), p(p_b(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j))\big) \\ &= d\big(\hbar(\boldsymbol{t}_j), p(p_a(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j)), p(p_b(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j))\big) \\ &= d\big(\hbar(\boldsymbol{t}_j), p(p_b(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j)), p(p_b(\boldsymbol{t}_j), p_v(\boldsymbol{t}_j))\big) = \hbar(\boldsymbol{t}_j). \end{aligned}$$

Moreover, using $w = p_u(\boldsymbol{t}_2) = p_v(\boldsymbol{t}_2)$, we have

$$\begin{aligned} h(\boldsymbol{t}_2) &= d\big(\hbar(\boldsymbol{t}_2), d\big(p(p_a(\boldsymbol{t}_2), p_v(\boldsymbol{t}_2)), p(p_a(\boldsymbol{t}_2), p_u(\boldsymbol{t}_2)), \hbar(\boldsymbol{t}_2)\big), p(p_b(\boldsymbol{t}_2), p_v(\boldsymbol{t}_2))\big) \\ &= d\big(\hbar(\boldsymbol{t}_2), d\big(p(p_a(\boldsymbol{t}_2), w), p(p_a(\boldsymbol{t}_2), w), \hbar(\boldsymbol{t}_2)\big), p(p_b(\boldsymbol{t}_2), w)\big) \\ &= d\big(\hbar(\boldsymbol{t}_2), \hbar(\boldsymbol{t}_2), p(p_b(\boldsymbol{t}_2), w)\big) \\ &= p(p_b(\boldsymbol{t}_2), w) = p(p_b(\boldsymbol{t}_2), p_u(\boldsymbol{t}_2)) = \hbar(\boldsymbol{t}_2). \end{aligned}$$

For every $\boldsymbol{x} \in A^k$ we have $p_a(\boldsymbol{x}) \ \mu \ p_b(\boldsymbol{x})$, and hence we get

$$\begin{aligned} &p(p_a(\boldsymbol{x}), p_u(\boldsymbol{x})) \ \mu \ p(p_b(\boldsymbol{x}), p_u(\boldsymbol{x})) = \hbar(\boldsymbol{x}), \\ &p(p_a(\boldsymbol{x}), p_v(\boldsymbol{x})) \ \mu \ p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x})). \end{aligned}$$

Consequently,

$$\begin{aligned} d\big(p\big(p_a(\boldsymbol{x}), p_v(\boldsymbol{x})\big), p\big(p_a(\boldsymbol{x}), p_u(\boldsymbol{x})\big), \hbar(\boldsymbol{x})\big) \ \mu \ d\big(p\big(p_b(\boldsymbol{x}), p_v(\boldsymbol{x})\big), \hbar(\boldsymbol{x}), \hbar(\boldsymbol{x})\big) \\ = p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x})), \end{aligned}$$

and therefore

$$\begin{aligned} h(\boldsymbol{x}) &= d\big(\hbar(\boldsymbol{x}), d\big(p(p_a(\boldsymbol{x}), p_v(\boldsymbol{x})), p(p_a(\boldsymbol{x}), p_u(\boldsymbol{x})), \hbar(\boldsymbol{x})\big), p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x}))\big) \\ &\mu \ d\big(\hbar(\boldsymbol{x}), p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x})), p(p_b(\boldsymbol{x}), p_v(\boldsymbol{x}))\big) = \hbar(\boldsymbol{x}). \end{aligned}$$

From this we deduce that $h(\boldsymbol{t}_1) \ \eta \ \hbar(\boldsymbol{t}_1)$, and thus, by applying the unary polynomial $z \mapsto d(p(b,u), d(p(a,v), z, p(b,u)), p(b,v))$ to the pair $(p(a,v), p(a,u))$ from η, we have

$$\begin{aligned} p(b,v) &= d\big(p(b,u), p(b,u), p(b,v)\big) \\ &= d\big(p(b,u), d(p(a,v), p(a,v), p(b,u)), p(b,v)\big) \\ &\eta \ d\big(p(b,u), d(p(a,v), p(a,u), p(b,u)), p(b,v)\big) \\ &= h_1(\boldsymbol{t}_1) \ \eta \ h_2(\boldsymbol{t}_1) = p(b,u). \end{aligned}$$

Hence $p(b,u) \ \eta \ p(b,v)$. This proves that $C(1,1,\alpha,\beta,\eta)$. Now, Lemma 4.1(a) implies $[\alpha,\beta] \leq \eta$. Since $\boldsymbol{t}_1 \neq \boldsymbol{t}_2$ we have $0_A < \beta$, thus $\mu \leq \beta$. As $\alpha = \mu$ is non-Abelian, Lemma 4.1(c) yields $\alpha = \mu = [\mu,\mu] \leq [\mu,\beta] = [\alpha,\beta] \leq \eta \leq \alpha$. This concludes the proof of $\alpha = \eta$.

Now we construct the interpolating function. By the induction hypothesis there are $p, q \in \mathrm{Pol}_k \mathbf{A}$ with image inside U, such that p interpolates l at $\{\boldsymbol{t}_2, \dots, \boldsymbol{t}_n\}$ and q interpolates l at $\{\boldsymbol{t}_1, \boldsymbol{t}_3, \dots, \boldsymbol{t}_n\}$. Since $U^2 \subseteq \mu$, we have that $p(\boldsymbol{x}) \ \mu \ q(\boldsymbol{x})$ for all $\boldsymbol{x} \in A^k$, and moreover that $p(\boldsymbol{t}_i) = l(\boldsymbol{t}_i) = q(\boldsymbol{t}_i)$ for every

$i \in \{3, \ldots, n\}$. Hence $(p(\boldsymbol{t}_1), q(\boldsymbol{t}_1)) \in \alpha$. Since $\alpha = \eta$ and $q(\boldsymbol{t}_1) = l(\boldsymbol{t}_1)$, we have that $(p(\boldsymbol{t}_1), l(\boldsymbol{t}_1)) \in \eta$. Therefore, there exist $p_2, p_3 \in \mathrm{Pol}_k\, \mathbf{A}$ such that

(1) $\forall i \in \{2, \ldots, n\} \colon p_2(\boldsymbol{t}_i) = p_3(\boldsymbol{t}_i)$;
(2) $\forall \boldsymbol{x} \in A^k \colon p_2(\boldsymbol{x})\ \mu\ p_3(\boldsymbol{x})$;
(3) $p_2(\boldsymbol{t}_1) = p(\boldsymbol{t}_1)$ and $p_3(\boldsymbol{t}_1) = l(\boldsymbol{t}_1)$.

Let $p_T \colon A^k \to A$ be defined by $p_T(\boldsymbol{x}) = d(p(\boldsymbol{x}), p_2(\boldsymbol{x}), p_3(\boldsymbol{x}))$ for all $\boldsymbol{x} \in A^k$. Clearly, $p_T \in \mathrm{Pol}_k\, \mathbf{A}$. Moreover, we have that for all $i \in \{2, \ldots n\}$

$$p_T(\boldsymbol{t}_i) = d(p(\boldsymbol{t}_i), p_2(\boldsymbol{t}_i), p_3(\boldsymbol{t}_i)) = p(\boldsymbol{t}_i) = l(\boldsymbol{t}_i).$$

Furthermore,

$$p_T(\boldsymbol{t}_1) = d(p(\boldsymbol{t}_1), p_2(\boldsymbol{t}_1), p_3(\boldsymbol{t}_1)) = d(p(\boldsymbol{t}_1), p(\boldsymbol{t}_1), l(\boldsymbol{t}_1)) = l(\boldsymbol{t}_1).$$

Moreover, we have that for all $\boldsymbol{x} \in A^k$

$$p_T(\boldsymbol{x}) = d(p(\boldsymbol{x}), p_2(\boldsymbol{x}), p_3(\boldsymbol{x}))\ \mu\ d(o, p_2(\boldsymbol{x}), p_2(\boldsymbol{x})) = o.$$

Thus, $p_T(\boldsymbol{x}) \in o/\mu = U$ and we can conclude that p_T has codomain U and interpolates l on T. □

The following proposition is a partial converse of Proposition 3.13. In particular, it states that every finite subdirectly irreducible algebra with a monolith of type **3** (which is non-Abelian by [25, Theorem 5.7]) and a Mal'cev polynomial is an equational domain with respect to its clone of polynomial operations.

Proposition 4.5. *Let* $\mathbf{A}$ *be a finite subdirectly irreducible algebra with a Mal'cev polynomial, let* μ *be the monolith and let us assume that* μ *is non-Abelian. Then there exist* $f \in \mathrm{Pol}_4\, \mathbf{A}$ *and* $a \in A$ *such that* $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = a\}$*, and* f_μ *is constant.*

Proof. Let U be an equivalence class of μ with at least two distinct elements a, b and let $f \colon A^4 \to U$ be defined by

$$f(x_1, x_2, x_3, x_4) = \begin{cases} a & \text{if } x_1 = x_2 \text{ or } x_3 = x_4; \\ b & \text{otherwise.} \end{cases}$$

Proposition 4.4 implies that $f \in \mathrm{Pol}_4\, \mathbf{A}$. From the definition of f we see that $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = a\}$, and f_μ has $a/\mu = U = b/\mu$ as its single value. □

Next, we determine what can be said about a Mal'cev algebra whose universal algebraic geometry contains all finite relations.

Proposition 4.6. *Let* $\mathbf{A}$ *be an algebra with a Mal'cev polynomial and assume that every three-element quaternary relation on* A *is an algebraic set with respect to* $\mathrm{Pol}\, \mathbf{A}$*. Then for all* $\alpha, \beta \in \mathrm{Con}\, \mathbf{A} \backslash \{0_A\}$ *we have* $[\alpha, \beta] > 0_A$*.*

Proof. Let d be the Mal'cev polynomial and let $\alpha, \beta \in \operatorname{Con} \mathbf{A} \backslash \{0_A\}$. We prove that $\neg C(\alpha, \beta; 0_A)$. To this end, take $(a, b) \in \alpha \backslash 0_A$ and $(u, v) \in \beta \backslash 0_A$ to form $B = \{(a, a, u, u), (a, a, u, v), (a, b, u, u)\}$, which does not contain (a, b, u, v). By our assumption, the quaternary relation B is algebraic with respect to $\operatorname{Pol} \mathbf{A}$. Hence Lemma 3.1 yields that there are quaternary polynomials $p, q \in \operatorname{Pol}_4 \mathbf{A}$ such that $p(a, b, u, v) \neq q(a, b, u, v)$ and $p|_B = q|_B$. We use these to define the binary polynomial operation f for all $x_1, x_2 \in A$ by

$$f(x_1, x_2) := d(p(a, b, u, v), p(a, x_1, u, x_2), q(a, x_1, u, x_2)).$$

For d is a Mal'cev operation, we readily verify

$$\begin{aligned}
f(a, u) &= d(p(a, b, u, v), p(a, a, u, u), q(a, a, u, u)) = p(a, b, u, v), \\
f(a, v) &= d(p(a, b, u, v), p(a, a, u, v), q(a, a, u, v)) = p(a, b, u, v), \\
f(b, u) &= d(p(a, b, u, v), p(a, b, u, u), q(a, b, u, u)) = p(a, b, u, v), \\
f(b, v) &= d(p(a, b, u, v), p(a, b, u, v), q(a, b, u, v)) = q(a, b, u, v).
\end{aligned}$$

Since $p(a, b, u, v) \neq q(a, b, u, v)$, we have that $\neg C(\alpha, \beta; 0_A)$; thus the definition of the commutator yields $[\alpha, \beta] \neq 0_A$, cf. also Lemma 4.1(a). □

The following proposition provides a condition on the commutator which is sufficient for equational additivity in Mal'cev algebras.

Proposition 4.7. *Let* $\mathbf{A}$ *be an algebra on a set* A *with a Mal'cev polynomial* $d \in \operatorname{Pol}_3 \mathbf{A}$. *If for all* $\alpha, \beta \in \operatorname{Con} \mathbf{A} \backslash \{0_A\}$ *we have* $[\alpha, \beta] > 0_A$, *then* $\operatorname{Pol} \mathbf{A}$ *is equationally additive.*

Proof. Let $n \in \mathbb{N}$, let $C, B \in \operatorname{Alg}_n \operatorname{Pol} \mathbf{A}$ and let $\boldsymbol{w} \in A^n \backslash (C \cup B)$. We prove that there exist a constant $\tau_{\boldsymbol{w}} \in A$ and a polynomial $p_{\boldsymbol{w}} \in \operatorname{Pol}_n \mathbf{A}$ such that $p_{\boldsymbol{w}}(\boldsymbol{w}) \neq \tau_{\boldsymbol{w}}$ and $p_{\boldsymbol{w}}(\boldsymbol{x}) = \tau_{\boldsymbol{w}}$ for all $\boldsymbol{x} \in C \cup B$. Since C and B are algebraic and $\boldsymbol{w} \notin C \cup B$, Lemma 3.1 applied to B and C, respectively, gives us $f_C, f_B, g_C, g_B \in \operatorname{Pol}_n \mathbf{A}$ such that $f_C|_C = g_C|_C$, $f_B|_B = g_B|_B$, $f_C(\boldsymbol{w}) \neq g_C(\boldsymbol{w})$ and $f_B(\boldsymbol{w}) \neq g_B(\boldsymbol{w})$. Hence the congruences generated by these respective pairs are non-trivial:

$$\alpha := \langle\{(f_C(\boldsymbol{w}), g_C(\boldsymbol{w}))\}\rangle_{\operatorname{Con} \mathbf{A}} \neq 0_A, \quad \beta := \langle\{(f_B(\boldsymbol{w}), g_B(\boldsymbol{w}))\}\rangle_{\operatorname{Con} \mathbf{A}} \neq 0_A.$$

Therefore, the assumption yields that

$$[\langle\{(f_C(\boldsymbol{w}), g_C(\boldsymbol{w}))\}\rangle_{\operatorname{Con} \mathbf{A}}, \langle\{(f_B(\boldsymbol{w}), g_B(\boldsymbol{w}))\}\rangle_{\operatorname{Con} \mathbf{A}}] = [\alpha, \beta] > 0_A.$$

Thus, Lemma 4.3 implies that there exists a polynomial $q \in \operatorname{Pol}_2 \mathbf{A}$ such that

$$q(g_C(\boldsymbol{w}), g_B(\boldsymbol{w})) \neq q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w}))$$

and $q(a_1, f_B(\boldsymbol{w})) = q(f_C(\boldsymbol{w}), a_2) = q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w}))$ holds for all $a_1, a_2 \in A$.

