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Aequationes Mathematicae

Odd strength spherical designs attaining the Fazekas–Levenshtein bound for covering and universal minima of potentials

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Abstract. We characterize the cases of existence of spherical designs of an odd strength attaining the Fazekas–Levenshtein bound for covering and prove some of their properties. We also find all universal minima of the potential of regular spherical configurations in two new cases: the demihypercube on S^d, $d \geq 4$, and the 2_{41} polytope on S^7 (which is dual to the E_8 lattice).

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1. Statement of the problem and review of known results

Let $S^d := \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1^2 + \ldots + x_{d+1}^2 = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$. Throughout the text, ω_N will denote a configuration of N pairwise distinct points on S^d and $\mathbf{x}_1, \ldots, \mathbf{x}_N$ will denote the points in ω_N. We will also call ω_N a spherical code. We call a function $g : [-1, 1] \to (-\infty, \infty]$ an *admissible potential function* if g is continuous on $[-1, 1)$ with $g(1) = \lim_{t \to 1^-} g(t)$ and differentiable in $(-1, 1)$. Additional assumption(s) on the derivative(s) of g will be further specified in each case. Define the g-potential of ω_N by

$$p^g(\mathbf{x}, \omega_N) := \sum_{i=1}^{N} g(\mathbf{x} \cdot \mathbf{x}_i), \quad \mathbf{x} \in S^d.$$

We consider the following extremal problem over the sphere.

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Problem 1.1. Find the quantity

$$P^g(\omega_N, S^d) := \min_{\mathbf{x}\in S^d} p^g(\mathbf{x}, \omega_N) \tag{1.1}$$

and points $\mathbf{x}^* \in S^d$ attaining the minimum in (1.1).

We have an important special case when the potential function g is absolutely monotone on $[-1, 1)$. Then $g(t) = f(2 - 2t)$ for some completely monotone function f on $(0, 4]$. Recall that a function g is called *absolutely monotone* or *completely monotone* on an interval I if $g^{(k)} \geq 0$ or $(-1)^k g^{(k)} \geq 0$ on I, respectively, for every $k \geq 0$. The function g is *strictly absolutely* or *strictly completely monotone* on I if the corresponding inequality is strict in the interior of I for all $k \geq 0$. The kernel

$$g(\mathbf{x} \cdot \mathbf{y}) = f\left(|\mathbf{x} - \mathbf{y}|^2\right) \tag{1.2}$$

with a strictly absolutely monotone g on $[-1, 1)$ includes the Riesz s-kernel for $s > 0$ and the Gaussian kernel. After adding an appropriate positive constant, it also includes the logarithmic kernel and the Riesz s-kernel for $-2 < s < 0$.

The problem about absolute extrema on the sphere of potentials of spherical codes was earlier solved by Stolarsky [23,24] and Nikolov and Rafailov [20,21] for Riesz s-kernels, $s \neq 0$, and sets of vertices of a regular N-gon on S^1 and of a regular simplex, regular cross-polytope, and cube inscribed in S^d. Hardin et al. [16] proved that absolute minima of the potential of a regular N-gon on S^1 with respect to a decreasing and convex function of the geodesic distance are attained at points of the dual regular N-gon.

Recently, results from [20,21,23,24] for $s > -2$, $s \neq 0$ (except for absolute maxima for the cube), were extended to kernels (1.2) with an absolutely monotone function g and spherical designs of the highest (in a certain sense) strength. In particular, for a regular simplex on S^d, absolute maxima are at its vertices and absolute minima are at the antipods of its vertices, see [7]. Absolute maxima with respect to kernel (1.2) were found in [5] for sharp spherical codes that are antipodal or are designs of an even strength (called by some authors "strongly sharp"). Absolute maxima appear to be independent of the potential function g when g is strictly absolutely monotone on $[-1, 1)$ (they are at points of the code itself). Such absolute maxima are called *universal maxima*.

The set of universal minima (defined in a similar way) of any spherical code that, for some $m \in \mathbb{N}$, is a $(2m - 1)$-design forming m distinct dot products with some point $\mathbf{z} \in S^d$, is also known (we will call such codes m-stiff). It is exactly the set of all such points $\mathbf{z}$, see talk [6][1] by the author for the proof or Lemma 3.5 in [5],[2] from which this result follows. We will call the set of such

[1]Talk [6] was given in January, 2022 at ESI and can be found in the ESI's YouTube account.
[2]Paper [5] has been available on ArXiv since March 2022.

points $\mathbf{z}$ the dual configuration. We restate this result here as Theorem 2.5. Immediate consequences of this result are universal minima of a regular $2m$-gon on S^1, of a regular cross-polytope and cube on S^d, and of the 24-cell on S^3, since finding the dual configuration is elementary for these codes. Stiff spherical codes are exactly the ones attaining the Fazekas–Levenshtein bound for covering [13, Theorem 2] in the case of an odd strength. One of the goals of this paper is a further study of such spherical codes.

Universal minima of any strongly sharp code are at the antipods of points of the code. This also follows from [5, Lemma 3.5]. Immediate consequences of this fact (other than the regular simplex) are universal minima of a regular $(2m+1)$-gon on S^1, the Schläffi configuration on S^5, and the McLaughlin configuration on S^{21}. Strongly sharp spherical codes attain the Fazekas–Levenshtein bound for covering [13, Theorem 2] for even strength designs (this case has been well-studied before).

Boyvalenkov et al. [9, Theorems 3.4 and 3.7] (see also [10, Theorem 1.4])[3] proved universal upper and lower bounds for the potential of a general spherical design. These bounds become sharp in the cases mentioned above: in the case of a minimum for stiff and strongly sharp configurations and in the case of a maximum for sharp antipodal and strongly sharp ones. The lower bound is an analogue of the Fazekas–Levenshtein bound for covering [13, Theorem 2].

Paper [9] also showed that the universal maxima of the 600-cell on S^3 are vertices of the 600-cell itself. The work [10] proved that a number of known sharp codes are also stiff. Then the result from [6] implies that their universal minima are at points of the dual configuration. Paper [10] further studies the dual configuration for each code and antipods of the two strongly sharp codes on S^5 and S^{21} mentioned above. The author in paper [2][4] found explicitly the sets of all universal minima for five more stiff configurations (which are not sharp) on spheres of different dimensions as well as for the 56-point kissing configuration on S^6, which is a known sharp code (paper [10] gives one universal minimum of this code).

Certain remarkable spherical configurations are not stiff or strongly sharp. Universal minima of the regular icosahedron and regular dodecahedron on S^2 were characterized in [3][5] as well as universal minima of the E_8 lattice on S^7. Furthermore, one universal minimum and the corresponding absolute minimum value of the potential were found in [10][6] for the Leech lattice on S^{23}. Papers [3][7] and [10] establish general theorems (different to a certain extent)

[3]Papers [9] and [10] have been on ArXiv since July and October 2022, respectively.
[4]On ArXiv since December 2022.
[5]On ArXiv since October 9, 2022. The proof for icosahedron was briefly discussed in talk [6] in January 2022.
[6]On Arxiv since October 31, 2022.
[7]The general "skip one add two" theorem from [3] was stated in talk [4] in August 2022, see Theorem 2.6.

for the so-called "skip one add two" case and use them to establish the above mentioned results.

Critical points of the total potential of finite configurations of charges were also analysed (see [1,14] and references therein). This work is related to the known Maxwell's conjecture. A more detailed review of known results on extrema of potentials of spherical codes can be found, for example, in [2].

In this paper, we characterize the cases of existence of odd strength spherical designs attaining the Fazekas–Levenshtein bound for covering [13, Theorem 2] and prove certain properties of such designs and of sets of their universal minima (see Sect. 6). We also characterize universal minima of regular spherical configurations in two new cases: the demihypercube on S^d, $d \geq 4$ (including the 16-point sharp code[8] on S^4) and the 2_{41} polytope on S^7 (it is dual to the E_8 lattice). For stiff spherical codes that have no antipodal pairs, we show that their universal minima are the same as the ones of their symmetrizations about the origin.

One important application of Problem 1.1 is the polarization problem on the sphere. Papers [5,7,9,16,20,23] that we mentioned when reviewing results on extrema of potentials solve some of its cases. A more comprehensive review of known work on polarization can be found, for example, in book [8, Chapter 14] with most recent results reviewed in, e.g., [5].

The paper is structured as follows. Section 2 contains the necessary preliminaries. In Sect. 3, we characterize universal minima of the d-demicube for $d \geq 5$. In Sect. 4, we find all universal minima of the 2_{41} polytope on S^7. Section 5 extends known results on universal minima for some sharp configurations that are non-tight designs to their symmetrizations. In Sect. 6, we characterize the cases of existence of odd strength spherical designs attaining the Fazekas–Levenshtein bound for covering (Theorem 6.8) and establish certain properties of such designs and of sets of their universal minima.

2. Preliminaries

In this section, we state definitions and known facts used further in the paper. Define

$$w_d(t) := \gamma_d (1-t^2)^{d/2-1},$$

where the constant γ_d is such that w_d is a probability density on $[-1,1]$. The *Gegenbauer orthogonal polynomials* corresponding to the sphere S^d in $\mathbb{R}^{d+1}$ are terms of the sequence $\{P_n^{(d)}\}_{n=0}^{\infty}$ of univariate polynomials such that $\deg P_n^{(d)} = n$, $n \geq 0$, and

[8]In [10], the universal minima of the 16-point sharp code on S^4 were found without a characterization.

$$\int_{-1}^{1} P_i^{(d)}(t)P_j^{(d)}(t)w_d(t)\,dt = 0, \quad i \neq j,$$

normalized so that $P_n^{(d)}(1) = 1$, $n \geq 0$ (see [25, Chapter 4] or, e.g., [8, Chapter 5]).