Let us now define the polynomial $p_{\boldsymbol{w}} \in \operatorname{Pol}_n \mathbf{A}$ for all $\boldsymbol{x} \in A^n$ by

$$p_{\boldsymbol{w}}(\boldsymbol{x}) = q(d(g_C(\boldsymbol{x}), f_C(\boldsymbol{x}), f_C(\boldsymbol{w})), d(g_B(\boldsymbol{x}), f_B(\boldsymbol{x}), f_B(\boldsymbol{w}))).$$

For every $\boldsymbol{c} \in C$ we have

$$\begin{aligned} p_{\boldsymbol{w}}(\boldsymbol{c}) &= q(d(g_C(\boldsymbol{c}), f_C(\boldsymbol{c}), f_C(\boldsymbol{w})), d(g_B(\boldsymbol{c}), f_B(\boldsymbol{c}), f_B(\boldsymbol{w}))) \\ &= q(f_C(\boldsymbol{w}), d(g_B(\boldsymbol{c}), f_B(\boldsymbol{c}), f_B(\boldsymbol{w}))) \\ &= q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w})), \end{aligned}$$

while for every $\boldsymbol{b} \in B$ we have

$$\begin{aligned} p_{\boldsymbol{w}}(\boldsymbol{b}) &= q(d(g_C(\boldsymbol{b}), f_C(\boldsymbol{b}), f_C(\boldsymbol{w})), d(g_B(\boldsymbol{b}), f_B(\boldsymbol{b}), f_B(\boldsymbol{w}))) \\ &= q(d(g_C(\boldsymbol{b}), f_C(\boldsymbol{b}), f_C(\boldsymbol{w})), f_B(\boldsymbol{w})) \\ &= q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} p_{\boldsymbol{w}}(\boldsymbol{w}) &= q(d(g_C(\boldsymbol{w}), f_C(\boldsymbol{w}), f_C(\boldsymbol{w})), d(g_B(\boldsymbol{w}), f_B(\boldsymbol{w}), f_B(\boldsymbol{w}))) \\ &= q(g_C(\boldsymbol{w}), g_B(\boldsymbol{w})) \neq q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w})). \end{aligned}$$

Therefore, setting $\tau_{\boldsymbol{w}} = q(f_C(\boldsymbol{w}), f_B(\boldsymbol{w}))$ we have that $p_{\boldsymbol{w}}(\boldsymbol{w}) \neq \tau_{\boldsymbol{w}}$, whereas for all $\boldsymbol{x} \in C \cup B$ the equality $p_{\boldsymbol{w}}(\boldsymbol{x}) = \tau_{\boldsymbol{w}}$ holds. Hence Lemma 3.1 yields that $C \cup B \in \mathrm{Alg}_n(\mathrm{Pol}\,\mathbf{A})$. □

Theorem 4.8. *Let* $\mathbf{A}$ *be an algebra with at least two elements and a Mal'cev polynomial. Then the following statements are equivalent:*

(a) $\mathrm{Pol}\,\mathbf{A}$ *is equationally additive.*
(b) *For all* $n \in \mathbb{N}$*, any finite subset of* A^n *belongs to* $\mathrm{Alg}(\mathrm{Pol}(\mathbf{A}))$*.*
(c) *Every three-element subset of* A^4 *belongs to* $\mathrm{Alg}(\mathrm{Pol}(\mathbf{A}))$*.*
(d) *For all* $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}\backslash\{0_A\}$ *we have* $[\alpha, \beta] > 0_A$*.*

If A *is finite, (a)–(d) are furthermore equivalent to the following:*

(e) $\mathbf{A}$ *is subdirectly irreducible and the monolith* μ *is non-Abelian.*
(f) *There exist* $f \in \mathrm{Pol}_4\,\mathbf{A}$ *and* $a \in A$ *such that* $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = a\}$ *and* f_γ *is constant for all* $\gamma \in \mathrm{Con}\,\mathbf{A}\backslash\{0_A\}$*.*

Proof. Since $\mathrm{Pol}\,\mathbf{A}$ contains all constant operations, for every $n \in \mathbb{N}$, every singleton $\{(a_1, \ldots, a_n)\}$ can be written as

$$\{\boldsymbol{x} \in A^n \mid c_{a_1}^{[n]}(\boldsymbol{x}) = e_1^{[n]}(\boldsymbol{x}) \wedge \cdots \wedge c_{a_n}^{[n]}(\boldsymbol{x}) = e_n^{[n]}(\boldsymbol{x})\}$$

and $\emptyset = \{x \in A \mid c_a(x) = c_b(x)\}$ where $a, b \in A$ are distinct elements. Therefore, (a) implies (b), and (c) is just a special case of (b). Proposition 4.6 proves that (c) implies (d), while Proposition 4.7 shows that (d) implies (a). If we assume that A is finite, then by Corollary 3.11 we have that (a) implies (e); moreover, Proposition 4.5 shows that (e) implies (f). From Theorem 3.4 we see that (f) implies (a). □

We now specify our results to Artinian rings. We remark that the commutator of two ideals as defined in Sect. 2 in the case of rings coincides with the classical ideal product (cf. [35, Exercise 4.156(12)]).

Corollary 4.9. *Let* $\mathbf{R}$ *be an Artinian ring with unity. Then* $\operatorname{Pol}\mathbf{R}$ *is equationally additive if and only if* $\mathbf{R}$ *is isomorphic to the ring of linear endomorphisms of a finite dimensional vector space over a division ring.*

Proof. If $\mathbf{R}$ consists of a single element, then $\operatorname{Pol}\mathbf{R}$ is equationally additive and $\mathbf{R}$ is isomorphic to the endomorphism ring of a zero-dimensional vector space. From now on let us assume that zero and unity in $\mathbf{R}$ are distinct. If $\mathbf{R}$ is the ring of linear transformations of a finite dimensional vector space over a division ring, then it is simple and non-Abelian. Hence it satisfies (d) of Theorem 4.8. Let us now assume that $\operatorname{Pol}\mathbf{R}$ is equationally additive. Then $\mathbf{R}$ satisfies (d) of Theorem 4.8, and therefore it is subdirectly irreducible and the monolithic ideal I satisfies $I \cdot I = I$. Since the Jacobson radical $\operatorname{rad}\mathbf{R}$ is nilpotent (cf. [29, Theorem 4.3]), we infer that $\operatorname{rad}\mathbf{R} = \{0\}$. Thus, $\mathbf{R}$ is primitive (cf. [29, Propositions 4.1 and 4.4]). Hence the Wedderburn–Artin Theorem yields that $\mathbf{R}$ is isomorphic to the ring of linear transformations of a finite dimensional vector space over a division ring. □

Corollary 4.9 entails that the equational domains among all Artinian rings with unity are simple. This is a consequence of the fact that for Artinian rings with unity condition (d) of Theorem 4.8 implies simplicity. In the case of near-rings (cf. [36, Definition 1.1]) this is not any more true. We provide an example of a finite near-ring that is an equational domain but not simple.

Corollary 4.10. *For a prime number* $p > 2$ *the near-ring* $\mathbf{N} = (C_0(\mathbb{Z}_{p^2}); +, \circ)$ *of zero-preserving congruence-preserving functions on* $\mathbb{Z}_{p^2}$ *is an equational domain but not simple.*

Proof. One readily verifies that $\mathbf{N}$ satisfies the assumptions of [5, Corollary 5.2], and therefore $\mathbf{N}$ is subdirectly irreducible, and its monolithic ideal is equal to $M = (0 : p\mathbb{Z}_{p^2}) \cap (p\mathbb{Z}_{p^2} : \mathbb{Z}_{p^2})$, that is, the ideal consisting of all the maps that send $p\mathbb{Z}_{p^2}$ to 0 and $\mathbb{Z}_{p^2}$ to $p\mathbb{Z}_{p^2}$. Next, we show that $[M, M] \neq 0$. To this end, we define $a, b, x \colon \mathbb{Z}_{p^2} \to \mathbb{Z}_{p^2}$ as follows. For every $n \in \mathbb{Z}_{p^2}$ we set

$$a(n) := \begin{cases} 0 & \text{if } n \in \{1, \ldots, p-1\} \cup p\mathbb{Z}_{p^2}, \\ kp & \text{if } n \in \{kp+1, \ldots, (k+1)p-1\}, \text{ with } k \in \{1, \ldots, p-1\}; \end{cases}$$

$$b(n) := pn;$$

$$x(n) := \begin{cases} n & \text{if } n \in p\mathbb{Z}_{p^2}, \\ n \bmod p & \text{if } n \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}. \end{cases}$$

We observe that $a, b \in M$, and $x \in C_0(\mathbb{Z}_{p^2})$. Hence $a \circ (b+x) - a \circ x \in [M, M]$ (cf. [7, Definition 2.1] and [30, Theorem 3.1]). Thus, we need only show that $a \circ (b+x) - a \circ x$ is not constantly zero. One readily verifies that $(a \circ x)(1) = 0$ and $(a \circ (b+x))(1) = p$. Hence $[M, M] \neq \{0\}$, and Theorem 4.8 yields that $\mathbf{N}$ is an equational domain. Since $\{0\} \subsetneq M \subsetneq C_0(\mathbb{Z}_{p^2})$, we obtain that $\mathbf{N}$ is not simple. □

5. The number of equationally additive constantive expansions of finite Abelian groups

In the present section we study the number of equationally additive constantive expansions of Abelian groups on finite sets. We begin with a lemma.

Lemma 5.1. *Let $l, m, p, q \in \mathbb{N}$, with p and q prime, and m square-free. For each $n \in \mathbb{N}$ we write $\mathbf{Z}_n := (\mathbb{Z}_n; +, -, 0)$ for the cyclic group of integers $\{0, \ldots, n-1\}$ modulo n. Let $\mathbf{H}$ be a finite group the centre of which contains a subgroup of order q^2. Then the following statements hold:*

(a) *There are only finitely many equationally additive clones that contain the clone* $\mathrm{Clo}(\mathbf{Z}_m)$.
(b) *The number of equationally additive clones containing* $\mathrm{Pol}(\mathbf{Z}_p \times \mathbf{Z}_p)$ *or* $\mathrm{Pol}(\mathbf{Z}_{p^2})$, *respectively, is finite.*
(c) *If $l \geq 3$, then there are exactly $\aleph_0$ equationally additive clones that contain* $\mathrm{Pol}(\mathbf{Z}_{p^l})$.
(d) *There are exactly $\aleph_0$ equationally additive clones above* $\mathrm{Pol}(\mathbf{H} \times \mathbf{Z}_{p^l})$.

Proof. In [22, Corollary 1.3] it is shown that for square-free m, there are only finitely many clones containing $\mathrm{Clo}(\mathbf{Z}_m)$; hence (a) follows (the assumption of equational additivity is not used for this).

We now prove (b). Let $\mathbf{V}$ be an expansion of $\mathbf{Z}_{p^2}$ or of $\mathbf{Z}_p \times \mathbf{Z}_p$ such that $\mathrm{Pol}\,\mathbf{V}$ is equationally additive. Then Corollary 3.11 implies that $\mathbf{V}$ is subdirectly irreducible and that the monolith U is non-Abelian. Then, either $\mathbf{V}$ is simple and non-Abelian, or $\mathrm{Id}\,\mathbf{V}$ is a three-element chain with a non-Abelian monolith. Thus, $\mathbf{V}$ satisfies the property (SC1) as defined in [7, p. 310], which for finite expanded groups is equivalent to the property (SC1) given in [27, p. 48], as was argued in [7, p. 310]. Hence, by [27, Lemma 21], $\mathrm{Id}\,\mathbf{V}$ satisfies (APMI) as defined in [7, p. 310 and Definition 8.1, p. 324]. Thus, [7, Corollary 11.3] yields that $\mathbf{V}$ is weakly polynomially rich, that is, according to [7, Definition 3.7], the clone of polynomial functions of $\mathbf{V}$ coincides with the clone of extended type preserving functions as defined in [7, Definition 3.4]. Moreover, under (APMI), [7, Corollary 11.7] yields that the clone of functions preserving the extended types of $\mathbf{V}$ is generated by the binary functions it contains. Therefore, we can infer that $\mathrm{Pol}\,\mathbf{V}$ is generated by $\mathrm{Pol}_2\,\mathbf{V}$, a subset of the p^{2p^4}-element set of binary operations on the carrier of $\mathbf{V}$ that contains addition, the two projections and all p^2 constants. Thus, $\mathrm{Pol}\,\mathbf{V}$ is one of at most $2^{p^{2p^4} - p^2 - 3}$ clones.

Next, we prove (c). To this end let $N := \langle p^{l-1} \rangle$. Clearly, N is normal, $N \cong \mathbf{Z}_p$, and $\mathbf{Z}_{p^l}/N \cong \mathbf{Z}_{p^{l-1}}$. We define $f\colon \mathbb{Z}_{p^l}^4 \to \mathbb{Z}_{p^l}$ for $x_1, x_2, x_3, x_4 \in \mathbb{Z}_{p^l}$ by $f(x_1, x_2, x_3, x_4) := 0$ if $x_1 = x_2$ or $x_3 = x_4$, and $f(x_1, x_2, x_3, x_4) := p^{l-1}$ otherwise. Furthermore, for each $i \in \mathbb{N}\backslash\{1\}$, we set $\mathbf{V}_i = (\mathbb{Z}_{p^l}; +, -, 0, f, h_i)$, where the map $h_i\colon \mathbb{Z}_{p^l}^i \to \mathbb{Z}_{p^l}$ is given for all $x_1,\dots,x_i \in \mathbb{Z}_{p^l}$ by the product $h_i(x_1,\dots,x_i) := p^{l-2}\prod_{j=1}^{i} x_j$. One readily checks that N is an ideal of $\mathbf{V}_i$ and that any ideal I of $\mathbf{V}_i$ with at least one element $a \neq 0$ must contain $f(a,0,0,0) - f(0,0,0,0) = p^{l-1}$. Hence N is the monolith of $\operatorname{Id}\mathbf{V}_i$. Since $\Delta^{(4)}_{\mathbb{Z}_{p^l}} = \{\boldsymbol{x} \in (\mathbb{Z}_{p^l})^4 \mid f(\boldsymbol{x}) = 0\}$, Theorem 3.4 yields that $\operatorname{Clo}\mathbf{V}_i$ is equationally additive, and, by Corollary 3.5, $\operatorname{Pol}\mathbf{V}_i$ is equationally additive, as well. The map $\phi\colon \mathbb{Z}_{p^{l-1}} \to \mathbb{Z}_{p^l}/N$ sending each $x \in \mathbb{Z}_{p^{l-1}}$ to $\phi(x) = x + N$ provides an isomorphism between the algebra $\mathbf{B}_i := (\mathbb{Z}_{p^{l-1}}; +, -, 0, c_0^{[4]}, p_i)$ and $\mathbf{V}_i/N$, where $c_0^{[4]}$ is the quaternary constant zero function and p_i is given by the same term as h_i, namely, $p_i(x_1,\dots,x_i) = p^{l-2}x_1\cdots x_i$ for $x_1,\dots,x_i \in \mathbb{Z}_{p^{l-1}}$. For each $i \in \mathbb{N}\backslash\{1\}$, let $\mathbf{A}_i = (\mathbb{Z}_{p^{l-1}}; +, -, 0, p_i)$, cf. [10, proof of Theorem 1.3]. Then, for all $i \in \mathbb{N}\backslash\{1\}$ we have that $\operatorname{Pol}\mathbf{B}_i = \operatorname{Pol}\mathbf{A}_i$. Moreover, in [10, proof of Theorem 1.3] it is argued that

$$\forall i,j \in \mathbb{N}\backslash\{1\}\colon \operatorname{Pol}\mathbf{A}_i = \operatorname{Pol}\mathbf{A}_j \iff i = j. \tag{5.1}$$

It is our goal to show this for $\operatorname{Pol}\mathbf{V}_i$ and $\operatorname{Pol}\mathbf{V}_j$, as well. To this end, let $i,j \in \mathbb{N}\backslash\{1\}$ be such that $\operatorname{Pol}\mathbf{V}_i = \operatorname{Pol}\mathbf{V}_j$. We show that $i = j$. Let $\psi(N)$ be the congruence associated to N as defined in Sect. 2, that is, the kernel of ϕ. If $\operatorname{Pol}\mathbf{V}_i = \operatorname{Pol}\mathbf{V}_j$, then $(\operatorname{Pol}\mathbf{V}_i)/\psi(N) = (\operatorname{Pol}\mathbf{V}_j)/\psi(N)$, since $\psi(N)$ does not depend on the choice of i and j. Therefore, equation (2.1) yields that $\operatorname{Pol}(\mathbf{V}_i/N) = \operatorname{Pol}(\mathbf{V}_j/N)$. Then, since ϕ does not depend on the choice of i and j, $\operatorname{Pol}\mathbf{B}_i = \operatorname{Pol}\mathbf{B}_j$, and hence $\operatorname{Pol}\mathbf{A}_i = \operatorname{Pol}\mathbf{A}_j$. Finally, condition (5.1) yields that $i = j$. Thus, the map $i \mapsto \operatorname{Pol}\mathbf{V}_i$ from $\mathbb{N}\backslash\{1\}$ to the set of clones on $\mathbb{Z}_{p^l}$ is injective. This proves that there are at least $\aleph_0$ distinct equationally additive clones that contain $\operatorname{Pol}\mathbf{Z}_{p^l}$ for $l \geq 3$. Since there are at most $\aleph_0$ constantive Mal'cev clones on a finite set [4, Theorem 5.3], $\aleph_0$ is the exact number.

It remains to prove (d). For a finite Abelian group $\mathbf{G}'$, for a $n \in \mathbb{N}$ and for an operation $h: H^n \to H$, we define $\iota_{\mathbf{G}'}(h): (\mathbf{H} \times \mathbf{G}')^n \to \mathbf{H} \times \mathbf{G}'$ by $\iota_{\mathbf{G}'}(h)((x_1,y_1),\dots,(x_n,y_n)) := (h(x_1,\dots,x_n), 0)$ for all $\boldsymbol{x} \in H^n$ and $\boldsymbol{y} \in (G')^n$. Let us set $\mathbf{V}_{\mathbf{G}',f',\mathcal{H}} = (H \times G'; +, -, (0,0), f', (\iota_{\mathbf{G}'}(h))_{h\in\mathcal{H}})$, where $\mathcal{H}$ is any set of operations on H and f' is a quaternary operation on $H \times G'$. For $n \in \mathbb{N}$, and for any $p \in \operatorname{Pol}_n \mathbf{V}_{\mathbf{G}',f',\mathcal{H}}$, let $\pi(p)\colon H^n \to H$ be the function that maps every $\boldsymbol{x} \in H^n$ to the projection of $p\binom{\boldsymbol{x}}{\boldsymbol{0}}$ to its first component. We set $\pi\big(\operatorname{Pol}\mathbf{V}_{\mathbf{G}',f',\mathcal{H}}\big) := \bigcup_{n\in\mathbb{N}}\{\pi(p) \mid p \in \operatorname{Pol}_n \mathbf{V}_{\mathbf{G}',f',\mathcal{H}}\}$; moreover let $c_{\mathbf{0}}^{[4]}$ denote the constant zero function on $H \times G'$ of arity four.

Next, we demonstrate that for each constantive clone $\mathcal{H}$ on H that contains the group operation of $\mathbf{H}$, and for each finite Abelian group $\mathbf{G}'$, we have

$$\mathcal{H} = \pi\big(\operatorname{Pol}\mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}}\big). \tag{5.2}$$

For each $h \in H$ we see that $\pi(\iota_{\mathbf{G}'}(h)) = h$, and, since $\iota_{\mathbf{G}'}(h)$ is a fundamental operation of $\mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}}$, we thus have $\mathcal{H} \subseteq \pi\big(\operatorname{Pol}\mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}}\big)$. For the opposite inclusion, we note that, since the second parameter of $\mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}}$ is constant, we can write this algebra as $\hat{\mathbf{H}} \times \hat{\mathbf{G}}$ where $\hat{\mathbf{H}} = (H; +, -, 0, c_0^{[4]}, (h)_{h\in\mathcal{H}})$, and $\hat{\mathbf{G}} = (G'; +, -, 0, c_0^{[4]}, \big(c_0^{[\operatorname{ar}(h)]}\big)_{h\in\mathcal{H}})$ and $\operatorname{ar}(h)$ denotes the arity of $h \in \mathcal{H}$. We now extend the signature of these algebras by all constant values of $H \times G'$ as follows. We define

$$\begin{aligned}\hat{\mathbf{H}}_+ &:= (H; +, -, 0, c_0^{[4]}, (h)_{h\in\mathcal{H}}, (a)_{(a,b)\in H\times G'}),\\ \hat{\mathbf{G}}_+ &:= (G'; +, -, 0, c_0^{[4]}, \big(c_0^{[\operatorname{ar}(h)]}\big)_{h\in\mathcal{H}}, (b)_{(a,b)\in H\times G'}),\end{aligned}$$

such that the term operations of $\hat{\mathbf{H}}_+ \times \hat{\mathbf{G}}_+$ become the polynomial operations of $\hat{\mathbf{H}} \times \hat{\mathbf{G}} = \mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}}$. We now use the homomorphism property of the projection π_H onto $\hat{\mathbf{H}}_+$. For every n-ary term t in the language of $\hat{\mathbf{H}}_+$ and $\boldsymbol{x} \in H^n$ we have

$$\pi\big(t^{\hat{\mathbf{H}}_+\times\hat{\mathbf{G}}_+}\big)(\boldsymbol{x}) = \pi_H\left(t^{\hat{\mathbf{H}}_+\times\hat{\mathbf{G}}_+}\left(\begin{smallmatrix}\boldsymbol{x}\\ \mathbf{0}\end{smallmatrix}\right)\right) = \pi_H\begin{pmatrix} t^{\hat{\mathbf{H}}_+}(\boldsymbol{x})\\ t^{\hat{\mathbf{G}}_+}(\mathbf{0})\end{pmatrix} = t^{\hat{\mathbf{H}}_+}(\boldsymbol{x}),$$

hence the operation $\pi\big(t^{\hat{\mathbf{H}}_+\times\hat{\mathbf{G}}_+}\big)$ coincides with the operation $t^{\hat{\mathbf{H}}_+} \in \operatorname{Clo}\hat{\mathbf{H}}_+$. Moreover, as $\mathcal{H}$ is a constantive clone on H including the addition of $\mathbf{H}$, we observe that $\mathcal{H} \subseteq \operatorname{Pol}\hat{\mathbf{H}} \subseteq \operatorname{Clo}\hat{\mathbf{H}}_+ \subseteq \mathcal{H}$. Thus, π maps every polynomial operation of $\mathbf{V}_{\mathbf{G}',c_{\mathbf{0}}^{[4]},\mathcal{H}} = \hat{\mathbf{H}} \times \hat{\mathbf{G}}$ into $\mathcal{H}$, proving equation (5.2).

Let $\mathbf{G} := \mathbf{H} \times \mathbf{Z}_{p^l}$ and consider the subgroup $N := \big\langle(0, p^{l-1})\big\rangle$ of $\mathbf{G}$. Again, N is normal, $N \cong \mathbf{Z}_p$, and by mapping $\boldsymbol{x} \in \mathbf{H} \times \mathbf{Z}_{p^{l-1}}$ to $\phi(\boldsymbol{x}) = \boldsymbol{x} + N$, we see that $\mathbf{G}/N \cong \mathbf{H} \times \mathbf{Z}_{p^{l-1}}$. For $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4 \in G$ set $f(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) := (0, 0)$ if $\boldsymbol{x}_1 = \boldsymbol{x}_2$ or $\boldsymbol{x}_3 = \boldsymbol{x}_4$, and $f(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) := (0, p^{l-1})$ otherwise; this defines $f\colon G^4 \to G$. In the proof of Theorem 6 from [26], Idziak constructs a strictly increasing infinite sequence $\mathcal{C}_3' \subsetneq \mathcal{C}_4' \subsetneq \mathcal{C}_5' \subsetneq \dots$ of clones on H, containing the group operation and all constants from H. For any constantive clone $\mathcal{H}$ on H, any $f'\colon G^4 \to G$ and $j \in \mathbb{N}$, we abbreviate $\mathbf{V}_{j,f',\mathcal{H}} := \mathbf{V}_{\mathbf{Z}_{p^j},f',\mathcal{H}}$. A routine check establishes that N is an ideal of the expanded groups $\mathbf{V}_{l,f,\mathcal{H}}$, for any choice of $\mathcal{H}$. As argued in the proof of (c), any ideal I of $\mathbf{V}_{l,f,\mathcal{H}}$ with $\mathbf{0} \neq \boldsymbol{a} \in I$ must contain the element $f(\boldsymbol{a}, \mathbf{0}, \mathbf{0}, \mathbf{0}) - f(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = (0, p^{l-1})$. Therefore, N is the monolith of $\mathbf{V}_{l,f,\mathcal{H}}$, and the map ϕ from above provides an isomorphism from $\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}}$ to $\mathbf{V}_{l,f,\mathcal{H}}/N$, where $c_{\mathbf{0}}^{[4]}$ is the quaternary constant zero function. By Theorem 3.4 and Corollary 3.5, $\operatorname{Pol}\mathbf{V}_{j,f,\mathcal{H}}$ is equationally additive for every $j \in \mathbb{N}$, since f and the constant with value $\mathbf{0} = (0, 0)$ allow us to define the algebraic set $\Delta^{(4)}_{H\times\mathbb{Z}_{p^j}}$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two constantive clones on H that contain the group operation of $\mathbf{H}$ and let us assume that $\operatorname{Pol}\mathbf{V}_{l,f,\mathcal{H}_1} = \operatorname{Pol}\mathbf{V}_{l,f,\mathcal{H}_2}$. We prove that $\mathcal{H}_1 = \mathcal{H}_2$. Since N does not depend on the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$, setting $\psi(N)$ to be the congruence associated to N (cf. Section 2 and the proof of (c)), we have that $(\mathrm{Pol}\,\mathbf{V}_{l,f,\mathcal{H}_1})/\psi(N) = (\mathrm{Pol}\,\mathbf{V}_{l,f,\mathcal{H}_2})/\psi(N)$. Thus, (2.1) yields that

$$\mathrm{Pol}(\mathbf{V}_{l,f,\mathcal{H}_1}/N) = \mathrm{Pol}(\mathbf{V}_{l,f,\mathcal{H}_2}/N).$$

Since for every constantive clone $\mathcal{H}$ on H that contains the group operation of $\mathbf{H}$, ϕ provides an isomorphism between $\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}}$ and $\mathbf{V}_{l,f,\mathcal{H}}/N$ that does not depend on the choice of $\mathcal{H}$, we infer that the polynomial clones $\mathrm{Pol}\,\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}_1} = \mathrm{Pol}\,\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}_2}$ coincide, and therefore (5.2) yields

$$\mathcal{H}_1 = \pi\big(\mathrm{Pol}\,\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}_1}\big) = \pi\big(\mathrm{Pol}\,\mathbf{V}_{l-1,c_{\mathbf{0}}^{[4]},\mathcal{H}_2}\big) = \mathcal{H}_2.$$

This means that the map $\mathcal{H} \mapsto \mathrm{Pol}\,\mathbf{V}_{l,f,\mathcal{H}}$, defined for constantive expansions $\mathcal{H}$ of $\mathrm{Clo}\,\mathbf{H}$, is injective. Thus, for any of the $\aleph_0$ examples given by Idziak, we have a distinct equationally additive clone $\mathrm{Pol}\,\mathbf{V}_{l,f,\mathcal{C}'_j} \supseteq \mathrm{Pol}(\mathbf{H} \times \mathbf{Z}_{p^l})$. As argued in the proof of (c), [4, Theorem 5.3] shows that the number of constantive equationally additive expansions of $\mathrm{Clo}(\mathbf{H} \times \mathbf{Z}_{p^l})$ cannot be larger than $\aleph_0$. $\square$

Theorem 5.2. *Let $\mathbf{G}$ be a finite Abelian group with m elements. If m is square-free or the square of a prime, the set of equationally additive clones containing* $\mathrm{Pol}\,\mathbf{G}$ *is finite. Otherwise, it is countably infinite.*

Proof. Consider the representation of $\mathbf{G}$ as a direct product of cyclic groups of prime power order.

First, we suppose that every factor of this product is of prime order. That is, $\mathbf{G} \cong \prod_{i=1}^{n} (\mathbf{Z}_{p_i})^{k_i}$ with $n \geq 0$, distinct primes $p_1, \dots, p_n$ and integers $k_1, \dots, k_n \in \mathbb{N}$. If $\mathbf{G}$ is trivial, i.e., $n = 0$, or $n \geq 1$ and $k_i = 1$ for all $i \in [n]$, then $m = \prod_{i=1}^{n} p_i$ is square-free and the result follows from Lemma 5.1(a). Otherwise, there is $i \in [n]$ such that $k_i \geq 2$, and no generality is lost in assuming $i = 1$.

As a subcase we consider the possibility that $n = 1$, i.e., $\mathbf{G} \cong \mathbf{Z}_{p_1}^{k_1}$ with $k_1 \geq 2$. If $k_1 = 2$, then $m = k_1^2$, and hence, by Lemma 5.1(b), there are only finitely many equationally additive clones containing $\mathrm{Pol}\,\mathbf{G}$. If, otherwise, $k_1 \geq 3$, then $\mathbf{G} \cong \mathbf{Z}_{p_1}^{k_1-1} \times \mathbf{Z}_{p_1}$, $m = p_1^{k_1}$ and $\mathbf{Z}_{p_1} \times \mathbf{Z}_{p_1} \times \{0\}^{k_1-3}$ is an Abelian subgroup of $\mathbf{Z}_{p_1}^{k_1-1}$ of order p_1^2. Then the result follows from Lemma 5.1(d). This finishes the subcase where $n = 1$.

The opposite possibility is that $n \geq 2$; in this subcase we represent $\mathbf{G}$ as $\mathbf{G} \cong \mathbf{Z}_{p_1}^{k_1} \times \mathbf{Z}_{p_2}^{k_2-1} \times \big(\prod_{i=3}^{n} \mathbf{Z}_{p_i}^{k_i}\big) \times \mathbf{Z}_{p_2}$, and $\mathbf{Z}_{p_1} \times \mathbf{Z}_{p_1} \times \{0\}^{k_1+\dots+k_n-3}$ is an Abelian subgroup of $\mathbf{Z}_{p_1}^{k_1} \times \mathbf{Z}_{p_2}^{k_2-1} \times \prod_{i=3}^{n} \mathbf{Z}_{p_i}^{k_i}$ of order p_1^2. Clearly, the order m of $\mathbf{G}$ is neither square-free ($k_1 \geq 2$) nor the square of a prime ($n \geq 2$) in this case. Again Lemma 5.1(d) shows that the result claimed by the theorem is true.