For a configuration $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset S^d$, let $\mathcal{I}(\omega_N)$ be the set of all $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} P_n^{(d)}(\mathbf{x}_i \cdot \mathbf{x}_j) = 0. \tag{2.1}$$

We call $\mathcal{I}(\omega_N)$ *the index set* of ω_N. Let σ_d be the d-dimensional area measure on the sphere S^d normalized to be a probability measure. A configuration ω_N is called a *spherical n-design* if, for every polynomial p on $\mathbb{R}^{d+1}$ of degree at most n,

$$\frac{1}{N}\sum_{i=1}^{N} p(\mathbf{x}_i) = \int_{S^d} p(\mathbf{x})\,d\sigma_d(\mathbf{x}), \tag{2.2}$$

see the paper by Delsarte et al. [12]. The maximal number n in this definition is called the *strength* of the spherical design ω_N.

We recall the following equivalent definitions of a spherical design. Let $\mathbb{P}_n$ denote the space of all univariate polynomials of degree at most n.

Theorem 2.1. (see [12,13] or, e.g., [8, Lemma 5.2.2 and Theorem 5.4.2]) *Let $d, n \geq 1$ and $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a point configuration on S^d. The following are equivalent:*

(i) *ω_N is a spherical n-design;*
(ii) $\{1, \ldots, n\} \subset \mathcal{I}(\omega_N)$;
(iii) *for every polynomial $q \in \mathbb{P}_n$, we have $p^q(\mathbf{y}, \omega_N) = \sum_{i=1}^{N} q(\mathbf{y} \cdot \mathbf{x}_i) = C$, $\mathbf{y} \in S^d$, where C is a constant.*

If item (iii) holds in the above theorem, then $C = a_0(q)N$, where

$$a_0(q) := \int_{-1}^{1} q(t)w_d(t)\,dt \tag{2.3}$$

is the 0-th Gegenbauer coefficient of polynomial q. For a given $m \in \mathbb{N}$ and a given configuration $\omega_N \subset S^d$, denote by $\mathcal{D}_m(\omega_N)$ the set of all points $\mathbf{z} \in S^d$ for which the set of dot products

$$D(\mathbf{z}, \omega_N) := \{\mathbf{z} \cdot \mathbf{x}_i : i = 1, \ldots, N\}$$

has at most m distinct elements.

Definition 2.2. We call a point configuration $\omega_N \subset S^d$ *m-stiff*, $d, m \geq 1$, if ω_N is a spherical $(2m-1)$-design and the set $\mathcal{D}_m(\omega_N)$ is non-empty. The set $\mathcal{D}_m(\omega_N)$ of a given m-stiff configuration ω_N is called *the dual configuration* for ω_N.

For a given $m \in \mathbb{N}$, a spherical $(2m-1)$-design on S^d attains the Fazekas–Levenshtein bound [13, Theorem 2] if and only if it is m-stiff. Following [11], we call a configuration $\omega_N \subset S^d$ *sharp* if, for some $m \in \mathbb{N}$, it is a $(2m-1)$-design and there are exactly m distinct values of the dot product between distinct points in ω_N. If, in addition, ω_N is a $2m$-design, we call it *strongly sharp*.

The next statement is part of the classification of quadratures in [12] corresponding to spherical designs of the highest strength; i.e., stiff, strongly sharp, or sharp antipodal codes. It is an immediate consequence of [13, Theorem 2]. For the reader's convenience, we mention that a proof of Proposition 2.3 can also be found, for example, in [2, Proposition 7.2] or [9, Corollary 3.9]. Let $\{\varphi_1, \dots, \varphi_m\}$ be the fundamental polynomials for the set $-1 < \kappa_1^m < \dots < \kappa_m^m < 1$ of zeros of the Gegenbauer polynomial $P_m^{(d)}$; that is, $\varphi_i \in \mathbb{P}_{m-1}$, $\varphi_i(\kappa_i^m) = 1$, and $\varphi_i(\kappa_j^m) = 0$, $j \neq i$, $i = 1, \dots, m$.

Proposition 2.3. *If ω_N is an m-stiff configuration on S^d, then for every $\mathbf{z} \in \mathcal{D}_m(\omega_N)$, the set $D(\mathbf{z}, \omega_N)$ contains exactly m distinct elements located in $(-1,1)$, which are $\kappa_1^m, \dots, \kappa_m^m$. Furthermore, the number of indices i such that $\mathbf{z} \cdot \mathbf{x}_i = \kappa_j^m$ does not depend on $\mathbf{z}$ and equals $a_0(\varphi_j)N$, $j = 1, \dots, m$.*

In particular, if $m = 2$, then for every $\mathbf{z} \in \mathcal{D}_2(\omega_N)$, we have $D(\mathbf{z}, \omega_N) = \left\{-\frac{1}{\sqrt{d+1}}, \frac{1}{\sqrt{d+1}}\right\}$.

Remark 2.4. In view of Proposition 2.3, an m-stiff configuration may exist on S^d for given $m, d \geq 1$, only if all the numbers $a_0(\varphi_i)$, $i = 1, \dots, m$, are positive rationals. These numbers are the weights of the Gauss–Gegenbauer quadrature for integral (2.3) (the nodes are $\kappa_1^m, \dots, \kappa_m^m$). This is, in fact, an equivalent condition for the existence, see Theorem 6.8.

We next restate the result proved in talk [6]. Its complete proof can also be found in [2, Theorem 4.3].

Theorem 2.5. *Let $m \geq 1$, $d \geq 1$, and g be an admissible potential function with a convex derivative $g^{(2m-2)}$ on $(-1,1)$. If $\omega_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ is an m-stiff configuration on the sphere S^d, then the potential*

$$p^g(\mathbf{x}, \omega_N) = \sum_{i=1}^{N} g(\mathbf{x} \cdot \mathbf{x}_i), \quad \mathbf{x} \in S^d,$$

attains its absolute minimum over S^d at every point of the set $\mathcal{D}_m(\omega_N)$.

If, in addition, $g^{(2m-2)}$ is strictly convex on $(-1,1)$, then $\mathcal{D}_m(\omega_N)$ contains all points of absolute minimum of the potential $p^g(\cdot, \omega_N)$ on S^d.

In fact, Theorem 2.5 is a special case of [5, Lemma 3.5] if one uses $-g$ as the potential function in Lemma 3.5 in [5] and lets q there be the Hermite interpolating polynomial for $-g$ at the m values of the dot product that a point $\mathbf{z}$ from $\mathcal{D}_m(\omega_N)$ forms with points of ω_N. Then q has degree at most

$2m-1$, which is the strength of the spherical design ω_N. Since $-g^{(2m-2)}$ is concave on $(-1,1)$, a standard argument counting sign changes for consecutive derivatives of the difference $-g(t)-q(t)$ shows that $q(t) \geq -g(t)$, $t \in [-1,1]$, with the inequality being strict at any non-interpolation point when $g^{(2m-2)}$ is strictly convex on $(-1,1)$. Now, the assumptions of Lemma 3.5 in [5] are satisfied and we obtain the assertion of Theorem 2.5.

We will also need the "skip one add two" result from [3, Theorem 3.1].

Theorem 2.6. *Let $d, m \in \mathbb{N}$, $m \geq 2$, and $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a point configuration on S^d whose index set $\mathcal{I}(\omega_N)$ contains numbers $1, 2, \ldots, 2m-3, 2m-1, 2m$. Assume that numbers $-1 < t_1 < t_2 < \ldots < t_m < 1$ are such that*

$$\sum_{i=1}^{m} t_i < t_m/2 \quad \text{and} \quad \sum_{i=1}^{m} t_i^2 - 2\left(\sum_{i=1}^{m} t_i\right)^2 < \frac{m(2m-1)}{4m+d-3}, \tag{2.4}$$

and that the set $\mathcal{D}$ of points $\mathbf{x}^* \in S^d$ with $D(\mathbf{x}^*, \omega_N) \subset \{t_1, \ldots, t_m\}$ is non-empty. Let g be an admissible potential function with non-negative derivatives $g^{(2m-2)}$, $g^{(2m-1)}$, and $g^{(2m)}$ on $(-1,1)$. Then, for every point $\mathbf{x}^* \in \mathcal{D}$,

$$\min_{\mathbf{x} \in S^d} \sum_{i=1}^{N} g(\mathbf{x} \cdot \mathbf{x}_i) = \sum_{i=1}^{N} g(\mathbf{x}^* \cdot \mathbf{x}_i). \tag{2.5}$$

If, in addition, $g^{(2m)} > 0$ on $(-1,1)$, then the absolute minimum in (2.5) is achieved only at points of the set $\mathcal{D}$.

We remark that proofs of Theorems 2.5 and 2.6 utilize the Delsarte-Yudin method (also known as the Delsarte or linear programming or polynomial method), see the work by Delsarte, Goethals, and Seidel [12] or by Yudin [26]. A detailed description of this approach and references to works using it can also be found, in particular, in [9–11,17–19] and in [8, Chapter 5].

We next find universal minima of the potential of new cases of regular spherical configurations that we were able to handle using available tools. The proofs of these results are given in Sects. 3–5.

3. The 16-point sharp code on S^4 and the demihypercube

Denote by $\omega_{2d}^* := \{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}$, $d \geq 2$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are vectors of the standard basis in $\mathbb{R}^d$, the set of vertices of the regular cross-polytope inscribed in S^{d-1} and let U_d be the set $\left\{\left(\pm\frac{1}{\sqrt{d}}, \ldots, \pm\frac{1}{\sqrt{d}}\right)\right\} \subset \mathbb{R}^d$ of vertices of the cube inscribed in S^{d-1}. It is not difficult to see that $\mathcal{D}_2(\omega_{2d}^*) = U_d$ and $\mathcal{D}_2(U_d) = \omega_{2d}^*$.

Let $\overline{\omega}^d$, $d \geq 2$, be the set of $N = 2^{d-1}$ points $\left(\pm\frac{1}{\sqrt{d}}, \ldots, \pm\frac{1}{\sqrt{d}}\right) \in U_d$ with an even number of minus signs. This configuration forms the set of vertices of

a *d-demicube* (also called the *demihypercube*). The set $\widetilde{\omega}^d$ of vectors from U_d with an odd number of minus signs is a reflection of $\overline{\omega}^d$ with respect to any of the coordinate hyperplanes; i.e., $\widetilde{\omega}^d$ is an isometric copy of $\overline{\omega}^d$. Therefore, it is sufficient to consider just $\overline{\omega}^d$. We have $\overline{\omega}^d \cup \widetilde{\omega}^d = U_d$ and the two sets are disjoint. For d odd, we have $\widetilde{\omega}^d = -\overline{\omega}^d$ with neither set containing antipodal pairs. For d even, each of the sets $\overline{\omega}^d$ and $\widetilde{\omega}^d$ is itself antipodal.

Observe that for $d = 2$, both configurations consist of one antipodal pair; i.e., they are 1-stiff. For $d = 3$, each one is a regular simplex inscribed in S^2 (strongly sharp and, hence, not stiff). For $d = 4$, the set $\overline{\omega}^d$ consists of eight points $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ with an even number of minus signs. Each set $\overline{\omega}^4$ and $\widetilde{\omega}^4$ is an isometric copy of a regular cross-polytope in $\mathbb{R}^4$; i.e., it is 2-stiff. For $d = 5$, configuration $\overline{\omega}^d$ consists of 16 points on S^4 of the form $\left(\pm\frac{1}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}\right)$ with an even number of minus signs. This is the well-known sharp $(5, 16, 1/5)$-code. It was described by Gossett [15]. The set $\widetilde{\omega}^5$ is the antipode of this code. The 2-stiffness property of $\overline{\omega}^5$ was observed in [10]. We show that the d-demicube is 2-stiff for any $d \geq 6$. We start with the following auxiliary statement.

Lemma 3.1. *Let ω_N be a non-empty subset of $U_d = \left\{-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right\}^d$, $d \geq 3$. Then ω_N is a 3-design if, and only if, N is even and for every set I of one, two, or three pairwise distinct indices, exactly half of the vectors in ω_N have an even number of negative coordinates with indices in I and exactly half have an odd number of negative coordinates with indices in I.*

Proof. Let $\omega_N \subset U_d$ be arbitrary. Using the notation $\mathbf{y} = (y_1, \ldots, y_d)$ for a point $\mathbf{y} \in \omega_N$, define

$$S_i := \sum_{\mathbf{y}\in\omega_N} y_i, \quad S_{i,j} := \sum_{\mathbf{y}\in\omega_N} y_i y_j, \quad \text{and} \quad S_{i,j,k} := \sum_{\mathbf{y}\in\omega_N} y_i y_j y_k.$$

Observe that $S_{i,j}$ and $S_{i,j,k}$ do not depend on permutations of indices and that $S_{i,i} = \frac{N}{d}$. When some two indices coincide, say $i = j$, we have

$$S_{i,j,k} = \sum_{\mathbf{y}\in\omega_N} \frac{y_k}{d} = \frac{1}{d} S_k. \tag{3.1}$$

Formula (3.1) holds even if $i = j = k$. Let $\mathbf{x} = (x_1, \ldots, x_d) \in S^{d-1}$ be any vector. Then

$$\sum_{\mathbf{y}\in\omega_N} \mathbf{x}\cdot\mathbf{y} = \sum_{\mathbf{y}\in\omega_N} \sum_{i=1}^{d} x_i y_i = \sum_{i=1}^{d} \sum_{\mathbf{y}\in\omega_N} x_i y_i = \sum_{i=1}^{d} x_i \sum_{\mathbf{y}\in\omega_N} y_i = \sum_{i=1}^{d} S_i x_i,$$

$$\sum_{\mathbf{y}\in\omega_N} (\mathbf{x}\cdot\mathbf{y})^2 = \sum_{\mathbf{y}\in\omega_N} \left(\sum_{j=1}^{d} x_j y_j\right)^2 = \sum_{\mathbf{y}\in\omega_N} \sum_{i=1}^{d} \sum_{j=1}^{d} x_i x_j y_i y_j$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{\mathbf{y}\in\omega_N} x_i x_j y_i y_j = \sum_{i=1}^{d} x_i^2 \sum_{\mathbf{y}\in\omega_N} y_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{d} x_i x_j \sum_{\mathbf{y}\in\omega_N} y_i y_j$$

$$= \frac{N}{d} + \sum_{\substack{i,j=1\\ i\neq j}}^{d} S_{i,j} x_i x_j,$$

and

$$\sum_{\mathbf{y}\in\omega_N} (\mathbf{x}\cdot\mathbf{y})^3 = \sum_{\mathbf{y}\in\omega_N}\left(\sum_{j=1}^{d} x_j y_j\right)^3 = \sum_{\mathbf{y}\in\omega_N}\sum_{i,j,k=1}^{d} x_i x_j x_k y_i y_j y_k$$

$$= \sum_{i,j,k=1}^{d} x_i x_j x_k \sum_{\mathbf{y}\in\omega_N} y_i y_j y_k = \sum_{i,j,k=1}^{d} S_{i,j,k} x_i x_j x_k.$$

The configuration ω_N will be a 3-design if, and only if, the three sums above are constant, see Theorem 2.1. If

$$S_i = 0 \text{ for all } i, \quad S_{i,j} = 0, \text{ for } i \neq j, \quad \text{and} \quad S_{i,j,k} = 0, \text{ for } i,j,k \text{ distinct} \tag{3.2}$$

then three sums above will be constant (one should also use (3.1)). Conversely, if all three sums above are constant, then the first one has the same value for every vector $\pm\mathbf{e}_i$, which is $\pm S_i$, $i = 1,\dots,d$. Then $S_i = 0$ for all i. For every vector $\mathbf{x} \in S^{d-1}$ with the ℓ-th coordinate being $1/\sqrt{2}$, the n-th coordinate being $\pm\frac{1}{\sqrt{2}}$, and the remaining coordinates being zero, $\ell \neq n$, the value of the second sum is $N/d \pm S_{\ell,n} = \text{const}$. This forces $S_{\ell,n} = 0$, $\ell \neq n$. For vector $\pm\mathbf{x}$, where the ℓ-th, n-th, and m-th coordinates of $\mathbf{x}$ equal $1/\sqrt{3}$, ℓ, n, m are pairwise distinct, and the remaining coordinates are zero, the third sum equals (use (3.1) and the fact that $S_i = 0$ for all i)

$$\pm\frac{1}{3\sqrt{3}} \sum_{i,j,k\in\{\ell,n,m\}} S_{i,j,k} = \pm\frac{6}{3\sqrt{3}} S_{\ell,n,m} = \text{const}.$$

Then $S_{\ell,n,m} = 0$. Thus, ω_N is a 3-design if, and only if, relations (3.2) hold.

In each sum S_i, $S_{i,j}$, and $S_{i,j,k}$ in (3.2), all terms have the same absolute values. Then the value of each sum in (3.2) equals that common absolute value times the difference between the number of positive and negative terms. Therefore, relations (3.2) hold if, and only if, each sum in (3.2) has an equal number of positive and negative terms. This, in turn, will hold if, and only if, for any set of indices $I = \{i\}$ or $\{i,j\}$, where $i \neq j$, or $\{i,j,k\}$, where i,j,k are pairwise distinct, the number of vectors in ω_N with an even number of negative components with indices in I equals the number of vectors in ω_N with an odd number of negative components with indices in I. This also forces N to be even. □

Lemma 3.2. *The d-demicube $\overline{\omega}^d$, $d \geq 4$, is 2-stiff.*

Proof. Since $\overline{\omega}^d$ is a subset of the set of vertices of a cube, it is contained in two parallel hyperplanes. Thus, it remains to show that $\overline{\omega}^d$ is a 3-design. Let I be any set of k pairwise distinct indices, where $k = 1, 2, 3$. A combination of signs of coordinates corresponding to I with an even number of negative ones can be chosen in 2^{k-1} different ways. For each of these combinations, the remaining $d-k$ positions can have 2^{d-k} different combinations of signs with 2^{d-k-1} of them having an even number of minus signs. Then the total number of vectors in $\overline{\omega}^d$ with an even number of negative coordinates corresponding to I will be $2^{k-1} \cdot 2^{d-k-1} = 2^{d-2}$. By a similar argument, the number of vectors in $\overline{\omega}^d$ with an odd number of negative coordinates corresponding to I will also be 2^{d-2}. Lemma 3.1 now implies that $\overline{\omega}^d$ is a 3-design and, hence, is 2-stiff. $\square$

We next find the dual configuration of the d-demicube. For $d = 2$, the d-demicube is a pair of antipodal vectors, which is 1-stiff. Its dual is the perpendicular pair of antipodal vectors. For $d = 3$, the d-demicube is a regular simplex, which is not stiff, since it is strongly sharp. For $d = 4$, the configuration $\overline{\omega}^d$ is a regular cross-polytope, and its dual is the corresponding cube inscribed in S^3. For $d \geq 5$, we have the following result.

Lemma 3.3. *For every $d \geq 5$, we have $\mathcal{D}_2(\overline{\omega}^d) = \omega^*_{2d}$.*

Since $\mathcal{D}_2(\omega^*_{2d}) = U_d$, $d \geq 2$, Lemma 3.3 shows that the inclusion $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ can be strict for an m-stiff configuration ω_N with $m \geq 2$ even if ω_N is antipodal. Furthermore, for $d \geq 5$ odd, it provides another example of a non-antipodal m-stiff configuration with $m \geq 2$ (non-antipodal 1-stiff configurations are easy to construct).