Second, we suppose that there is a prime p in the representation of $\mathbf{G}$ with a cyclic factor $\mathbf{Z}_{p^l}$ where $l \geq 2$; hence m is not square-free. If that factor is the only one in the representation, then $\mathbf{G} \cong \mathbf{Z}_{p^l}$. The case where $l = 2$ and $m = p^2$ is solved by Lemma 5.1(b), and the case where $l \geq 3$, $m = p^l$, is handled by Lemma 5.1(c). Now let us assume that more factors appear in the decomposition, being either cyclic groups the order of which is a power of the same prime p or of another prime. This means there are a prime q, not necessarily distinct from p, an exponent $k \geq 1$ and an Abelian group $\mathbf{G}'$ such that $\mathbf{G} \cong \mathbf{Z}_{p^l} \times \mathbf{G}' \times \mathbf{Z}_{q^k}$. Then the order m is neither square-free, nor the square of a single prime. Moreover, $\langle \{p^{l-2}\} \rangle \times \{0_{\mathbf{G}'}\}$ is an Abelian subgroup of order p^2 of $\mathbf{Z}_{p^l} \times \mathbf{G}'$, and Lemma 5.1(d) shows that the number of equationally additive clones containing $\operatorname{Pol} \mathbf{G}$ is $\aleph_0$. $\square$

6. Characterization of equationally additive Boolean clones

In this section we shall describe which clones from Post's lattice are equationally additive (see also Fig. 1). This hence answers which algebras on the set $\{0, 1\}$ are equational domains. From Theorem 3.4 we know that equational additivity is equivalent to $\Delta_A^{(4)}$ being algebraic. For the two-element set we shall see that we can get along with a ternary relation instead of $\Delta_{\{0,1\}}^{(4)}$.

Lemma 6.1. *For any set A the relations $\Delta_A^{(4)}$ and*

$$\Delta_A^{(3)} := \left\{ (x_1, x_2, x_3) \in A^3 \;\middle|\; x_1 = x_2 \vee x_2 = x_3 \right\}$$

are primitive positively definable from each other, namely for all elements $x_1, \ldots, x_4 \in A$ we have

$$\begin{aligned}
(x_1, x_2, x_3) \in \Delta_A^{(3)} &\Leftrightarrow (x_1, x_2, x_2, x_3) \in \Delta_A^{(4)}, \\
(x_1, x_2, x_3, x_4) \in \Delta_A^{(4)} &\Leftrightarrow \exists y_1, y_2 \in A\colon \\
&\qquad (x_1, x_2, y_1) \in \Delta_A^{(3)} \wedge (y_1, x_3, x_4) \in \Delta_A^{(3)} \wedge \\
&\qquad (x_2, x_1, y_2) \in \Delta_A^{(3)} \wedge (y_2, x_3, x_4) \in \Delta_A^{(3)}.
\end{aligned}$$

Proof. It is obvious that $\Delta_A^{(3)}$ can be obtained by identifying arguments in $\Delta_A^{(4)}$. For the second equivalence, we take any tuple $(x_1, x_2, x_3, x_4) \in \Delta_A^{(4)}$. If $x_1 = x_2$, then we let $y_1 = y_2 = x_3$, and the right-hand side is satisfied. If $x_1 \neq x_2$, then $x_3 = x_4$ because $(x_1, x_2, x_3, x_4) \in \Delta_A^{(4)}$, and in this case we let $y_1 = x_2$ and $y_2 = x_1$ to satisfy the right-hand side. Now conversely, suppose that there are elements $y_1, y_2 \in A$ such that

$$(x_1, x_2, y_1), (y_1, x_3, x_4), (x_2, x_1, y_2), (y_2, x_3, x_4) \in \Delta_A^{(3)}.$$

In order to get a contradiction, we assume that $(x_1, x_2, x_3, x_4) \notin \Delta_A^{(4)}$, that is, $x_1 \neq x_2$ and $x_3 \neq x_4$. From the definition of $\Delta_A^{(3)}$ it follows that $x_2 = y_1$

and $x_1 = y_2$, and $y_1 = x_3$ and $y_2 = x_3$, wherefore $x_1 = y_2 = x_3 = y_1 = x_2$, contradicting the choice of x_1 and x_2. $\square$

The following is a folklore fact from clone theory.

Corollary 6.2. (cf. [13, Lemma 6.1.17] and [41, Lemma 1.3.1]) *For any set A we have*

$$\mathsf{Pol}\{\Delta_A^{(3)}\} = \mathsf{Pol}\{\Delta_A^{(4)}\} = \mathrm{Clo}\left(A; A^A\right),$$

i.e., the polymorphism clone of $\Delta_A^{(3)}$ coincides with that of $\Delta_A^{(4)}$, which is the clone of all essentially at most unary operations.

Proof. The first equality follows directly from Lemma 6.1, the second one is proved in [41, Lemma 1.3.1a)]. $\square$

Let us note that in the context of $A = \{0, 1\}$ the ternary Boolean relation $\Delta_A^{(3)}$ has become known in theoretical computer science under the pseudonym $\mathsf{dup}^3 = \{0,1\}^3 \setminus \{(0,1,0),(1,0,1)\}$ [15, Table 1, p. 61], the polymorphism clone of which is the clone N generated by all unary operations.

The Mal'cev condition considered in the following lemma will appear again in the characterization of the equationally additive Boolean clones in Theorem 6.5.

Lemma 6.3. *Any variety $\mathfrak{V}$ admitting the Mal'cev condition*

$$\begin{aligned} f(x,x,y) &\approx x \approx f(x,y,x),\\ f(y,x,x) &\approx f(x,y,f(y,x,x)) \end{aligned}$$

is congruence distributive.

Proof. By assumption there is a ternary term f in the language of $\mathfrak{V}$ such that the above identities are universally satisfied in $\mathfrak{V}$. Based on f we can define the following five ternary terms over the language of $\mathfrak{V}$ by substitution:

$$\begin{aligned} f_0(x,y,z) &:= x,\\ f_1(x,y,z) &:= f(x,y,f(z,x,x)),\\ f_2(x,y,z) &:= f(z,x,x),\\ f_3(x,y,z) &:= f(z,x,y), \text{ and}\\ f_4(x,y,z) &:= z. \end{aligned}$$

These form a sequence of Jónsson terms for $\mathfrak{V}$: The equations $f_i(x,y,x) \approx x$ for $0 \leq i \leq 4$ follow from the identity $f(x,x,y) \approx x \approx f(x,y,x)$, as does the condition $f_0(x,x,y) \approx x \approx f(x,x,f(y,x,x)) \approx f_1(x,x,y)$. The subsequent condition $f_1(x,y,y) \approx f(x,y,f(y,x,x)) \approx f(y,x,x) \approx f_2(x,y,y)$ follows from the second part of the Mal'cev condition, while the next two identities $f_2(x,x,y) \approx f(y,x,x) \approx f_3(x,x,y)$ are trivially fulfilled. The final part, that is, $f_3(x,y,y) \approx f(y,x,y) \approx y \approx f_4(x,y,y)$, follows from the first line of the assumed Mal'cev condition. Since the Jónsson identities for $f_0,\dots,f_4$ hold in $\mathfrak{V}$, the variety is congruence distributive, see [16, Theorem 12.6]. □

In [45], Tóth and Waldhauser explore necessary conditions for a relation to be the solution set of finitely many equations from a given clone $\mathcal{C}$. Since the complement of a finitary relation on a finite set is finite, every algebraic set that can be given as the solution set of an infinite system of $\mathcal{C}$-equations, can also be described by a finite subset of these equations: we use one equation to exclude each point of the complement (cf. Lemma 3.1). Hence, on a finite set A, the solution sets from [45] are exactly the algebraic sets in our sense. Tóth and Waldhauser investigate whether a relation is algebraic for $\mathcal{C}$ in terms of the *centralizer* clone $\mathcal{C}^* = \bigcup_{n\in\mathbb{N}} \operatorname{Hom}((A;\mathcal{C})^n,(A;\mathcal{C}))$, consisting of all functions that commute with all the operations in $\mathcal{C}$. With respect to the Boolean domain, Tóth and Waldhauser prove a characterization that can be rephrased in our terminology as follows:

Theorem 6.4. [45, Theorem 4.1] *For every clone $\mathcal{F}$ on the set $\{0,1\}$ we have $\operatorname{Alg}\mathcal{F} = \mathsf{Inv}(\mathcal{F}^*)$, where $\mathcal{F}^*$ is the centralizer clone of $\mathcal{F}$.*

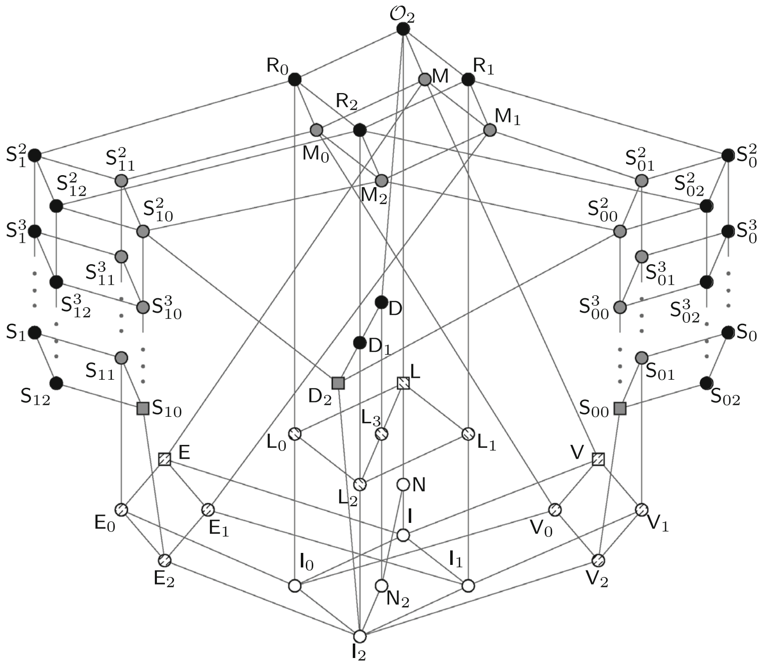
Theorem 6.4 implies that $\operatorname{Alg}\mathcal{F}^{**} = \mathsf{Inv}(\mathcal{F}^{***}) = \mathsf{Inv}(\mathcal{F}^{*}) = \operatorname{Alg}\mathcal{F}$ for every Boolean clone $\mathcal{F}$, since the tricentralizer and the centralizer of a set of operations coincide. Thus, to determine whether a Boolean clone is equationally additive, it suffices to consider its bicentral closure $\mathcal{F}^{**}$; in other words, considering all Boolean centralizer clones provides the complete picture. There are precisely 25 centralizer clones on $\{0,1\}$. They were originally presented by Kuznecov [34, p. 27], but the arguments given there remain rather sketchy. A complete description can be found in [24].

The following theorem characterizes which Boolean clones are equationally additive. The result is illustrated in Fig. 1, where also the identifiers for Boolean clones used in the theorem are clarified. With the exception of the top clone $\mathcal{O}_2$, we will denote Boolean clones by the standard symbols given in [17, Figure 2, p. 8], where explicit generating systems are listed, too.

Theorem 6.5. *Let $\mathcal{F}$ be a clone on $\{0,1\}$ and $g,h,p,t,t^\partial\colon \{0,1\}^3 \to \{0,1\}$ be the ternary Boolean operations given for $x,y,z \in \{0,1\}$ by the following rules:*

$$\begin{aligned}
h(x,y,z) &:= (x\wedge y)\vee(x\wedge z)\vee(y\wedge z),\\
g(x,y,z) &:= (x+y+z) \bmod 2,\\
p(x,y,z) &:= (x\wedge z)\vee(x\wedge\overline{y}\wedge\overline{z})\vee(\overline{x}\wedge\overline{y}\wedge z) \text{ where } \overline{x} = (1+x) \bmod 2,\\
t(x,y,z) &:= x\vee(y\wedge z),\\
t^\partial(x,y,z) &:= x\wedge(y\vee z),
\end{aligned}$$

that is, h is the Boolean majority operation, g the Boolean minority (Mal'cev) operation and p the Pixley operation. Then the following facts are equivalent.



○ clones of TCT-type **1** ⊘ clones of TCT-type **2** ⊘ clones of TCT-type **5**
● clones of TCT-type **3** ● clones of TCT-type **4**
●/● equationally additive clones
■ minimal equationally additive clones
▨/▨ maximal non-equationally additive clones

FIGURE 1. Lattice of Boolean clones labelled according to [17]; equationally additive clones forming the order filter shown by the completely filled nodes

(a) *$\mathcal{F}$ is equationally additive.*
(b) $\Delta^{(4)}_{\{0,1\}} \in \mathrm{Alg}_4\,\mathcal{F}$.
(c) $\Delta^{(3)}_{\{0,1\}} \in \mathrm{Alg}_3\,\mathcal{F}$.
(d) $\mathcal{F}^* \subseteq \mathsf{N}$, *that is, the centralizer of $\mathcal{F}$ is essentially at most unary.*
(e) $\mathsf{D}_1 \subseteq \mathcal{F}^{**}$, *that is, the bicentralizer of $\mathcal{F}$ contains all self-dual conservative operations.*
(f) $g, h \in \mathcal{F}^{**}$.
(g) $p \in \mathcal{F}^{**}$.
(h) $\mathsf{D}_2 \subseteq \mathcal{F}$ *or* $\mathsf{S}_{00} \subseteq \mathcal{F}$ *or* $\mathsf{S}_{10} \subseteq \mathcal{F}$.
(i) $h \in \mathcal{F}$ *or* $t \in \mathcal{F}$ *or* $t^\partial \in \mathcal{F}$.
(j) *Neither* $\mathcal{F} \subseteq \mathsf{E}$ *nor* $\mathcal{F} \subseteq \mathsf{V}$, *nor* $\mathcal{F} \subseteq \mathsf{L}$.