Proof of Lemma 3.3. Every vector $\pm\mathbf{e}_i \in \omega^*_{2d}$ forms only dot products $\frac{1}{\sqrt{d}}$ and $-\frac{1}{\sqrt{d}}$ with points from $\overline{\omega}^d$; i.e., it belongs to $\mathcal{D}_2(\overline{\omega}^d)$. Choose any $\mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{D}_2(\overline{\omega}^d)$. Assume to the contrary that $\mathbf{x}$ has at least three nonzero coordinates. Let k be the number of strictly negative components in $\mathbf{x}$. If k is even, we choose a vector $\mathbf{z} = (z_1, \ldots, z_d) \in \overline{\omega}^d$ with $-\frac{1}{\sqrt{d}}$ in all positions corresponding to strictly negative components in $\mathbf{x}$ and $\frac{1}{\sqrt{d}}$ in all other positions. If k is odd, we choose $\mathbf{z} \in \overline{\omega}^d$ with $-\frac{1}{\sqrt{d}}$ in all but one positions corresponding to strictly negative components of $\mathbf{x}$ and $\frac{1}{\sqrt{d}}$ in all the other positions. Then the dot product $\mathbf{x} \cdot \mathbf{z} = x_1 z_1 + \ldots + x_d z_d$ has at most one strictly negative term and at least two other strictly positive terms, which we denote by $x_i z_i$ and $x_j z_j$. Since $d \geq 5$, we can choose two disjoint pairs of positions in $\mathbf{z}$ one containing z_i and the other one containing z_j with both pairs avoiding the position corresponding to the possible negative term in $\mathbf{x} \cdot \mathbf{z}$. Changing the sign of the coordinates of $\mathbf{z}$ in the first pair of positions, we keep $\mathbf{z}$ in $\overline{\omega}^d$ and strictly decrease the dot product $\mathbf{x} \cdot \mathbf{z}$. Changing the sign of the coordinates of

the new vector $\mathbf{z}$ in the second pair of positions, we keep the resulting vector in $\overline{\omega}^d$ and further decrease the dot product $\mathbf{x} \cdot \mathbf{z}$. This shows that $\mathbf{x}$ forms at least three distinct dor products with points of $\overline{\omega}^d$ contradicting its choice.

Therefore, $\mathbf{x}$ has at most two non-zero components. Assume to the contrary that $\mathbf{x}$ has exactly two non-zero components, say x_ℓ and x_n. Then $\mathbf{x}$ forms dot products $\frac{\pm x_\ell \pm x_n}{\sqrt{d}}$ with vectors from $\overline{\omega}^d$ and at least three of them are distinct. Thus, $\mathbf{x}$ has one non-zero component. Since $\mathbf{x}$ is on S^{d-1}, this component must be ± 1; that is, $\mathbf{x} \in \omega^*_{2d}$. Thus, $\mathcal{D}_2(\overline{\omega}^d) = \omega^*_{2d}$. □

We are ready to characterize universal minima of the d-demicube for $d \geq 5$.

Theorem 3.4. *Let $d \geq 5$ and g be an admissible potential function with a convex derivative g'' on $(-1, 1)$. Then the potential $p^g(\cdot, \overline{\omega}^d)$ of d-demicube $\overline{\omega}^d$ attains its absolute minimum over S^{d-1} at every point of cross-polytope ω^*_{2d}.*

If, in addition, g'' is strictly convex on $(-1, 1)$, then ω^*_{2d} contains all points of absolute minimum of the potential $p^g(\cdot, \overline{\omega}^d)$ on S^{d-1}.

In the case $d = 5$, the first paragraph of Theorem 3.4 follows from the results of [10].

Proof. Since $\overline{\omega}^d$ is 2-stiff, by Theorem 2.5, the potential $p^g(\cdot, \overline{\omega}^d)$ attains its absolute minimum over S^{d-1}, $d \geq 5$, at points of the set $\mathcal{D}_2(\overline{\omega}^d)$, which, by Lemma 3.3, equals ω^*_{2d}. If g'' is strictly convex on $(-1, 1)$, then, by Theorem 2.5, the set $\mathcal{D}_2(\overline{\omega}^d) = \omega^*_{2d}$ contains all absolute minima of $p^g(\cdot, \overline{\omega}^d)$ over S^{d-1}. □

4. The 2_{41} polytope on S^7

Recall that *the E_8 lattice* is the set (lattice in $\mathbb{R}^8$) of vectors in $\mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8$ whose coordinates sum to an even integer. Let $\overline{\omega}_{240}$ be the set of minimal length non-zero vectors of the E_8 lattice normalized to lie on S^7. The configuration $\overline{\omega}_{240}$ consists of $4\binom{8}{2} = 112$ vectors with 6 zero coordinates and two coordinates being $\pm 1/\sqrt{2}$ and $2^7 = 128$ vectors with all eight coordinates $\pm\frac{1}{2\sqrt{2}}$ and even number of "$-$" signs (this part is the 8-demicube). For brevity, we will also call $\overline{\omega}_{240}$ the E_8 lattice.

The 2_{41} polytope on S^7 (the name is due to Coxeter), denoted here by $\overline{\omega}_{2160}$, is the set of $N = 2160$ vectors on S^7 that includes $16\binom{8}{4} = 1120$ vectors with 4 zero coordinates and 4 coordinates being $\pm 1/2$ (let us call them type I vectors), 16 vectors with 7 zero coordinates and one coordinate being ± 1 (let us call them type II vectors), and $8\left(\binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7}\right) = 1024$ vectors with 7 coordinates being $\pm 1/4$, one coordinate being $\pm 3/4$, and an odd number of negative coordinates (call them type III vectors). One can verify directly that equality (2.1) holds for $d = 7$ and $n \in \{1, \ldots, 7, 9, 10\}$. Indeed,

since $\overline{\omega}_{2160}$ is antipodal, (2.1) holds trivially for every n odd. For $n = 2, 4, 6, 10$, we have

$$\sum_{\mathbf{x}\in\overline{\omega}_{2160}} \sum_{\mathbf{y}\in\overline{\omega}_{2160}} P_n^{(7)}(\mathbf{x}\cdot\mathbf{y}) = 4320(P_n^{(7)}(1) + 64P_n^{(7)}(3/4) \\ + 280P_n^{(7)}(1/2) + 448P_n^{(7)}(1/4) + 287P_n^{(7)}(0)) = 0. \tag{4.1}$$

The code $\overline{\omega}_{2160}$ is a 7-design (and not an 8-design). However, it is not stiff, since $\mathcal{D}_4(\overline{\omega}_{2160}) = \emptyset$ as the following lemma suggests.

Lemma 4.1. *For every vector $\mathbf{x} \in S^7$, the set $D(\mathbf{x}, \overline{\omega}_{2160})$ has at least five distinct elements. The only vectors $\mathbf{x} \in S^7$ such that $D(\mathbf{x}, \overline{\omega}_{2160})$ has exactly five distinct elements are those in $\overline{\omega}_{240}$. For each $\mathbf{x} \in \overline{\omega}_{240}$, we have $D(\mathbf{x}, \overline{\omega}_{2160}) = \left\{0, \pm\frac{1}{2\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right\}$.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_8) \in \mathcal{D}_5(\overline{\omega}_{2160})$ be arbitrary. If non-zero coordinates of $\mathbf{x}$ had at least three distinct absolute values, then $\mathbf{x}$ would form at least 6 distinct dot products with vectors of type II. Therefore, non-zero coordinates of $\mathbf{x}$ have at most two distinct absolute values.

Assume to the contrary that non-zero coordinates of $\mathbf{x}$ have exactly two distinct absolute values. Denote them by $0 < b < c$. Then $\mathbf{x}$ forms each of the dot products $\pm b, \pm c$ with vectors of type II. If $\mathbf{x}$ formed with some vector $\mathbf{z} \in \overline{\omega}_{2160}$ a positive dot product u distinct from b and c, since $\overline{\omega}_{2160}$ is antipodal, there would be a sixth dot product $-u$, contradicting the assumption that $\mathbf{x} \in \mathcal{D}_5(\overline{\omega}_{2160})$. Thus, $D(\mathbf{x}, \overline{\omega}_{2160})$ contains only two positive dot products: b and c. Let k coordinates of $\mathbf{x}$ have absolute value b and ℓ coordinates have absolute value c. If it were that $\ell \geq 2$, then $\mathbf{x}$ would form positive dot products $\frac{2c+b+v}{2}$ and $\frac{b+v}{2}$ with two appropriately chosen vectors of type I, where $v \geq 0$ is the absolute value of one of the coordinates of $\mathbf{x}$. Then $\frac{2c+b+v}{2} = c$ forcing $b \leq 0$. Thus, $\ell = 1$. If it were that $k \geq 3$, then $\mathbf{x}$ would form positive dot products $\frac{c+3b}{2}$ and $\frac{c+b}{2}$ with two vectors of type I. This would force $\frac{c+b}{2} = b$; that is, $c = b$. Thus, $k \leq 2$. We can now take vector $\mathbf{z}$ to be of type III with the coordinate $\pm 3/4$ corresponding to a zero coordinate of $\mathbf{x}$ such that $\mathbf{x}$ forms positive dot products $\mathbf{x}\cdot\mathbf{z} = \frac{c+kb}{4}$ and $\frac{c+(k-2)b}{4}$. Then $c = \frac{c+kb}{4} \leq \frac{c+2b}{4} < \frac{3c}{4}$, which is a contradiction.

Thus, all non-zero coordinates of $\mathbf{x}$ have the same absolute value, which we denote by a. Let n be the number of non-zero coordinates of $\mathbf{x}$. If $n = 1$, then $a = 1$ and $\mathbf{x}$ forms nine dot products, $0, \pm 1/4, \pm 1/2, \pm 3/4, \pm 1$, with points of $\overline{\omega}_{2160}$. If $n = 3$, then $\mathbf{x}$ forms seven dot products, $0, \pm a/2, \pm a, \pm 3a/2$, with type I vectors. If now $4 \leq n \leq 7$, then $\mathbf{x}$ forms nine dot products $0, \pm a/2, \pm a, \pm 3a/2, \pm 2a$ with type I vectors. Therefore, $n = 2$ or 8.

If $n = 2$, then $a = 1/\sqrt{2}$ and $\mathbf{x} \in \overline{\omega}_{240}$. Finally, if $n = 8$, then every coordinate of $\mathbf{x}$ is $\pm\frac{1}{2\sqrt{2}}$. Assume to the contrary that $\mathbf{x}$ has an odd number of negative coordinates. Then for every vector $\mathbf{z}$ of type III, $\mathbf{x}\cdot\mathbf{z}$ is a sum of

seven signed terms $\frac{1}{8\sqrt{2}}$ and one signed term $\frac{3}{8\sqrt{2}}$ with an even total number of minus signs. Then $\mathbf{x}\cdot\mathbf{z}$, in particular, has six values $\pm 2w, \pm 6w, \pm 10w$, where $w = \frac{1}{8\sqrt{2}}$. Therefore, the coordinates of $\mathbf{x}$ have an even number of minus signs and $\mathbf{x} \in \overline{\omega}_{240}$.

Thus, if $\mathbf{x} \notin \overline{\omega}_{240}$ then $\mathbf{x}$ forms more than five distinct dot products with points of $\overline{\omega}_{2160}$. One can also verify directly that every $\mathbf{x} \in \overline{\omega}_{240}$ forms exactly five distinct dot products with points of $\overline{\omega}_{2160}$, which are $0, \pm\frac{1}{2\sqrt{2}}, \pm\frac{1}{\sqrt{2}}$. □

We are now ready to state the main result of this section.

Theorem 4.2. *Let $\overline{\omega}_{2160} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{2160}\}$ be the 2_{41} polytope on S^7 and g be an admissible potential function with non-negative derivatives $g^{(8)}$, $g^{(9)}$, and $g^{(10)}$ on $(-1,1)$. Then, for every point $\mathbf{x}^* \in \overline{\omega}_{240}$,*

$$\min_{\mathbf{x}\in S^7} \sum_{i=1}^{2160} g(\mathbf{x}\cdot\mathbf{x}_i) = \sum_{i=1}^{2160} g(\mathbf{x}^*\cdot\mathbf{x}_i). \tag{4.2}$$

If, in addition, $g^{(10)} > 0$ on $(-1,1)$, then the absolute minimum in (4.2) is achieved only at points of the set $\overline{\omega}_{240}$.

Proof. We have $\{1,2,3,4,5,6,7,9,10\} \subset \mathcal{I}(\overline{\omega}_{2160})$ in view of (4.1). Applying Theorem 2.6 with $d = 7$, $m = 5$, $\{t_1, \ldots, t_5\} = \left\{0, \pm\frac{1}{2\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right\}$ we have $\mathcal{D} = \overline{\omega}_{240}$ and equality (4.2) holds for every $\mathbf{x}^* \in \overline{\omega}_{240}$. If $g^{(10)} > 0$ on $(-1,1)$, then, by Theorem 2.6, (4.2) holds only for $\mathbf{x}^* \in \overline{\omega}_{240}$. □

5. Symmetrizations of sharp codes that are non-tight designs

In this section, we discuss one simple method which allows us to construct some new stiff configurations and obtain their universal minima.

Lemma 5.1. *Let $\omega_N \subset S^d$ be an m-stiff configuration, $m, d \geq 1$, which does not contain an antipodal pair. Let $\omega'_N := \omega_N \cup (-\omega_N)$ be its symmetrization. Then ω'_N is also m-stiff with $\mathcal{D}_m(\omega'_N) = \mathcal{D}_m(\omega_N)$.*

Proof. The configurations ω_N and $-\omega_N$ are disjoint and both are $(2m-1)$-designs. Then their union ω'_N has $2N$ points and is also a $(2m-1)$-design. We immediately have $\mathcal{D}_m(\omega'_N) \subset \mathcal{D}_m(\omega_N)$. If $\mathbf{x} \in \mathcal{D}_m(\omega_N)$ then, by Proposition 2.3, $\mathbf{x}$ forms one of the dot products $\kappa_1^m, \ldots, \kappa_m^m$ with any point from ω_N. Since $\kappa_1^m, \ldots, \kappa_m^m$ are zeros of $P_m^{(d)}$, they are symmetric about the origin. Since $-\mathbf{x} \in \mathcal{D}_m(\omega_N)$, for every $\mathbf{y} \in -\omega_N$, we have $\mathbf{x}\cdot\mathbf{y} = -\mathbf{x}\cdot(-\mathbf{y}) \in \{\kappa_1^m, \ldots, \kappa_m^m\}$ because $-\mathbf{y} \in \omega_N$. Then $\mathbf{x} \in \mathcal{D}_m(\omega'_N)$; that is, $\mathcal{D}_m(\omega_N) = \mathcal{D}_m(\omega'_N)$. Since $\mathcal{D}_m(\omega_N) \neq \emptyset$, we have $\mathcal{D}_m(\omega'_N) \neq \emptyset$; that is, ω'_N is m-stiff. □

We remark that Lemmas 5.1 and 3.2 immediately imply Lemma 3.3 for $d \geq 5$ odd, since, in this case, $\overline{\omega}^d$ has no antipodal pair, $U_d = \overline{\omega}^d \cup (-\overline{\omega}^d)$, and $\mathcal{D}_2(U_d) = \omega^*_{2d}$. Lemma 5.1 also applies to the following two cases.

Table 1 in [11] contains known sharp codes on the sphere. Among them, the following codes are not tight designs. Vertices of an $(N-1)$-dimensional regular simplex in $\mathbb{R}^d$, $3 \leq N \leq d$ (we denote this code by T_N), vertices of the demihypercube $\overline{\omega}^5$ on S^4, the 112- and 162-point sharp codes on S^{20}, the Higman–Sims configuration and the 891-point sharp code on S^{21}, as well as isotropic subspaces that appear for an infinite sequence of dimensions.

Since T_N is 1-stiff, so is the code $T_{2N} := T_N \cup (-T_N)$. By Proposition 2.3 and Theorem 2.5, the set of universal minima of T_{2N} is the intersection of the sphere S^{d-1} with the orthogonal complement of T_N. The symmetrization $\overline{\omega}_5 \cup (-\overline{\omega}_5)$ is the cube U_5 whose universal minima are at the vertices of the cross-polytope in $\mathbb{R}^5$. We will not consider the 891-point sharp code[9] or the sequence of isotropic subspaces, since the sets of their universal minima are not known.[10] The remaining three codes will be split into two cases.

Case I. The Higman–Sims configuration, denoted by $\overline{\omega}_{100}$, is a 100-point 3-design on S^{21}, where distinct points form only dot products $-4/11$ and $1/11$ with each other (see, e.g., [11, Table 1]). According to [10], this configuration is 2-stiff. Paper [10] finds 176 pairs of antipodal vectors on S^{21}, where each vector forms only dot products $\frac{1}{\sqrt{22}}$ and $-\frac{1}{\sqrt{22}}$ with vectors from $\overline{\omega}_{100}$. Denote this set of $2 \cdot 176 = 352$ vectors by $\overline{\omega}_{352}$. We have $\overline{\omega}_{352} \subset \mathcal{D}_2(\overline{\omega}_{100})$. Each of these vectors is a universal minimum of $\overline{\omega}_{100}$ (see Theorem 2.5). Since no dot product in $\overline{\omega}_{100}$ is -1, the set $\overline{\omega}_{100}$ does not contain an antipodal pair.

By Lemma 5.1, the symmetrized Higman–Sims configuration $\overline{\omega}_{200} := \overline{\omega}_{100} \cup (-\overline{\omega}_{100})$ is 2-stiff with $\mathcal{D}_2(\overline{\omega}_{200}) = \mathcal{D}_2(\overline{\omega}_{100}) \supset \overline{\omega}_{352}$.

Case II. Two sharp codes on S^{20} can be derived from the McLaughlin configuration $\overline{\omega}_{275} \subset S^{21}$, which is strongly sharp (see, e.g., [11, Table 1]). Fix a point $\mathbf{x} \in \overline{\omega}_{275}$. It forms dot product $1/6$ with 162 points from $\overline{\omega}_{275}$. Let $\omega^{\mathbf{x}}_{162}$ denote the set of these 162 points. Point $\mathbf{x}$ forms dot product $-1/4$ with the remaining set of 112 points from $\overline{\omega}_{275}$, which we denote by $\omega^{\mathbf{x}}_{112}$.

We apply homotheties to $\omega^{\mathbf{x}}_{162}$ and $\omega^{\mathbf{x}}_{112}$ to scale them to $\widetilde{S}^{20} := S^{21} \cap H$, where H is the 21-dimensional linear subspace of $\mathbb{R}^{22}$ orthogonal to $\mathbf{x}$. Denote the resulting configurations by $\overline{\omega}_{162}$ and $\overline{\omega}_{112}$, respectively. Both $\overline{\omega}_{162}$ and $\overline{\omega}_{112}$ are 3-designs. They are known sharp configurations, see [11, Table 1] ($\overline{\omega}_{112}$ is in fact the isotropic subspace with $q = 3$). Since any vector from $\omega^{\mathbf{x}}_{162}$ and any vector from $\omega^{\mathbf{x}}_{112}$ form only dot products $1/6$ or $-1/4$ with each other, any vector from $\overline{\omega}_{162}$ and any vector from $\overline{\omega}_{112}$ form only dot products

[9] Jointly with the authors of paper [10], we are currently working to find universal minima of the 891-point sharp code.