(k) *There is a ternary operation* $f \in \mathcal{F}^{[3]}$ *realizing the Mal'cev condition* $f(x,x,y) \approx x \approx f(x,y,x)$ *and* $f(y,x,x) \approx f(x,y,f(y,x,x))$.
(l) *The algebra* $(\{0,1\};\mathcal{F})$ *is of TCT-type* **3** *(Boolean algebra) or* **4** *(Boolean lattice).*
(m) *The algebra* $(\{0,1\};\mathcal{F})$ *is not of TCT-types* **1** *(group action),* **2** *(vector space) or* **5** *(semilattice).*
(n) *The algebra* $(\{0,1\};\mathcal{F})$ *generates a congruence distributive variety.*

Proof. Points (a) and (b) are equivalent by Theorem 3.4. Let ρ be one among $\Delta^{(4)}_{\{0,1\}}$ or $\Delta^{(3)}_{\{0,1\}}$. By Theorem 6.4 we have $\rho \in \operatorname{Alg}\mathcal{F} = \mathsf{Inv}\,\mathcal{F}^*$ if and only if $\mathcal{F}^* \subseteq \mathsf{Pol}\{\rho\} = \mathsf{N}$, where the last equality follows from Corollary 6.2. Hence, each of (b) and (c) is equivalent to (d). The latter is certainly equivalent to $\mathcal{F}^{**} \supseteq \mathsf{N}^* = \mathsf{D}_1$ because the centralizer of N is the centralizer of the negation and the two Boolean constants, that is, the intersection of the clone of self-dual operations with the clones of zero- and one-preserving functions, in other words, the clone D_1 of self-dual conservative operations. Thus, (d) and (e) are equivalent. Since D_1 is generated by $\{g,h\}$ (it is the join of the minimal clones L_2 and D_2 generated by g and h, respectively) or $\{p\}$ (cf. [17, Figure 2, p. 8]), statement (e) is equivalent to each of (f) and (g). Now for $\mathcal{G} = \mathsf{D}_2$ the least centralizer clone above $\mathcal{G}$ is $\mathcal{G}^{**} = \mathsf{D}_1$, for $\mathcal{G} \in \{\mathsf{S}_{00}, \mathsf{S}_{10}\}$, it is the clone $\mathcal{G}^{**} = \mathsf{Pol}\{\{0\},\{1\}\} \supseteq \mathsf{D}_1$ of conservative operations, cf. [24, Figure 5, p. 3158]. Therefore, from $\mathcal{G} \subseteq \mathcal{F}$, i.e. (h), we obtain $\mathsf{D}_1 \subseteq \mathcal{G}^{**} \subseteq \mathcal{F}^{**}$, i.e. (e). If $\mathcal{F}$ does not satisfy (h), then, according to Post's lattice, there is $\mathcal{G} \in \{\mathsf{E},\mathsf{V},\mathsf{L}\}$ such that $\mathcal{F} \subseteq \mathcal{G}$. From [24, Figure 5] we see that $\mathcal{G}$ is a centralizer clone, wherefore $\mathcal{F}^{**} \subseteq \mathcal{G}^{**} = \mathcal{G}$. This means that (e) fails, as $\mathsf{D}_1 \not\subseteq \mathcal{G}$, and hence, (e) and (h) are equivalent. Moreover, (h) and (j) are equivalent as E, V and L are the maximal elements in the complement of the order-filter of equationally additive clones in Post's lattice described by its minimal elements in (h). Furthermore, we infer from [17, Figure 2, p. 8] that the clones D_2, S_{00} and S_{10} are generated by the Boolean majority operation h, t and t^{∂}, respectively. Therefore, condition (h) is equivalent to (i).

We have now established that statements (a)–(j) are all equivalent. As our next step we shall show that (i) and (k) are equivalent. For this let us first assume the truth of (i) and let f be h, t or t^{∂}, respectively. If $f = h$, then $f(x,x,y) \approx x \approx f(x,y,x)$ and $f(y,x,x) \approx x \approx f(x,y,x) \approx f(x,y,f(y,x,x))$ are trivial. Otherwise, the conditions stated in (k) follow from the idempotence, commutativity and absorption laws for lattices; for example, for $f = t$ we have $f(y,x,x) \approx y \vee x \approx x \vee y \approx x \vee (y \wedge (y \vee x)) \approx f(x,y,f(y,x,x))$, and dually for $f = t^{\partial}$. Conversely, if we have a ternary operation f on $\{0,1\}$ subject to (k), then the equation $f(x,x,y) \approx x \approx f(x,y,x)$ uniquely determines the values of f on 6 out of the 8 argument triples. Thus there are in total $2^{8-6} = 4$ possible ternary Boolean operations f satisfying this condition. These are h, t, t^∂ and $e_1^{[3]}$, however, by the second part of (k), f cannot be the first projection, wherefore (i) follows.

Subsequently we will prove that (h) implies (l) and (by its contrapositive) that (m) implies (j). Since, clearly, (l) and (m) are equivalent, this will show that all the statements (a)–(m) are equivalent. Finally, we will show that (k) implies (n) and, by contradiction, that (n) implies (j), and the proof will be finished. Let us also note that the equivalence of statements (h) and (n), which appears as a part of our theorem is already known from the literature; to our knowledge it was first proved in [1, Proposition 2.1].

To prove that (h) implies (l), let $\mathcal{G} \in \{\mathsf{D}_2, \mathsf{S}_{00}, \mathsf{S}_{10}\}$ and suppose $\mathcal{G} \subseteq \mathcal{F}$. Then the polynomial expansion of $\mathcal{G}$ obtained by joining the Boolean clone I of all constant operations is the maximal Boolean clone of monotone operations $\mathsf{M} = \mathcal{G} \vee \mathsf{I} \subseteq \mathcal{F} \vee \mathsf{I}$. Therefore, $\mathcal{F} \vee \mathsf{I} \in \{\mathsf{M}, \mathcal{O}_2\}$, and the TCT-type of $\mathcal{F}$ is $\mathbf{4}$ if $\mathcal{F} \vee \mathsf{I} = \mathsf{M}$, or $\mathbf{3}$ if $\mathcal{F} \vee \mathsf{I} = \mathcal{O}_2$. This shows that (h) implies (l). Conversely, let us assume the negation of (j), that is, that $\mathcal{F} \subseteq \mathcal{G}$ for some $\mathcal{G} \in \{\mathsf{V}, \mathsf{E}, \mathsf{L}\}$. Then $\mathcal{G} \supseteq \mathsf{I}$, wherefore $\mathcal{F} \vee \mathsf{I} \subseteq \mathcal{G} \vee \mathsf{I} = \mathcal{G}$. Hence, $\mathcal{F}$ is polynomially equivalent to a semilattice, a vector space over $\mathsf{GF}(2)$, or—if it is essentially at most unary—a group action, which is the negation of (m).

Lastly, to show that (k) implies (n) let $f \in \mathcal{F}^{[3]}$ be an operation as claimed in (k) and let $\mathfrak{V}$ be the variety generated by $(\{0,1\}; \mathcal{F})$. Due to (k) the $\mathfrak{V}$-term $f(x,y,z)$ shows that $\mathfrak{V}$ admits the Mal'cev condition from Lemma 6.3, hence $\mathfrak{V}$ is congruence distributive, i.e., (n) holds. For the converse, we assume now (n) together with the negation of (j), which would imply that $\mathcal{F}$ and hence one of E, V or L would have a sequence of Jónsson operations. Therefore, $(\{0,1\}; \wedge, c_0, c_1)$, $(\{0,1\}; \vee, c_0, c_1)$ or $(\{0,1\}; +, c_0, c_1)$ would generate a congruence distributive variety, which is false, as in each case the congruence lattice of the square of the respective algebra already fails to be distributive. This contradiction shows that (n) entails (j). $\square$

The equivalence of statements (a) and (l) of Theorem 6.5 will be widely used in the subsequent sections.

Corollary 6.6. *Let $\mathbf{A}$ be an algebra on a two-element set. Then* $\operatorname{Clo}\mathbf{A}$ *is equationally additive if and only if* $\mathtt{typ}(\mathbf{A}) \in \{\mathbf{3}, \mathbf{4}\}$.

Knowing that the Boolean clones D_2, S_{00}, S_{10} and all clones above them are equationally additive, Theorem 3.4 tells that $\Delta^{(4)}_{\{0,1\}}$ is an algebraic set, hence definable as a solution set of some system of equations. In the following remark, we exhibit an explicit system of equations defining $\Delta^{(4)}_{\{0,1\}}$.

Remark 6.7. The clone D_2 is generated by the Boolean majority operation h, and every clone in the principal filter generated by this clone is equationally additive, since for all $x_1, x_2, x_3, x_4 \in \{0,1\}$ we have (cf. [12])

$$(x_1, x_2, x_3, x_4) \in \Delta^{(4)}_{\{0,1\}} \iff h(x_3, x_4, x_1) = h(x_3, x_4, x_2).$$

With respect to the clones S_{00} and S_{10}, we infer from [17, Figure 2, p. 8] that they are generated by the ternary functions t and t^{∂} (cf. Theorem 6.5), respectively, which are given for arbitrary elements $x, y, z \in \{0,1\}$ by the rules $t(x,y,z) = x \vee (y \wedge z)$ and $t^{\partial}(x,y,z) = x \wedge (y \vee z)$. The clones S_{00} and S_{10} are dual to each other, and for both of them, i.e., for $\tau \in \{t, t^{\partial}\}$, we have for all $x_1, x_2, x_3, x_4 \in \{0,1\}$ that (cf. [12])

$$\begin{aligned}(x_1,x_2,x_3,x_4) \in \Delta^{(4)}_{\{0,1\}} \iff \; & \tau(x_3,x_4,x_1) = \tau(x_3,x_4,x_2), \text{ and} \\ & \tau(x_4,x_3,x_1) = \tau(x_4,x_3,x_2).\end{aligned}$$

Computing the four-generated free algebra in the variety generated by the algebra $\mathbf{A} = (\{0,1\}; \tau)$, we find that there are exactly 53 quaternary term operations of $\mathbf{A}$, cf. [12]. One can check that for every pair of quaternary term operations $f, g \in (\mathsf{S}_{00})^{[4]}$ that agree on the 12 quadruples in $\Delta^{(4)}_{\{0,1\}}$, they also agree on at least one of the four elements of $\{0,1\}^4 \backslash \Delta^{(4)}_{\{0,1\}}$, see also [12]. It is hence impossible to define $\Delta^{(4)}_{\{0,1\}}$ by a single equation of the form $f(x_1,x_2,x_3,x_4) = g(x_1,x_2,x_3,x_4)$ over S_{00}.

In the following, we prove that the characterization of equational domains given in Corollary 6.6 carries over to finite E-minimal algebras as defined in [25, Definition 2.14]. We recall that a finite algebra is *E-minimal* if it has at least two elements and every unary idempotent polynomial is constant or the identity operation. Finite non-trivial p-groups provide prominent examples of such algebras. In [25, Theorem 4.32] it is proved that the prime quotients of every E-minimal algebra all have the same type. Hence one can associate to each E-minimal algebra one of the five types of minimal algebras introduced in Sect. 2.

We fix some notation that will only be used in the proof of the following lemma. For a set A, $n \in \mathbb{N}$, $i \in \{1,\dots,n\}$, $f\colon A^n \to A$ and $\boldsymbol{a} \in A^{n-1}$ we define the unary polynomial $f_i^{\boldsymbol{a}}\colon A \to A$ by $f_i^{\boldsymbol{a}}(x) = f(a_1,\dots,a_{i-1},x,a_i,\dots,a_{n-1})$ for all $x \in A$.

Lemma 6.8. *Let* $\mathbf{A}$ *be a subdirectly irreducible (finite) E-minimal algebra of type* $\mathbf{1}$. *Then* $\operatorname{Clo}\mathbf{A}$ *is not equationally additive.*

Proof. Since $\mathbf{A}$ is E-minimal, we have $k := |A| \geq 2$. Without loss of generality, let us assume that $A = \{1,\dots,k\}$ and the monolith μ of $\mathbf{A}$ has the form $\mu = \langle\{(1,2)\}\rangle_{\operatorname{Con}\mathbf{A}}$. Since $\mathbf{A}$ has type $\mathbf{1}$, [33, Theorem 4.4] implies that for all $n \in \mathbb{N}$ and for all $f \in \operatorname{Clo}_n\mathbf{A}$ (exactly) one of the following two statements holds:

(1) for each $i \in \{1,\dots,n\}$ and every $\boldsymbol{a} \in A^{n-1}$ we have $f_i^{\boldsymbol{a}}(1) = f_i^{\boldsymbol{a}}(2)$, or
(2) there is $j \in \{1,\dots,n\}$ such that for each $\boldsymbol{a} \in A^{n-1}$ the function $f_j^{\boldsymbol{a}}$ induces a permutation on A and $f_i^{\boldsymbol{a}}(1) = f_i^{\boldsymbol{a}}(2)$ for all $i \in \{1,\dots,n\}\backslash\{j\}$.

Let $f, g \in \mathrm{Clo}_4\,\mathbf{A}$ be such that $f|_{\Delta_A^{(4)}} = g|_{\Delta_A^{(4)}}$. We prove that

$$f(2,1,2,1) = g(2,1,2,1).$$

First we observe that for all $i \in \{1,\dots,4\}$ if $f_i^{(1,1,1)}(1) \neq f_i^{(1,1,1)}(2)$, then $g_i^{(1,1,1)}(1) \neq g_i^{(1,1,1)}(2)$ and vice versa: in fact, we have

$$\begin{aligned}&g_i^{(1,1,1)}(1) = g(1,1,1,1) = f(1,1,1,1) = f_i^{(1,1,1)}(1) \neq \\ &f_i^{(1,1,1)}(2) = f(1,\dots,\underset{i}{2},\dots,1) = g(1,\dots,\underset{i}{2},\dots,1) = g_i^{(1,1,1)}(2).\end{aligned}$$

Thus, either both f and g satisfy (1) or they both satisfy (2) with the same $j \in \{1,\dots,n\}$. If both f and g satisfy (1) or both satisfy (2) with $j \neq 1$, then we have

$$\begin{aligned}f(2,1,2,1) = f_1^{(1,2,1)}(2) = f_1^{(1,2,1)}(1) = f(1,1,2,1) = g(1,1,2,1)\\ = g_1^{(1,2,1)}(1) = g_1^{(1,2,1)}(2) = g(2,1,2,1).\end{aligned}$$

If both f and g satisfy (2) with $j = 1$, then we have

$$\begin{aligned}f(2,1,2,1) = f_3^{(2,1,1)}(2) = f_3^{(2,1,1)}(1) = f(2,1,1,1) = g(2,1,1,1)\\ = g_3^{(2,1,1)}(1) = g_3^{(2,1,1)}(2) = g(2,1,2,1).\end{aligned}$$

This concludes the proof of the fact that $\Delta_A^{(4)}$ is not an algebraic set. Therefore, Theorem 3.4 yields that $\mathrm{Clo}\,\mathbf{A}$ is not equationally additive. □

Lemma 6.9. *Let* $\mathbf{A}$ *be a (finite) E-minimal algebra. Then* $\mathbf{A}$ *generates a congruence distributive variety if and only if* $\mathbf{A}$ *has TCT-type* **3** *or* **4**.

Proof. If $\mathbf{A}$ is of type **3** or **4**, then [25, Lemma 4.29] yields that $|A| = 2$ and the result follows from the equivalence of (l) and (n) in Theorem 6.5. If $\mathbf{A}$ is of type **1**, **2**, or **5**, then [25, Theorem 8.6] yields that $\mathbf{A}$ does not generate a congruence distributive variety. □

Lemma 6.10. *Let* $\mathbf{A}$ *be a (finite) E-minimal algebra. Then* $\mathrm{Clo}\,\mathbf{A}$ *is equationally additive if and only if* $\mathbf{A}$ *has type* **3** *or* **4**.

Proof. $\mathbf{A}$ being E-minimal implies $|A| \geq 2$. If $\mathbf{A}$ has type **3**, **4** or **5**, then [25, Lemma 4.29] yields that $|A| = 2$ and the equivalence follows from Corollary 6.6.

The opposite case is that $\mathbf{A}$ is a finite E-minimal algebra of type **1** or **2**. This contradicts $\mathbf{A}$ having type **3** or **4**, hence, to fulfil the stated equivalence, we have to prove that $\mathrm{Clo}\,\mathbf{A}$ fails to be equationally additive. Since $2 \leq |A| < \aleph_0$, Proposition 3.8 implies that $\mathbf{A}$ is subdirectly irreducible. If $\mathbf{A}$ has type **1**, then Lemma 6.8 directly states that $\mathrm{Clo}\,\mathbf{A}$ is not equationally additive. Therefore, the case that is still to be discussed is that of a (finite non-trivial) subdirectly irreducible E-minimal algebra $\mathbf{A}$ of type **2**. Let μ be its monolith. Now [25, Theorem 13.9] implies that $\mathbf{A}$ is Mal'cev, and by [25, Theorem 4.32(2)] all its prime quotients have type **2**. In particular, we have $\mathtt{typ}(0_A, \mu) = \mathbf{2}$, and hence

[25, Theorem 5.7(3)] yields $[\mu, \mu] = 0_A$, i.e., that μ is Abelian. Thus, by Corollary 3.11, $\operatorname{Pol} \mathbf{A}$ cannot be equationally additive; therefore, by Corollary 3.5, $\operatorname{Clo} \mathbf{A}$ cannot be either. □

Theorem 6.11. *For a (finite) E-minimal algebra* $\mathbf{A}$ *the following statements are equivalent:*

(a) $\operatorname{Clo} \mathbf{A}$ *is equationally additive;*
(b) $\mathbf{A}$ *is of type* $\mathbf{3}$ *or* $\mathbf{4}$*;*
(c) $\mathbf{A}$ *generates a congruence distributive variety.*

Proof. The equivalence of (a) and (b) follows from Lemma 6.9. The equivalence of (b) and (c) follows from Lemma 6.10. □

7. Characterization of the equationally additive clones of self-dual operations

Let $A = \{0, 1, 2\}$ and let the permutation $\zeta_3 = (0\,1\,2)$ be the cyclic shift of the three elements of A. An operation $f\colon A^n \to A$ with $n \in \mathbb{N}$ is called *self-dual* if $f \in \{\zeta_3\}^*$, that is, if it commutes with ζ_3, in other words, if ζ_3 is an automorphism of the algebra $(A; f)$. The ideal of the lattice of clones on the three-element set A generated by the centralizer clone $\{\zeta_3\}^*$ is fully described in [46, Figure 2, p. 260]. In the present section we will stay with the notation introduced in [46, Section 1], and we will describe all equationally additive clones of self-dual operations on A.

Let $f_\pi^\infty\colon A^3 \to A$ be defined as follows: For each $\boldsymbol{x} = (x_1, x_2, x_3) \in A^3$ let

$$f_\pi^\infty(\boldsymbol{x}) = \begin{cases} x_2 & \text{if } \boldsymbol{x} \in \{(0,1,1),(1,2,2),(2,0,0)\}, \\ x_1 & \text{otherwise.} \end{cases}$$

With a quick glance at its operation table (cf. also [12]), one verifies that this operation coincides with the function introduced under the same name in [46, p. 265]. According to [46, Theorem 8, p. 266], the operation f_π^∞ generates the clone $\mathbf{a}_\infty\pi_\infty$, defined on page 261 of [46]. The dual $\mathbf{A}_\infty\pi_\infty$ of this clone with respect to the transposition $\sigma\colon A \to A$ switching 0 and 1 (cf. [46, pp. 255, 259, 261]) is given by applying this switch to every tuple of every relation defining $\mathbf{a}_\infty\pi_\infty$ as a polymorphism clone. It follows from this that $\mathbf{A}_\infty\pi_\infty$ arises as an isomorphic copy of $\mathbf{a}_\infty\pi_\infty$ by conjugating every operation in $\mathbf{a}_\infty\pi_\infty$ using the transposition σ. As a consequence $\mathbf{A}_\infty\pi_\infty$ is generated by $(f_\pi^\infty)^*\colon A^3 \to A$, given for all $\boldsymbol{x} = (x_1, x_2, x_3) \in A^3$ by

$$\begin{aligned}(f_\pi^\infty)^*(\boldsymbol{x}) &= \sigma(f_\pi^\infty(\sigma^{-1}(x_1), \sigma^{-1}(x_2), \sigma^{-1}(x_3))) \\ &= \begin{cases} x_2 & \text{if } \boldsymbol{x} \in \{(1,0,0),(0,2,2),(2,1,1)\}, \\ x_1 & \text{otherwise.} \end{cases}\end{aligned}$$

For use in the proof of Theorem 7.6, we observe that both f_π^∞ and $(f_\pi^\infty)^*$ are idempotent, that is, they equal the identity operation id_A when all three arguments are identified.

Lemma 7.1. *All clones on* $A = \{0,1,2\}$ *containing* $\mathbf{a}_\infty\pi_\infty$ *or* $\mathbf{A}_\infty\pi_\infty$ *from* [46, p. 261] *are equationally additive.*

Proof. Let $f \in \{f_\pi^\infty, (f_\pi^\infty)^*\}$. Moreover, let $S \subseteq A^4$ be the solution set of the following system of equations

$$\begin{cases} f(x_1,x_2,x_3) \approx f(x_1,x_2,x_4) \\ f(x_2,x_1,x_3) \approx f(x_2,x_1,x_4) \\ f(x_3,x_4,x_1) \approx f(x_3,x_4,x_2) \\ f(x_4,x_3,x_1) \approx f(x_4,x_3,x_2). \end{cases}$$

Using a computer (cf. [12]), one readily verifies that $S = \Delta_A^{(4)}$, whence $\Delta_A^{(4)}$ is algebraic over any clone containing f. Thus, Theorem 3.4 yields that every clone containing f is equationally additive. As, by [46, Theorem 8] (proved as [46, Theorem 30, p. 304]), f_π^∞ generates $\mathbf{a}_\infty\pi_\infty$, and hence $(f_\pi^\infty)^*$ generates $\mathbf{A}_\infty\pi_\infty$, the statement of the lemma follows. □

Corollary 7.2. *On a set* A *with* $|A| = 3$ *there are exactly* $2^{\aleph_0}$ *distinct equationally additive clones of self-dual operations.*

Proof. Combining the definition of $\mathbf{a}_\infty\pi_\infty$ on p. 261 of [46] with [46, Theorem 16, p. 269] (proved as Theorem 38, p. 313), we infer that there are exactly $2^{\aleph_0}$ distinct clones of self-dual operations on $\{0,1,2\}$ that contain the clone $\mathbf{a}_\infty\pi_\infty$. Therefore, the result follows from Lemma 7.1 and the fact that there are only countably many finitary operations on a finite set, thus no more than $2^{\aleph_0}$ subsets (clones) on A. □

Following [38], we say that clones $\mathcal{C}$ and $\mathcal{D}$ on the same set A are *algebraically equivalent*, denoted by $\mathcal{C} \sim_{\mathrm{alg}} \mathcal{D}$, if $\operatorname{Alg}\mathcal{C} = \operatorname{Alg}\mathcal{D}$. It was shown in [28] that on the three-element set there are 18 maximal clones (cf. [41, Table 4, p. 111]). Following [41, Definition 4.3.12], we define $\mathcal{L}$ as the clone of polymorphisms of $\{(a,b,c,d) \in \{0,1,2\}^4 \mid a+b = c+d \mod 3\}$.

Corollary 7.3. *Let* $A = \{0,1,2\}$, *and let* $\mathcal{C}$ *be a maximal clone on* A *that is not the clone* $\mathsf{Pol}\{\preceq\}$ *of monotone operations with respect to some bounded (linear) order* $\preceq$ *on* A. *Then the number of algebraically inequivalent subclones of* $\mathcal{C}$ *is*

(a) *finite, if* $\mathcal{C} = \mathcal{L}$, *the clone of (affine) linear operations;*
(b) *at most countable, if* $\mathcal{C} = \{\zeta_3\}^*$*;*
(c) *continuum, otherwise.*

Proof. In [20, Theorem 15] it is proved that below the clone of linear operations on any set of prime cardinality there are only finitely many clones at all (see also [20, Figure 3, p. 121] for the case $|A| = 3$), hence (a) follows.

In [38] (see also [9]) it is shown that on a finite set there are only finitely many equationally additive clones up to algebraic equivalence. Lemma 7.1 proves that all clones of self-dual operations on $\{0,1,2\}$ above $\mathbf{a}_\infty\pi_\infty$ or its dual $\mathbf{A}_\infty\pi_\infty$ are equationally additive, hence split into finitely many algebraic equivalence classes. In [46] it is proved that there are exactly $\aleph_0$ clones of self-dual operations that are neither above $\mathbf{a}_\infty\pi_\infty$ nor $\mathbf{A}_\infty\pi_\infty$, see [46, Figure 2, p. 260] and the description on page 261 of [46]. Therefore, there are at most countably many algebraically inequivalent clones of self-dual operations on $\{0,1,2\}$, as claimed in (b).

In [10, Proposition 5.4] it is proved that the $2^{\aleph_0}$ clones from [41, 3.1.4 Hauptsatz(ii), p. 79] are algebraically inequivalent. In [21, § 1, proof of Theorem 1] the authors show how to find a conjugate of the clones from [31] below each of the remaining maximal clones. Except for the case of monotone operations with respect to some bounded order, their argument also works for the family of clones defined in [41, 3.1.4 Hauptsatz(ii), p. 79]. Hence (c) follows. □

We now work towards the description of the equationally additive clones of self-dual operations on $A = \{0,1,2\}$. We first prove that equational additivity is hereditary with respect to restriction of the base set.

Lemma 7.4. *A clone $\mathcal{C}$ on a set X is equationally additive if and only if for every $B \subseteq X$ that is invariant under $\mathcal{C}$ the restriction $\mathcal{C}|_B := \{f|_B \mid f \in \mathcal{C}\}$ is equationally additive.*

Proof. Clearly, if restrictions to invariant subsets are equationally additive, then $\mathcal{C} = \mathcal{C}|_X$ itself is equationally additive. For the converse let $B \subseteq X$ belong to $\mathsf{Inv}\,\mathcal{C}$ and let $\mathcal{C}$ be equationally additive. By Theorem 3.4 there is an index set I and there are two families $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ of operations from $\mathcal{C}^{[4]}$ such that $\Delta_X^{(4)} = \{\boldsymbol{x} \in X^4 \mid \forall i \in I\colon p_i(\boldsymbol{x}) = q_i(\boldsymbol{x})\}$. The equality $\Delta_B^{(4)} = \Delta_X^{(4)} \cap B^4$ implies that $\Delta_B^{(4)} = \{\boldsymbol{b} \in B^4 \mid \forall i \in I\colon p_i|_B(\boldsymbol{b}) = q_i|_B(\boldsymbol{b})\}$, and thus Theorem 3.4 shows that $\mathcal{C}|_B$ is equationally additive. □

The following lemma will help to show that certain clones of self-dual operations on $A = \{0,1,2\}$ fail to be equationally additive. It can easily be verified based on the generators of the clones provided in [46, Theorems 6 and 7, p. 265 et seq.]. For the aid of the reader, we define these and a few auxiliary operations. By mnr_B and maj_B we denote the unique ternary minority and majority operation on an at most two-element set B, respectively (on $B = \{0,1\}$ we have $\mathrm{mnr}_B = g$ and $\mathrm{maj}_B = h$ as defined in Theorem 6.5). For all $x,y,z \in A$ we set:

$$a(x,y) := 2x + 2y + 1 \bmod 3$$

$$\mathrm{plus}_0(x,y,z) := \begin{cases} \mathrm{mnr}_{\{x,y,z\}}(x,y,z) & \text{if } |\{x,y,z\}| \le 2 \\ x + 1 \bmod 3 & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

$$m(x,y,z) := \begin{cases} \mathrm{maj}_{\{x,y,z\}}(x,y,z) & \text{if } |\{x,y,z\}| \le 2 \\ x & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

$$\mathrm{ps}(x,y,z) := \begin{cases} x & \text{if } |\{x,y,z\}| \le 2 \\ y & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

$$\mathrm{right}(x,y) := 2(x^2 + x + xy + y + y^2) \bmod 3$$

$$\mathrm{left}(x,y) := x^2 + 2x + xy + 2y + y^2 \bmod 3$$

right(x,y)	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

left(x,y)	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

According to [46, Theorem 6, p. 265], a generates $\mathbf{SL}$, m generates $\mathbf{TN}$, and plus_0 generates $\mathbf{L}_2$; moreover, by [46, Theorem 7, p. 266], $\{\mathrm{right}, \mathrm{ps}\}$ generates $\mathbf{aP}$. The clone $\mathbf{AP}$ is the dual of $\mathbf{aP}$ under the transposition $\sigma\colon A \to A$ swapping 0 and 1 (cf. [46, pp. 255, 259]). It is thus generated by the operations $\mathrm{ps}^* = \mathrm{ps}$ and $(\mathrm{right})^* = \sigma \circ \mathrm{right} \circ (\sigma^{-1} \times \sigma^{-1} \times \sigma^{-1}) = \mathrm{left}$.

It is evident from the given definition of the generators that all of these clones except for $\mathbf{SL}$ preserve every subset of $\{0,1,2\}$, i.e., that they are conservative.

Lemma 7.5. *Let* $\mathbf{aP}, \mathbf{AP}$ *be defined as in* [46, p. 259], *and let* $\mathbf{L}_2, \mathbf{SL}$ *be defined as in* [46, p. 256]. *Then the following facts about these clones on* $A = \{0,1,2\}$ *hold:*

(a) *The clones* $\mathbf{aP}$, $\mathbf{AP}$ *and* $\mathbf{L}_2$ *have* $B = \{0,1\}$ *as an invariant subset and* $\mathbf{aP}|_B = \mathsf{V}_2$, $\mathbf{AP}|_B = \mathsf{E}_2$, $\mathbf{L}_2|_B = \mathsf{L}_2$ *(cf. Fig. 1 for the notation).*

(b) *The clone generated by* $\mathbf{SL}$ *and all constant operations on* A *is the clone of polynomial operations of the* $\mathsf{GF}(3)$*-vector space* $\mathbb{Z}_3$.

Proof. (a) This follows by a brief inspection of the generating functions provided above: we have $\mathrm{right}|_B = \vee$, $\mathrm{left}|_B = \wedge$, $\mathrm{ps}|_B = e_1^{[3]}$ and $\mathrm{plus}_0|_B = \mathrm{mnr}_B = g$, where g is the Boolean minority operation as given in Theorem 6.5.

(b) We have $a(a(x,1), a(0,y)) = a(2x+3, 2y+1) = x + y \bmod 3$ for all $x, y \in A$; hence $\mathbf{SL}$ and the clone generated by addition modulo 3 have the same constantive expansion (the same polynomial operations). Therefore, $(A; \mathbf{SL})$ is polynomially equivalent to the $\mathsf{GF}(3)$-vector space $\mathbb{Z}_3$. ∎

We are now ready to prove that the characterization of equational additivity found to be true in Theorem 6.5(n) for Boolean clones persists in the interval of clones of self-dual operations on $\{0,1,2\}$.