[10] Except for two first codes in the sequence of isotropic subspaces.

$\frac{1}{\sqrt{21}}$ or $-\frac{1}{\sqrt{21}}$. Then both configurations are 2-stiff with $\overline{\omega}_{112} \subset \mathcal{D}_2(\overline{\omega}_{162})$ and $\overline{\omega}_{162} \subset \mathcal{D}_2(\overline{\omega}_{112})$ (this was observed in [10]).

Since any two distinct points in $\overline{\omega}_{275}$ form dot products $1/6$ or $-1/4$ with each other, any two distinct points in $\overline{\omega}_{162}$ form dot products $1/7$ or $-2/7$ and any two distinct points in $\overline{\omega}_{112}$ form dot products $1/9$ or $-1/3$; i.e., both configurations do not contain antipodal pairs. Let $\overline{\omega}_{324} := \overline{\omega}_{162} \cup (-\overline{\omega}_{162})$ and $\overline{\omega}_{224} := \overline{\omega}_{112} \cup (-\overline{\omega}_{112})$ be their symmetrization about the origin. It is not difficult to see that $\overline{\omega}_{224} \subset \mathcal{D}_2(\overline{\omega}_{162})$ and $\overline{\omega}_{324} \subset \mathcal{D}_2(\overline{\omega}_{112})$. Each point of $\overline{\omega}_{224}$ is a universal minimum of $\overline{\omega}_{162}$ and each point of $\overline{\omega}_{324}$ is a universal minimum of $\overline{\omega}_{112}$ (see Theorem 2.5).

By Lemma 5.1, both $\overline{\omega}_{324}$ and $\overline{\omega}_{224}$ are 2-stiff with

$$\mathcal{D}_2(\overline{\omega}_{324}) = \mathcal{D}_2(\overline{\omega}_{162}) \supset \overline{\omega}_{224} \quad \text{and} \quad \mathcal{D}_2(\overline{\omega}_{224}) = \mathcal{D}_2(\overline{\omega}_{112}) \supset \overline{\omega}_{324}.$$

The conclusions in Cases I and II and Theorem 2.5 imply the following.

Proposition 5.2. *Let g be an admissible potential function with a convex derivative g'' on $(-1,1)$. Then*

(i) *every point of the configuration $\overline{\omega}_{352}$ is a point of absolute minimum over S^{21} of the potential $p^g(\cdot, \overline{\omega}_{200})$;*
(ii) *every point of the configuration $\overline{\omega}_{224}$ is a point of absolute minimum over $\widetilde{S}^{20}$ of the potential $p^g(\cdot, \overline{\omega}_{324})$;*
(iii) *every point of the configuration $\overline{\omega}_{324}$ is a point of absolute minimum over $\widetilde{S}^{20}$ of the potential $p^g(\cdot, \overline{\omega}_{224})$.*

6. Certain properties of general stiff configurations

Every time we have a stiff configuration, in view of Theorem 2.5, we automatically have its universal minima (the dual configuration). Moreover, every stiff configuration attains the Fazekas–Levenshtein bound for covering [13, Theorem 2]. Therefore, it is important to study stiff codes and their duals in general. In this section, we characterize 1-stiff configurations on S^d, m-stiff configurations on S^1, and their duals and also prove some basic properties of stiff configurations and their duals. We call the point $\mathbf{c} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i$ *the center of mass* of a configuration $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$.

Proposition 6.1. *Let $d \geq 1$. A configuration $\omega_N \subset S^d$, $N \geq 1$, is 1-stiff if and only if its center of mass is at the origin and ω_N is contained in a d-dimensional linear subspace of $\mathbb{R}^{d+1}$.*

Proof. The proposition follows from the fact that a point configuration is a spherical 1-design if and only if its center of mass is located at the origin and the fact that a hyperplane containing ω_N also contains its center of mass. □

We next describe the dual of a 1-stiff configuration.

Proposition 6.2. *Let $\omega_N \subset S^d$, $d \geq 1$, be a 1-stiff configuration. Then $\mathcal{D}_1(\omega_N) = L^\perp \cap S^d$, where L is the linear subspace of $\mathbb{R}^{d+1}$ spanned by ω_N. If $k := \dim L \leq d-1$, then $\mathcal{D}_1(\omega_N)$ is a sphere in a $(d+1-k)$-dimensional subspace of $\mathbb{R}^{d+1}$. If $k = d$, then $\mathcal{D}_1(\omega_N) = \{\mathbf{a}, -\mathbf{a}\}$ for some $\mathbf{a} \in S^d$, which is a 1-stiff configuration.*

Proof. Let $\mathbf{z}$ be any vector in $\mathcal{D}_1(\omega_N)$. By Proposition 2.3, we have $\mathbf{z} \cdot \mathbf{y} = 0$ for every $\mathbf{y} \in \omega_N$, since 0 is the only root of $P_1^{(d)}$. Then $\mathbf{z} \perp L$; i.e., $\mathbf{z} \in L^\perp \cap S^d$. If $\mathbf{z}$ is any vector in $L^\perp \cap S^d$, then it forms only one dot product (which is 0) with any point from ω_N; that is, $\mathbf{z} \in \mathcal{D}_1(\omega_N)$. The rest of Proposition 6.2 follows immediately. □

We now charactirize stiff configurations on S^1.

Proposition 6.3. *For every $m \geq 1$, a configuration on S^1 is m-stiff if and only if it is a regular $2m$-gon.*

We remark that the regular $2m$-gon $\widetilde{\omega}_{2m}$ on S^1 is antipodal and its dual $\mathcal{D}_m(\widetilde{\omega}_{2m})$ is another regular $2m$-gon with $\mathcal{D}_m(\mathcal{D}_m(\widetilde{\omega}_{2\,m})) = \widetilde{\omega}_{2\,m}$.

Proof of Proposition 6.3. If $\omega_N = \widetilde{\omega}_{2m}$ then it is a $(2m-1)$-design and the midpoint $\mathbf{y}$ of the arc joining any two neighboring vertices forms m distinct values of dot products with points from ω_N; i.e., ω_N is m-stiff.

By Proposition 2.3, the point $\mathbf{y}$ forms dot products $\kappa_1^m, \dots, \kappa_m^m$ (zeros of $P_m^{(1)}$) with points of $\widetilde{\omega}_{2m}$, each with frequency $2 = 2ma_0(\varphi_i)$, $i = 1, \dots, m$, where $\{\varphi_1, \dots, \varphi_m\}$ is the fundamental system of polynomials for the nodes κ_i^m. We have

$$a_0(\varphi_i) = \frac{1}{m} \quad \text{and} \quad \kappa_i^m = \cos\frac{(2i-1)\pi}{2m}, \qquad i = 1, \dots, m. \tag{6.1}$$

Assume that ω_N is m-stiff. Let $\mathbf{z}$ be any point in $\mathcal{D}_m(\omega_N)$. By Proposition 2.3, point $\mathbf{z}$ forms dot products $\kappa_1^m, \dots, \kappa_m^m$ with points of ω_N, where κ_i^m, $i = 1, \dots, m$, are the zeros of $P_m^{(1)}$. Then ω_N is contained in the set of points of intersection of S^1 with m parallel lines; that is, $\#\omega_N \leq 2m$. By Proposition 2.3 and (6.1), the frequency M_i of the dot product κ_i^m is $M_i = Na_0(\varphi_i) = N/m \leq 2$, $i = 1, \dots, m$. Hence, frequencies M_i are equal and each of the m parallel lines contains the same number of points from ω_N (one or two). Assume to the contrary that each M_i equals 1. Then ω_N has only m points, which means that any point $\mathbf{y} \in \omega_N$ forms at most m distinct values of the dot product with points of ω_N; i.e., $\mathbf{y} \in \mathcal{D}_m(\omega_N)$. One of the dot products is 1, while by Proposition 2.3, these dot products must be zeros κ_i^m of $P_m^{(1)}$ none of which is 1. This contradiction shows that $M_i = 2$, $i = 1, \dots, m$, and, hence $\#\omega_N = 2m$. Vector $\mathbf{z}$ forms each of the angles $\frac{(2i-1)\pi}{2m} = \arccos \kappa_i^m$ with exactly two points from ω_N, $i = 1, \dots, m$. Then ω_N is a regular $2m$-gon. □

We next prove the following basic statement.

Proposition 6.4. *For any configuration $\omega_N \subset S^d$, $d \geq 1$, with $\mathcal{D}_m(\omega_N) \neq \emptyset$, $m \geq 1$, the set $\mathcal{D}_m(\omega_N)$ is antipodal.*

If ω_N is m-stiff, then $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$. If, in addition, $\mathcal{D}_m(\omega_N)$ is finite and m-stiff, then $\mathcal{D}_m(\mathcal{D}_m(\mathcal{D}_m(\omega_N))) = \mathcal{D}_m(\omega_N)$.

Though the dual $\mathcal{D}_m(\omega_N)$ is antipodal, this is not always true for the m-stiff configuration ω_N itself. For example, the d-demicube $\overline{\omega}^d$ on S^{d-1} is not antipodal for any $d \geq 3$ odd.