Theorem 7.6. *For a clone $\mathcal{C} \subseteq \{\zeta_3\}^*$ on $A = \{0,1,2\}$ the following statements are equivalent.*

(a) *$\mathcal{C}$ is equationally additive;*
(b) *$\mathcal{C}$ contains one of the clones $\mathbf{a}_\infty\pi_\infty$, $\mathbf{A}_\infty\pi_\infty$ or* **TN** *(cf.* [46, Figure 2, p. 260]*);*
(c) *$(A;\mathcal{C})$ generates a congruence distributive variety.*

Proof. Let $\mathcal{C} \subseteq \{\zeta_3\}^*$. If $\mathbf{a}_\infty\pi_\infty \subseteq \mathcal{C}$, or $\mathbf{A}_\infty\pi_\infty \subseteq \mathcal{C}$, then Lemma 7.1 yields that $\mathcal{C}$ is equationally additive. Next, we prove that **TN** is equationally additive. To this end, let $m\colon A^3 \to A$ be defined as in [46, p. 264], cf. above; it is evident from its definition that m is a majority operation. According to [46, Theorem 6], we have that m generates **TN**. Moreover, it is easy to show via a computer (cf. [12]) that the solution set of the following system of equations is $\Delta_A^{(4)}$:

$$\begin{cases} m(x_1,x_2,x_3) \approx m(x_1,x_2,x_4) \\ m(x_2,x_1,x_3) \approx m(x_2,x_1,x_4). \end{cases}$$

Therefore, if $\mathbf{TN} \subseteq \mathcal{C} \subseteq \{\zeta_3\}^*$, then $\mathcal{C}$ is equationally additive by Theorem 3.4 and Corollary 3.5. Hence (b) implies (a).

Next, we prove that (a) implies (b). According to [46, Figure 2], **aP**, **AP**, $\mathbf{L}_2$ and **SL** are those clones of self-dual operations that are maximal with respect to not containing either of the clones $\mathbf{a}_\infty\pi_\infty$, $\mathbf{A}_\infty\pi_\infty$ or **TN**. Hence, as a consequence of Corollary 3.5, it suffices to prove that **aP**, **AP**, $\mathbf{L}_2$ and **SL** are not equationally additive. If **SL** were equationally additive, then so would be its constantive expansion, which, by Lemma 7.5(b), coincides with the clone of polynomial functions of the $\mathsf{GF}(3)$-vector space $\mathbb{Z}_3$. Since the vector space $\mathbb{Z}_3$ is simple and has a Mal'cev (term) operation, Corollary 3.11 says that equational additivity of its polynomial clone requires the vector space to be a non-Abelian algebra, which it is not (cf. [25, Exercise 3.2(2)]). Therefore, **SL** cannot be equationally additive. By Lemma 7.5(a), the clones **aP**, **AP** and $\mathbf{L}_2$ have $B := \{0,1\}$ as an invariant subset and $\mathbf{aP}|_B = \mathsf{V}_2$, $\mathbf{AP}|_B = \mathsf{E}_2$ and $\mathbf{L}_2|_B = \mathsf{L}_2$; each of these Boolean clones fails to be equationally additive by Theorem 6.5(j). Hence, by Lemma 7.4, none of **aP**, **AP** or $\mathbf{L}_2$ can be equationally additive. This establishes the equivalence of (a) and (b).

The fact that (b) implies (c) follows from the fact that the clones $\mathbf{a}_\infty\pi_\infty$, $\mathbf{A}_\infty\pi_\infty$ and **TN** have Jónsson operations, as argued in [14]: Namely, in the proof of [14, Proposition 5.3] it is shown how one can derive a sequence of five quasi-Jónsson operations from $f_\pi^\infty \in \mathbf{a}_\infty\pi_\infty$; since f_π^∞ is idempotent, these are actually Jónsson operations. The exact same can be done using $(f_\pi^\infty)^* \in \mathbf{A}_\infty\pi_\infty$. As observed above, the generator m of **TN** is a majority operation (and thus gives rise to a sequence of three Jónsson operations).

Finally, we show that (c) implies (b). To this end it suffices to prove that for all clones $\mathcal{D}$ below one of the clones **aP**, **AP**, $\mathbf{L}_2$, or **SL**, the algebra $(A;\mathcal{D})$ does not generate a congruence distributive variety. Now let $\mathcal{C} \in \{\mathbf{aP}, \mathbf{AP}, \mathbf{L}_2, \mathbf{SL}\}$ and assume that $(A; \mathcal{D})$ would generate a congruence distributive variety for some clone $\mathcal{D} \subseteq \mathcal{C}$. Then there would be a sequence of Jónsson operations in $\mathcal{D}$ and hence in $\mathcal{C}$. If $\mathcal{C} \in \{\mathbf{aP}, \mathbf{AP}, \mathbf{L}_2\}$, then $B := \{0, 1\}$ is invariant for $\mathcal{C}$ by Lemma 7.5(a), and by restricting the Jónsson operations to B we would obtain Jónsson operations in $\mathcal{C}|_B$. Hence $(B; \mathcal{C}|_B)$ would generate a congruence distributive variety, thus Theorem 6.5 excludes that $\mathcal{C}|_B \subseteq \mathsf{V}, \mathsf{E}$ or L. However, Lemma 7.5(a) shows that exactly the latter is the case since $\mathcal{C}|_B \in \{\mathsf{V}_2, \mathsf{E}_2, \mathsf{L}_2\}$. Therefore, the only possible remaining case is $\mathcal{D} \subseteq \mathcal{C} = \mathbf{SL}$ and hence $(A; \mathbf{SL})$ would generate a congruence distributive variety. But $(A; \mathbf{SL})$ is polynomially equivalent to the vector space $\mathbb{Z}_3$ by Lemma 7.5(b), and thus $(A; \mathbf{SL})^2$ is polynomially equivalent to $\mathbb{Z}_3^2$. Hence we obtain $\operatorname{Con}\big((A; \mathbf{SL})^2\big) = \operatorname{Con}(\mathbb{Z}_3^2)$, which fails to be distributive. This contradiction shows that our assumption is impossible and thus (b) follows. $\square$

8. The number of equationally additive clones on finite sets

In this section we investigate the cardinality of the order filter of equationally additive clones on a finite set. Our first basic observation is that the number of equationally additive clones on a set always is a lower bound for the number of equationally additive clones on any superset.

Lemma 8.1. *For sets $A \subseteq B$ there are at least as many equationally additive clones on B as on A.*

Proof. On any set the clone of all finitary operations is equationally additive, therefore the case $A = \emptyset$ is settled. If $A = B$ the statement is also evident. Therefore, from now on let us assume that $\emptyset \neq A \subsetneq B$. Let us choose elements $a \in A$ and $b \in B \setminus A$ and let us denote the set of all clones on A and B by $\mathfrak{L}_A$ and $\mathfrak{L}_B$, respectively. If, for $n \in \mathbb{N}$, $f\colon A^n \to A$ and $u\colon B^n \setminus A^n \to B$ are functions, then we denote by $f \oplus u\colon B^n \to B$ the operation defined from these two functions by the obvious case distinction. Clearly, any function constructed in this way preserves A, and conversely, any function $g\colon B^n \to B$ preserving A can be split up in this form. We employ the well-known injection $\Phi\colon \mathfrak{L}_A \to \mathfrak{L}_B$ between the clone lattices (cf. [41, 3.3.3 Einbettungssatz]), which is defined for every $\mathcal{F} \in \mathfrak{L}_A$ as

$$\Phi(\mathcal{F}) = \bigcup_{n \in \mathbb{N}} \{f\colon B^n \to B \mid f[A^n] \subseteq A, f|_{A^n} \in \mathcal{F}\}.$$

Letting $c_b^{[n]}\colon B^n \setminus A^n \to B$ be the constant n-ary function with value b, we observe for any $f\colon A^n \to A$ that $f \in \mathcal{F}$ if and only if $f \oplus c_b^{[n]} \in \Phi(\mathcal{F})$. Hence we see that Φ is injective, since for any n-ary function f separating clones $\mathcal{F}, \mathcal{G} \in \mathfrak{L}_A$ we also have $f \oplus c_b^{[n]}$ separating $\Phi(\mathcal{F})$ and $\Phi(\mathcal{G})$.

The proof will be done once we have shown that Φ preserves equational additivity. To this end assume that $\mathcal{F} \in \mathfrak{L}_A$ is equationally additive, and that, according to Theorem 3.4, there is some index set I and there are functions $f_i, g_i \in \mathcal{F}^{[4]}$ for $i \in I$ such that $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid \forall i \in I\colon f_i(\boldsymbol{x}) = g_i(\boldsymbol{x})\}$. Define $u\colon B^4 \backslash A^4 \to B$ by $u(\boldsymbol{b}) = b$ if $\boldsymbol{b} \in \Delta_B^{(4)}$ and by $u(\boldsymbol{b}) = a$ otherwise. Let $e_1^{[4]}$ denote the quaternary projection on A to the first coordinate. Then it is not hard to verify that

$$\Delta_B^{(4)} = \left\{\boldsymbol{x} \in B^4 \;\middle|\; e_1^{[4]} \oplus u(\boldsymbol{x}) = e_1^{[4]} \oplus c_b^{[4]}(\boldsymbol{x}) \wedge \forall i \in I\colon f_i \oplus c_b^{[4]}(\boldsymbol{x}) = g_i \oplus c_b^{[4]}(\boldsymbol{x})\right\}.$$

Thus, $\Delta_B^{(4)}$ is algebraic over the clone $\Phi(\mathcal{F})$, and hence $\Phi(\mathcal{F})$ is equationally additive by Theorem 3.4. $\square$

The theory presented in Sect. 6, in particular Theorem 6.5 and Fig. 1, shows that the number of equationally additive clones on any two-element set is countably infinite. The next step to take is investigating equationally additive clones on (at least) three-element carrier sets. In combination with Lemma 8.1, Corollary 7.2 shows that there are precisely continuum many equationally additive clones on any finite set with at least three elements. In the following we focus on the number of clones on finite sets that are equationally additive *and* contain all constant (unary) operations. In the subsequent proposition we start again by first considering three-element carrier sets. After that we shall exploit a construction by Ágoston, Demetrovics and Hannák from [2] to cover the general case.

Proposition 8.2. *On the set $A = \{0, 1, 2\}$ there are exactly $2^{\aleph_0}$ distinct equationally additive constantive clones.*

Proof. Let $f\colon A^3 \to A$ be defined as follows. For all $\boldsymbol{x} = (x_1, x_2, x_3) \in A^3$ we set

$$f(\boldsymbol{x}) = \begin{cases} 2 & \text{if } \boldsymbol{x} \in \{(0,2,0), (0,1,1), (1,2,2)\}, \\ x_1 & \text{otherwise.} \end{cases}$$

Furthermore, let $S \subseteq A^4$ be the solution set of the following system of equations

$$\begin{cases} f(x_1, x_2, x_3) \approx f(x_1, x_2, x_4) \\ f(x_2, x_1, x_3) \approx f(x_2, x_1, x_4) \\ f(x_3, x_4, x_1) \approx f(x_3, x_4, x_2) \\ f(x_4, x_3, x_1) \approx f(x_4, x_3, x_2). \end{cases}$$

Using a computer one quickly verifies that $\Delta_A^{(4)} = S$, cf. [12]. Thus, Theorem 3.4 yields that every clone that contains f is equationally additive.

We now prove that there are $2^{\aleph_0}$ constantive clones that contain f. For each $a \in A$, let $c_a\colon A \to A$ be the unary constant function with constant value a. For $k \in \mathbb{N}$ and $i \in [k]$, we set $\boldsymbol{e}_i^k$ to be the element of $\{0,1\}^k$ with 1 in the i-th component and 0 elsewhere. Moreover, we define the 'forbidden' set $B_k := \{\boldsymbol{r} \in \{0,1\}^k \mid 3 \leq \mathrm{wt}(\boldsymbol{r}) \leq k-1\}$, where $\mathrm{wt}(\boldsymbol{r})$ denotes the number of occurrences of 1 in $\boldsymbol{r}$, and we set $\rho_k := A^k \backslash (\{\boldsymbol{e}_1^k\} \cup B_k)$. For $n \in \mathbb{N}$, $n \geq 2$ we define $f_n\colon A^n \to A$ as follows. For all $\boldsymbol{x} \in A^n$ we let

$$f_n(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathrm{wt}(\boldsymbol{x}) = n, \text{i.e., } \boldsymbol{x} = \mathbf{1} := (1, \dots, 1), \\ 0 & \text{if } \boldsymbol{x} \in \{\boldsymbol{e}_1^n, \dots, \boldsymbol{e}_n^n\}, \\ 2 & \text{otherwise.} \end{cases}$$

As a first step, we prove that

$$\forall k \geq 2\colon \quad \{f\} \cup \{c_a \mid a \in A\} \subseteq \mathsf{Pol}(\{\rho_k\}). \tag{8.1}$$

Since ρ_k is reflexive, clearly $\{c_a \mid a \in A\} \subseteq \mathsf{Pol}(\{\rho_k\})$. Let $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \in \rho_k$. Then, the component-wise action of f on $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3)$ yields a tuple $\boldsymbol{z}$ satisfying for all $i \in [k]$ the condition $\boldsymbol{z}(i) = \boldsymbol{z}_1(i)$ or $\boldsymbol{z}(i) = 2$. If there is $i \in [k]$ with $\boldsymbol{z}(i) = 2$, then $\boldsymbol{z} \notin \{\boldsymbol{e}_1^k\} \cup B_k$, and therefore, $\boldsymbol{z} \in \rho_k$. Otherwise, $\boldsymbol{z} = \boldsymbol{z}_1 \in \rho_k$.

Next, we show that

$$\forall n \geq 2\colon \quad f_n \notin \mathsf{Pol}(\{\rho_{n+1}\}). \tag{8.2}$$

For $i \in [n]$, let $\boldsymbol{z}_i$ be the element of $\{0,1\}^{n+1}$ with 1 in its first and $(i+1)$-st component, and 0 elsewhere. For each $i \in \{1, \dots, n\}$ we have $\mathrm{wt}(\boldsymbol{z}_i) = 2$, thus $\boldsymbol{z}_i \in \rho_{n+1}$. Moreover, f_n acting component-wise on these tuples $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$ yields $f_n(\boldsymbol{z}_1, \dots, \boldsymbol{z}_n) = \boldsymbol{e}_1^{n+1} \notin \rho_{n+1}$. This proves that f_n does not preserve ρ_{n+1}.

Third, we verify that

$$\forall n \geq 2 \, \forall k \in \mathbb{N} \backslash \{n+1\}\colon \quad f_n \in \mathsf{Pol}(\{\rho_k\}). \tag{8.3}$$

For this we show that each of the tuples in $B_k \cup \{\boldsymbol{e}_1^k\}$ can only be obtained by the component-wise action of f_n on a sequence $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$ of tuples in A^k that contains at least one member outside ρ_k, that is, in $B_k \cup \{\boldsymbol{e}_1^k\}$. We assume $n \geq 2$ and split the proof into two cases according to the value of k.

Case $1 \leq k \leq n$. Let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n \in A^k$ be such that $f_n(\boldsymbol{z}_1, \dots, \boldsymbol{z}_n) = \boldsymbol{e}_1^k$, and let Z be the $(k \times n)$-matrix whose columns are the tuples $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$. As 1 has a unique preimage under f_n, the first row of Z is $\mathbf{1}$. Moreover, for all $2 \leq i \leq k$ there exists $l_i \in [n]$ such that the i-th row of Z is the tuple $\boldsymbol{e}_{l_i}^n$. Since $k-1 < n$, the matrix Z' obtained from Z by removing the first row, contains a column whose entries are all 0. Thus, $\boldsymbol{e}_1^k \in \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_n\}$, and so $\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_n\} \nsubseteq \rho_k$. If $k \geq 4$, then we also have to consider any $\boldsymbol{r} \in B_k$ with $\mathrm{wt}(\boldsymbol{r}) = w$, where $3 \leq w \leq k-1$, and we let $\boldsymbol{v}_1, \dots \boldsymbol{v}_n \in A^k$ be such that $f(\boldsymbol{v}_1, \dots \boldsymbol{v}_n) = \boldsymbol{r}$. Let V be the $(k \times n)$-matrix whose columns are $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Then V has exactly w rows whose entries are all 1. Let V' be the $((k-w) \times n)$-matrix obtained by removing those rows from V. For all $i \in [k-w]$ there exists $l_i \in [n]$ such that the i-th row of V' is the tuple $\boldsymbol{e}^n_{l_i}$. Since $k-w \leq k-3 < n$, there is a column $\boldsymbol{v}'$ in V' whose entries are all zero, and therefore, there exists $j \in [n]$ such that the column $\boldsymbol{v}_j \in \{0,1\}^k$ of V satisfies $\mathrm{wt}(\boldsymbol{v}_j) = w$. Since $3 \leq w \leq k-1$, we have $\boldsymbol{v}_j \in B_k$, and thus $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \nsubseteq \rho_k$.

Case $k \geq n+2$. Let $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n \in A^k$ be such that $f_n(\boldsymbol{z}_1, \dots, \boldsymbol{z}_n) = \boldsymbol{e}^k_1$, and let Z be the $(k \times n)$-matrix whose columns are the tuples $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$. As above, the first row of Z equals $\mathbf{1}$. Moreover, for all $2 \leq i \leq k$ there exists $l_i \in [n]$ such that the i-th row of Z is the tuple $\boldsymbol{e}^n_{l_i}$. Since $n \leq k-2 < k-1$, the matrix Z' obtained from Z by removing the first row, contains a column $\boldsymbol{z}' \in \{0,1\}^{k-1}$ with $\mathrm{wt}(\boldsymbol{z}') \geq 2$. If $\mathrm{wt}(\boldsymbol{z}') \leq k-2$, then Z contains a column $\boldsymbol{z} \in B_k$, i.e., $\boldsymbol{z} \notin \rho_k$. If $\mathrm{wt}(\boldsymbol{z}') = k-1$, then all the rows of Z' are of the form $\boldsymbol{e}^n_l$ for the same $l \in [n]$, and since $n \geq 2$, Z' has a column whose entries are all zeros. Thus, Z has a column equal to $\boldsymbol{e}^k_1 \notin \rho_k$. Since $k \geq n+2 \geq 4$, we additionally have to consider any $\boldsymbol{r} \in B_k$ with $\mathrm{wt}(\boldsymbol{r}) = w$, where $3 \leq w \leq k-1$, and we let $\boldsymbol{v}_1, \dots \boldsymbol{v}_n \in A^n$ be such that $f(\boldsymbol{v}_1, \dots \boldsymbol{v}_n) = \boldsymbol{r}$. Let V be the $(k \times n)$-matrix whose columns are $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Then V has w rows whose entries are all 1. Setting V' to be the $((k-w) \times n)$-matrix obtained by removing these rows from V, we have that for all $i \in [k-w]$ there exists $l_i \in [n]$ such that the i-th row of V' is $\boldsymbol{e}^n_{l_i}$. Thus, $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \{0,1\}^k$. Since $k-w \geq 1$ and $n \geq 2$, there is $j \in [n] \backslash \{l_1\}$, for which the entry of $\boldsymbol{v}_j$ in the first row of V that is distinct from $\mathbf{1}$ equals 0. Hence, $3 \leq w \leq \mathrm{wt}(\boldsymbol{v}_j) \leq k-1$, and so $\boldsymbol{v}_j \in B_k$, i.e., $\boldsymbol{v}_j \notin \rho_k$.

We are now ready to prove the statement of the theorem. We abbreviate $N := \mathbb{N} \backslash \{1,2\}$, and denote by $\mathfrak{L}_f$ the lattice of all constantive clones on A that contain f. Then we define $\Phi \colon \mathcal{P}(N) \to \mathfrak{L}_f$ as follows. For all $I \in \mathcal{P}(N)$ we let $\Phi(I) = \mathsf{Pol}(\{\rho_i \mid i \in I\})$. Equation (8.1) ensures that this function is well defined. We argue that Φ induces an order embedding of the lattice $(\mathcal{P}(N), \supseteq)$ into $(\mathfrak{L}_f, \subseteq)$. Clearly, Φ is compatible with the inclusion orders. To show that it also reflects them, take $I, J \subseteq N$ with $\Phi(J) \subseteq \Phi(I)$ and consider any $\iota \in I$. We thus have $\Phi(\{\iota\}) \supseteq \Phi(I) \supseteq \Phi(J)$. If $\iota \notin J$, equivalently, $J \subseteq N \backslash \{\iota\}$, then we would have $\Phi(J) \supseteq \Phi(N \backslash \{\iota\})$, therefore $f_{\iota-1} \in \Phi(N \backslash \{\iota\}) \subseteq \Phi(\{\iota\})$ by (8.3), but this would contradict (8.2). Hence $\iota \in J$, that is, we have demonstrated $I \subseteq J$. As every order embedding is injective, this proves that $|\mathfrak{L}_f| = 2^{\aleph_0}$, and the statement follows. $\square$

Theorem 8.3. *On a finite set A with at least three elements there are exactly $2^{\aleph_0}$ distinct equationally additive constantive clones.*

Proof. Let $A = \{0, \dots, n\}$ with $n \geq 2$. The case $n = 2$ is a consequence of Proposition 8.2. Thus we only consider the case $n \geq 3$. We define $f \colon A^4 \to A$ by

$$f(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \Delta^{(4)}_A, \\ n & \text{otherwise.} \end{cases}$$

TABLE 1. Numbers of clones on a finite non-trivial set A

$\lvert A\rvert$	All	Constantive	Equationally additive	Constantive equationally additive
2	$\aleph_0$ [42]	7 [42]	$\aleph_0$ (Fig. 1)	2 (Fig. 1)
≥ 3	$2^{\aleph_0}$ [31]	$2^{\aleph_0}$ [2]	$2^{\aleph_0}$	$2^{\aleph_0}$ (Theorem 8.3)

For each $i \in \mathbb{N}\backslash\{1,2\}$ we define $h_i\colon A^i \to A$ by

$$h_i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\{j \in [i]\colon x_j = 1\}| = 1 \text{ and } |\{j \in [i]\colon x_j = 2\}| = i-1, \\ & \text{or } |\{j \in [i]\colon x_j = 2\}| = 1 \text{ and } |\{j \in [i]\colon x_j = 1\}| = i-1; \\ 0 & \text{otherwise.} \end{cases}$$

For each $I \subseteq \mathbb{N}\backslash\{1,2\}$ we define $\mathbf{A}_I$ as the algebra $(A; \{f\} \cup \{h_i \mid i \in I\})$. Let $Z := A\backslash\{n\}$. Following [2], for each $i \in \mathbb{N}\backslash\{1,2\}$ we define $g_i\colon Z^i \to Z$ by

$$g_i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\{j \in [i]\colon x_j = 1\}| = 1 \text{ and } |\{j \in [i]\colon x_j = 2\}| = i-1, \\ & \text{or } |\{j \in [i]\colon x_j = 2\}| = 1 \text{ and } |\{j \in [i]\colon x_j = 1\}| = i-1; \\ 0 & \text{otherwise.} \end{cases}$$

For each $I \subseteq N\backslash\{1,2\}$ we define $\mathbf{Z}_I$ as the algebra $(Z; \{g_i \mid i \in I\})$. In [2] it was proved that

$$\forall I, J \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})\colon \operatorname{Pol}\mathbf{Z}_I = \operatorname{Pol}\mathbf{Z}_J \iff I = J. \tag{8.4}$$

For each $I \subseteq \mathbb{N}\backslash\{1,2\}$ we define $\mathbf{Z}_I^0$ as the algebra $(Z; \{c_0^{[4]}\} \cup \{g_i \mid i \in I\})$, where $c_0^{[4]}$ is the constant 0-function of arity 4. Clearly, $c_0^{[4]} \in \operatorname{Pol}_4\mathbf{Z}_I$ for all $I \subseteq \mathbb{N}\backslash\{1,2\}$. Thus, we have that for all $I \subseteq N\backslash\{1,2\}$ the algebras $\mathbf{Z}_I$ and $\mathbf{Z}_I^0$ are polynomially equivalent. Therefore, (8.4) yields that

$$\forall I, J \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})\colon \operatorname{Pol}\mathbf{Z}_I^0 = \operatorname{Pol}\mathbf{Z}_J^0 \iff I = J. \tag{8.5}$$

Let $I \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})$. Since $\Delta_A^{(4)} = \{\boldsymbol{x} \in A^4 \mid f(\boldsymbol{x}) = 0\}$, Theorem 3.4 yields that the clone $\operatorname{Pol}\mathbf{A}_I$ is equationally additive. Moreover, Proposition 3.13 yields that $\mathbf{A}_I$ is subdirectly irreducible and that $\mu = \langle\{(0,n)\}\rangle_{\operatorname{Con}\mathbf{A}_I}$ is the monolithic congruence of $\mathbf{A}_I$. Next, we show that $\mu = 0_A \cup \{(0,n),(n,0)\}$. Clearly, $S := 0_A \cup \{(0,n),(n,0)\}$ is an equivalence relation on A that contains the generators of μ and that is minimal with this property. Thus, it suffices to show that S is a subalgebra of $\mathbf{A}_I \times \mathbf{A}_I$. To this end let $\boldsymbol{a}, \boldsymbol{b} \in A^4$ with $\boldsymbol{a} \equiv_S \boldsymbol{b}$. Since $f(\boldsymbol{a}) \in \{0,n\}$ and $f(\boldsymbol{b}) \in \{0,n\}$ and $\{0,n\} \times \{0,n\} \subseteq S$, we have $(f(\boldsymbol{a}), f(\boldsymbol{b})) \in S$. This proves that S is closed under the component-wise action of f. Let $i \in I$ and let $\boldsymbol{c}, \boldsymbol{d} \in A^i$ with $\boldsymbol{c} \neq \boldsymbol{d}$ and $\boldsymbol{c} \equiv_S \boldsymbol{d}$. We show that $(h_i(\boldsymbol{c}), h_i(\boldsymbol{d})) \in S$. Since for all $z \in Z\backslash\{0\}$ the equivalence class of z modulo S is a singleton, we have that $\boldsymbol{c} \neq \boldsymbol{d}$ and $\boldsymbol{c} \equiv_S \boldsymbol{d}$ together yield that there exists $\ell \in [i]$ such that $\boldsymbol{c}(\ell), \boldsymbol{d}(\ell) \in \{0,n\}$ and $\boldsymbol{c}(\ell) \neq \boldsymbol{d}(\ell)$. Then the definition of h_i yields $h_i(\boldsymbol{c}) = 0 = h_i(\boldsymbol{d})$, and therefore $(h_i(\boldsymbol{c}), h_i(\boldsymbol{d})) = (0,0) \in S$. Thus, S is closed under the component-wise action of f and of h_i for all $i \in I$, and therefore is a subalgebra of $\mathbf{A}_I \times \mathbf{A}_I$. Hence $\mu = 0_A \cup \{(0,n),(n,0)\}$.

Next, we prove that $\mathbf{A}_I/\mu \cong \mathbf{Z}_I^0$. Define $\phi\colon A \to Z$ by $\phi(x) = x$ for all $x \in Z$ and $\phi(n) = 0$. We show that ϕ is a surjective homomorphism from $\mathbf{A}_I$ to $\mathbf{Z}_I^0$. To this end, let $\boldsymbol{b} \in A^4$, let $i \in I$, and let $\boldsymbol{a} \in A^i$. As $f(\boldsymbol{b}) \in \{0,n\}$, we have $\phi(f(\boldsymbol{b})) = 0 = c_0^{[4]}(\phi(b_1), \phi(b_2), \phi(b_3), \phi(b_4))$. Next, we demonstrate that $\phi(h_i(\boldsymbol{a})) = g_i(\phi(a_1), \ldots, \phi(a_i))$. We split the proof into two cases. Assuming

$\boldsymbol{a} \in Z^i$, we have, as the image of h_i is a subset of Z, that ϕ is the identity, and thus $\phi(h_i(\boldsymbol{a})) = h_i(\boldsymbol{a}) = g_i(\boldsymbol{a}) = g_i(\phi(a_1), \ldots, \phi(a_i))$. If, otherwise, $\boldsymbol{a} \notin Z^i$, then there is $j \in [i]$ such that $a_j = n$ and hence $\phi(a_j) = \phi(n) = 0$. Thus, we have $h_i(\boldsymbol{a}) = 0 = g_i(\phi(a_1), \ldots, \phi(a_i))$, and ϕ is a homomorphism. Since the kernel of ϕ is clearly equal to μ and ϕ is surjective, the first homomorphism theorem (cf. [16, Theorem 6.12]) yields $\mathbf{A}_I/\mu \cong \mathbf{Z}_I^0$.

Finally, we prove that

$$\forall I, J \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})\colon \operatorname{Pol}\mathbf{A}_I = \operatorname{Pol}\mathbf{A}_J \iff I = J. \tag{8.6}$$

To this end, let $I, J \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})$. Clearly, if $I = J$, then $\operatorname{Pol}\mathbf{A}_I = \operatorname{Pol}\mathbf{A}_J$. For the opposite implication, let us assume that $\operatorname{Pol}\mathbf{A}_I = \operatorname{Pol}\mathbf{A}_J$. Then (2.1) yields that $\operatorname{Pol}(\mathbf{A}_I/\mu) = (\operatorname{Pol}\mathbf{A}_I)/\mu = (\operatorname{Pol}\mathbf{A}_J)/\mu = \operatorname{Pol}(\mathbf{A}_J/\mu)$. Hence, since for all $L \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})$, ϕ induces an isomorphism between $\mathbf{Z}_L^0$ and $\mathbf{A}_L/\mu$ that is independent of L, we have $\operatorname{Pol}(\mathbf{Z}_I^0) = \operatorname{Pol}(\mathbf{Z}_J^0)$. Thus, we have $I = J$ by (8.5).

By (8.6), $\{\operatorname{Pol}\mathbf{A}_I \mid I \in \mathcal{P}(\mathbb{N}\backslash\{1,2\})\}$ is a set of distinct equationally additive constantive clones on A of cardinality $|\mathcal{P}(\mathbb{N}\backslash\{1,2\})| = 2^{\aleph_0}$. $\square$

We remark that the clones constructed in Theorem 8.3 all have the same universal algebraic geometry, namely $\bigcup_{n\in\mathbb{N}} \mathcal{P}(A^n)$. We summarize our knowledge of the number of equationally additive clones on finite sets in Table 1.

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Declarations

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Erhard Aichinger, Mike Behrisch and Bernardo Rossi
Institut für Algebra
Johannes Kepler Universität Linz
Altenberger Str. 69
4040 Linz
Austria
e-mail: erhard@algebra.uni-linz.ac.at

Bernardo Rossi
e-mail: bernardo.rossi@jku.at

Mike Behrisch
Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8–10
1040 Wien
Austria
e-mail: behrisch@logic.at

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