In view of Proposition 6.4, the equality $\omega_N = \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ with an m-stiff configuration ω_N implies that ω_N is antipodal. However, the inclusion $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ can be sometimes strict (even when both ω_N and $\mathcal{D}_m(\omega_N)$ are antipodal and m-stiff). This is the case, for example, for $\omega_N = \overline{\omega}^d$ and any $d \geq 5$ in view of Lemma 3.3 and the fact that the dual of the cross-polytope ω_{2d}^* is the whole cube U_d (for $d \geq 5$ even, both $\overline{\omega}^d$ and $\mathcal{D}_2(\overline{\omega}^d) = \omega_{2d}^*$ are antipodal and 2-stiff). At the same time, every stiff configuration from [2, Table 3] coincides with the dual of its dual.

Some other examples of non-antipodal stiff configurations ω_N are given in [10]. Since, by Proposition 6.4, the duals of their duals are antipodal, the inclusion $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ is strict as well.

Proof of Proposition 6.4. Let $\omega_N \subset S^d$ be arbitrary with $\mathcal{D}_m(\omega_N) \neq \emptyset$. For any point $\mathbf{z} \in \mathcal{D}_m(\omega_N)$, the point $-\mathbf{z}$ also forms at most m distinct dot products with points of ω_N; that is $-\mathbf{z} \in \mathcal{D}_m(\omega_N)$ and the set $\mathcal{D}_m(\omega_N)$ is antipodal. Choose any point $\mathbf{z}$ in an m-stiff ω_N. For any point $\mathbf{y} \in \mathcal{D}_m(\omega_N)$, by Proposition 2.3, we have $\mathbf{y} \cdot \mathbf{z} \in \{\kappa_1^m, \ldots, \kappa_m^m\}$; that is, $\mathbf{z}$ forms at most m distinct dot products with points of $\mathcal{D}_m(\omega_N)$. Then $\mathbf{z} \in \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ and $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$. Assume additionally that $X := \mathcal{D}_m(\omega_N)$ is finite and m-stiff. The inclusion $\omega_N \subset \mathcal{D}_m(\mathcal{D}_m(\omega_N))$ implies that $X = \mathcal{D}_m(\omega_N) \supset \mathcal{D}_m(\mathcal{D}_m(X))$. Since X is m-stiff, we have the opposite inclusion. □

We say that a point set in $\mathbb{R}^{d+1}$ is *in general position* if it is not contained in any hyperplane. One can construct plenty of examples of m-stiff configurations, $m \geq 2$, whose dual is not in general position. For instance, start with the cube U_3 inscribed in S^2 and let α_1 and α_2 be the parallel planes containing two paralel facets of U_3. For a given $n \geq 2$, we rotate the cube U_3 about the axis ℓ perpendicular to the planes α_1 and α_2 and passing through the origin at angles $\frac{\pi k}{2n}$, $k = 0, 1, \ldots, n-1$, and let ω_N be the union of the resulting n cubes. Then ω_N is a 3-design as a disjoint union of finitely many 3-designs. Since ω_N is still contained in planes α_1 and α_2, it is 2-stiff. However, its dual is $\mathcal{D}_2(\omega_N) = \{\mathbf{a}, -\mathbf{a}\}$, where $\mathbf{a}$ is a unit vector parallel to the axis ℓ. The dual is not in general position. It is also only 1-stiff. This example can, of course, be extended to other dimensions and other initial configurations.

Proposition 6.5. *Let $\omega_N \subset S^d$, $d \geq 1$, be an m-stiff configuration, $m \geq 2$. Then*

(i) *$\mathcal{D}_m(\omega_N)$ contains at most m^{d+1} points;*
(ii) *if $\mathcal{D}_m(\omega_N)$ is in general position, then ω_N contains at most m^{d+1} points;*
(iii) *if $\mathcal{D}_m(\omega_N)$ is not in general position, then $\mathcal{D}_m(\omega_N)$ is 1-stiff and, hence, not m-stiff.*

Proposition 6.5 implies that an m-stiff configuration on S^d, $m \geq 2$, cannot have more than m^{d+1} universal minima. This cardinality bound can be achieved: take ω_N to be the set of vertices of a regular cross-polytope inscribed in S^d. It is 2-stiff and its dual is a cube inscribed in S^d, which has exactly 2^{d+1} vertices. If the universal minima of an m-stiff configuration on S^d are in general position, then the configuration itself contains at most m^{d+1} points. This bound is attained by the set of 2^{d+1} vertices of a cube inscribed in S^d, which is 2-stiff. Its dual is the set of vertices of a $(d+1)$-dimensional regular cross-polytope.

For $m \geq 3$, there may exist an m-stiff configuration on S^d with more than 2^{d+1} points in it or in its dual. For example, the 24-cell on S^3 has $N = 24 > 2^4$ points (its dual is another 24-cell), or the dual of the symmetrized Schläffi configuration on S^5, which has $N = 72 > 2^6$ points, see [2]. All these configurations are 3-stiff.

If the universal minima are not in general position, then no upper bound depending only on m and d can be written for the cardinality of an m-stiff configuration ω_N, $d \geq 2$ (when ω_N exists for those m and d), see Proposition 6.7 below.

Another interesting question related to Proposition 6.5 is about the general assumptions under which the dual of a given m-stiff configuration, $m \geq 2$, is in general position and whether this is sufficient for the dual to be also m-stiff. The dual is m-stiff with the same m, for example, for every stiff configuration mentioned in [2, Table 3] and for the d-demicube, $d \geq 4$.

To establish Proposition 6.5, we need the following auxiliary statement.

Lemma 6.6. *Suppose $m, d \geq 1$ and $X \subset S^d$ is a finite set in general position. Then for any m-element subset $\Lambda \subset [-1, 1]$, there are at most m^{d+1} points $\mathbf{z} \in S^d$ such that $D(\mathbf{z}, X) \subset \Lambda$.*

Proof. Since X is in general position, it contains a linearly independent subset $\{\mathbf{y}_1, \ldots, \mathbf{y}_{d+1}\}$. Let $\mathbf{z} \in S^d$ be any point with $D(\mathbf{z}, X) \subset \Lambda$. Then $\mathbf{z} \cdot \mathbf{y}_j = \alpha_j$, $j = 1, \ldots, d+1$, where $\alpha_1, \ldots, \alpha_{d+1} \in \Lambda$. Consequently, $\mathbf{z}$ is a solution to a linear system with a fixed non-singular coefficient matrix and vector of right-hand sides $(\alpha_1, \ldots, \alpha_{d+1}) \in \Lambda^{d+1}$. Each such system has a unique solution. Since there are m^{d+1} possible vectors of right-hand sides for these systems, there are at most m^{d+1} points $\mathbf{z} \in S^d$ with $D(\mathbf{z}, X) \subset \Lambda$. □

Proof of Proposition 6.5. Since ω_N is at least a 3-design, it is in general position. If it were not, then ω_N would not be a 2-design: for the polynomial

$p(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{v} - \alpha)^2$, where $\mathbf{x} \cdot \mathbf{v} = \alpha$ is an equation of the hyperplane containing ω_N, the average of p over ω_N would be zero while the average of p over S^d would be positive. By Proposition 2.3, for every $\mathbf{z} \in \mathcal{D}_m(\omega_N)$, we have $D(\mathbf{z}, \omega_N) \subset \Lambda_m := \{\kappa_1^m, \ldots, \kappa_m^m\}$, where κ_j^m's are zeros of $P_m^{(d)}$. Then, by Lemma 6.6, we have $\#\mathcal{D}_m(\omega_N) \leq m^{d+1}$.

Assume that $\mathcal{D}_m(\omega_N)$ is in general position. Since ω_N is m-stiff, by Proposition 2.3, for any $\mathbf{z} \in \omega_N$ and $\mathbf{y} \in \mathcal{D}_m(\omega_N)$, we have $\mathbf{z} \cdot \mathbf{y} \in \Lambda_m$. Then $D(\mathbf{z}, \mathcal{D}_m(\omega_N)) \subset \Lambda_m$, $\mathbf{z} \in \omega_N$. By Lemma 6.6, we have $N \leq m^{d+1}$.

Assume that $\mathcal{D}_m(\omega_N)$ is not in general position; i.e., contained in some hyperplane H. By Proposition 6.4, it is antipodal. Then its center of mass is at the origin; i.e., H is a d-dimensional linear subspace of $\mathbb{R}^{d+1}$. By Proposition 6.1, $\mathcal{D}_m(\omega_N)$ is 1-stiff. Since $\mathcal{D}_m(\omega_N)$ is contained in one hyperplane, by the above argument, it cannot be a 3-design. Then $\mathcal{D}_m(\omega_N)$ cannot be m-stiff for any $m \geq 2$. □

The statement below implies that the set of possible cardinalities of m-stiff configurations on S^d, $d \geq 2$ (provided that an m-stiff configuration exists on S^d), forms an additive semigroup (in particular, it is not bounded above). This is not the case for $d = 1$ in view of Proposition 6.3. In the case $m = 1$ and $d \geq 2$, this semigroup is the set of all integers $N \geq 2$, see Proposition 6.1.

Proposition 6.7. *Suppose that for given $m \geq 1$ and $d \geq 2$, there exist m-stiff configurations on S^d of cardinalities N_1 and N_2. Then there is an m-stiff configuration on S^d of cardinality $N_1 + N_2$.*

Proposition 6.7 and Bézout's identity imply that if, for a given pair (m, d), $d \geq 2$, the cardinalities of two m-stiff configurations on S^d have the greatest common divisor δ, then for any sufficiently large multiple N of δ, there exists an m-stiff configuration on S^d of cardinality N.

Proof. Let $\omega_{N_i} \subset S^d$ be an m-stiff configuration of cardinality N_i, $i = 1, 2$, and let $\mathbf{z}_i \in \mathcal{D}_m(\omega_{N_i})$, $i = 1, 2$, be chosen so that $\mathbf{z}_1 \neq \mathbf{z}_2$ (if it happens that $\mathbf{z}_1 = \mathbf{z}_2$, we choose $-\mathbf{z}_2$ instead of $\mathbf{z}_2$). We will construct an m-stiff configuration of cardinality $N_1 + N_2$. By Proposition 2.3, both vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ form only dot products $\kappa_1^m, \ldots, \kappa_m^m$ with points from the corresponding configuration ω_{N_i}. Let H be the d-dimensional subspace of $\mathbb{R}^{d+1}$, which is the perpendicular bisector for the line segment $[\mathbf{z}_1, \mathbf{z}_2]$. Let $r_H : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be the reflection transformation about the subspace H and let U be its matrix (U is orthogonal). Then for every $\mathbf{x} \in \omega'_{N_1} := r_H(\omega_{N_1})$, there is $\mathbf{y} \in \omega_{N_1}$ such that $\mathbf{x} = U\mathbf{y}$ and

$$\mathbf{x} \cdot \mathbf{z}_2 = U\mathbf{y} \cdot \mathbf{z}_2 = \mathbf{y} \cdot U^T \mathbf{z}_2 = \mathbf{y} \cdot \mathbf{z}_1 = \kappa_i^m \quad \text{for some} \quad i = 1, \ldots, m. \tag{6.2}$$

The set ω'_{N_1} is m-stiff. Denote by $\mathbf{a} \in S^d$ a vector perpendicular to $\mathbf{z}_2$, not contained in any subspace $\operatorname{span}\{\mathbf{w} - \mathbf{v}\}$, where $\mathbf{w} \in \omega_{N_2}$ and $\mathbf{v} \in \omega'_{N_1}$, and $\mathbf{a}$ is not perpendicular to any vector from ω_{N_2}. Such a vector $\mathbf{a}$ exists, since

all these conditions delete a set of the $(d-1)$-dimensional measure zero from $S^d \cap \{\mathbf{z}_2\}^\perp$.

Then $L = \{\mathbf{a}\}^\perp$ is disjoint with ω_{N_2}. Furthermore, $\mathbf{z}_2 \in L$ and the configuration $\omega''_{N_1} := r_L(\omega'_{N_1})$ is disjoint with ω_{N_2}. If it were not, then L would be the perpendicular bisector for a line segment whose one endpoint $\mathbf{w}_1$ is in ω_{N_2} and the other endpoint $\mathbf{v}_1$ is in ω'_{N_1}. We have $\mathbf{w}_1 \neq \mathbf{v}_1$, since, otherwise, $\mathbf{w}_1 \in L$ and, hence, $\mathbf{a} \perp \mathbf{w}_1$. Then $\mathbf{w}_1 - \mathbf{v}_1$ is a non-zero vector perpendicular to L and $\mathbf{a} \in \operatorname{span}\{\mathbf{w}_1 - \mathbf{v}_1\}$ contradicting the choice of $\mathbf{a}$.

Let V be the matrix of the reflection transformation r_L. Then for every $\mathbf{z} \in \omega''_{N_1}$, there is $\mathbf{x} \in \omega'_{N_1}$ such that $\mathbf{z} = V\mathbf{x}$ and using (6.2) we have

$$\mathbf{z} \cdot \mathbf{z}_2 = V\mathbf{x} \cdot \mathbf{z}_2 = \mathbf{x} \cdot V^T \mathbf{z}_2 = \mathbf{x} \cdot \mathbf{z}_2 = \kappa_i^m \quad \text{for some} \quad i = 1, \dots, m.$$

Thus, $\mathbf{z}_2 \in \mathcal{D}_m(\omega''_{N_1} \cup \omega_{N_2})$ and $\omega''_{N_1} \cup \omega_{N_2}$ is a disjoint union of two $(2m-1)$-designs. Then it is also a $(2m-1)$-design, and, hence is an m-stiff configuration of cardinality $N_1 + N_2$. □

We conclude this section with the following important existence result for codes attaining the Fazekas–Levenshtein bound for covering, see [13, Theorem 2]. Recall that for a given $m \in \mathbb{N}$, a spherical $(2m-1)$-design on S^d attains the Fazekas–Levenshtein bound if and only if it is m-stiff. Any m-stiff configuration ω_N on S^d is a set of nodes of a cubature (with equal weights) exact on multivariate polynomials of degree up to $2m-1$. It gives a rise to the unique univariate quadrature for integration with weight $w_d(t)$ with m nodes and algebraic degree of precision $2m-1$. It is called the Gauss–Gegenbauer quadrature. Its weights are rational and positive, since they equal the frequencies of the m dot products formed by each universal minimum point of ω_N with points of ω_N. Here we establish the converse: if the Gauss–Gegenbauer quadrature with m nodes for integration with weight $w_d(t)$ has positive and rational weights, then one can construct an m-stiff configuration on S^d.

Recall that $\{\varphi_1, \dots, \varphi_m\}$ are the fundamental polynomials for the set of zeros $\kappa_1^m, \dots, \kappa_m^m$ of the m-th Gegenbauer polynomial $P_m^{(d)}$.

Theorem 6.8. *For every $m, d \geq 1$, there exists an m-stiff configuration $\omega_N \subset S^d$ if and only if every 0-th Gegenbauer coefficient $a_0(\varphi_k)$, $k = 1, \dots, m$, is a positive rational.*

The Gauss–Gegenbauer quadrature is the quadrature

$$a_0(h) = \int_{-1}^{1} h(t) w_d(t)\, dt \approx \sum_{i=1}^{m} \beta_i h(\kappa_i^m), \tag{6.3}$$

where $\beta_i = a_0(\varphi_i)$, $i = 1, \dots, m$. It is exact for all polynomials $h \in \mathbb{P}_{2m-1}$. Theorem 6.8 can be restated as follows: there extsis an m-stiff configuration on a given sphere S^d if and only if all the coefficients of the corresponding Gauss–Gegenbauer quadrature are positive rationals.

Remark 6.9. The coefficients of the Gauss–Gegenbauer quadrature are positive rationals for $d = 1$ and any $m \geq 1$. In this case, vertices of a regular $2m$-gon on S^1 form an m-stiff configuration. The Gauss–Gegenbauer quadrature also has positive rational coefficients (for all $d \geq 1$) when $m = 1$ (there is only one coefficient that equals 1), $m = 2$ (both coefficients equal 1/2), and $m = 3$. When $d \geq 2$ and $m \geq 4$, the Gauss–Gegenbauer quadrature may also have positive rational coefficients. For example, this happens for $d = 22$ and $m = 4$, when the 4600-point tight 7-design on S^{22} appears to be 4-stiff, see [10].

We will need the following statement which immediately follows from the results of Seymour and Zaslavsky [22, Main Theorem].

Lemma 6.10. *Let n and d be arbitrary positive integers. Then for every cardinality N sufficiently large, there exists an N-point spherical n-design on S^d.*

Proof of Theorem 6.8. Assume that for a given pair $m, d \geq 1$, there exists an m-stiff configuration on S^d. Then by Proposition 2.3, each frequency $a_0(\varphi_i)N$, $i = 1, \ldots, m$, is a positive integer; that is, each $a_0(\varphi_i)$ is a positive rational.

Assume now that for a given pair $m, d \geq 1$, each coefficient $a_0(\varphi_i)$, $i = 1, \ldots, m$, is a positive rational. We will construct an m-stiff configuration on S^d.

Case I: $d = 1$. An m-stiff configuration on S^1 is a regular $2m$-gon.

Case II: $m = 1$. There exists a 1-stiff configuration on S^d, $d \geq 1$, in view of Proposition 6.1. When $d = 1$, it is an antipodal pair and, when $d \geq 2$, it is, for example, a regular d-simplex inscribed in the intersection of a d-dimensional linear subspace of $\mathbb{R}^{d+1}$ with S^d.

The remaining case ($d, m \geq 2$) will be handled using the following lemma. For a given $u \in [-1, 1]$, define the mapping $\pi_u : S^{d-1} \to S^d$ by

$$\pi_u(\mathbf{z}) := u\mathbf{e}_{d+1} + \sqrt{1 - u^2}(\mathbf{z}, 0), \quad \mathbf{z} \in S^{d-1},$$

where we recall that $\mathbf{e}_{d+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$. The mapping π_u is injective for $u \in (-1, 1)$. □

Lemma 6.11. *Let $d \geq 2$ and $n \geq 1$. Suppose positive rationals $\gamma_1, \ldots, \gamma_\nu$ and reals $-1 < t_1 < \ldots < t_\nu < 1$ are such that*

$$\int_{-1}^{1} h(t) w_d(t)\, dt = \sum_{i=1}^{\nu} \gamma_i h(t_i) \tag{6.4}$$

holds for every polynomial $h \in \mathbb{P}_n$. Suppose also that X_i, $i = 1, \ldots, \nu$, are spherical n-designs on S^{d-1} such that $\frac{\#X_1}{\gamma_1} = \ldots = \frac{\#X_\nu}{\gamma_\nu}$. Denote $N := \#X_1 + \ldots + \#X_\nu$ and let

$$\omega_N := \pi_{t_1}(X_1) \cup \ldots \cup \pi_{t_\nu}(X_\nu).$$

Then ω_N is an n-design on S^d contained in the union of ν parallel hyperplanes.

Proof. By item (iii) of Theorem 2.1, it is sufficient to show that

$$p_\ell(\mathbf{x},\omega_N) := \sum_{i=1}^{N} (\mathbf{x}\cdot\mathbf{x}_i)^\ell = Nc_{\ell,d}, \quad \mathbf{x}\in S^d, \quad \ell = 1,\dots,n, \tag{6.5}$$

where $c_{\ell,d} := \int_{-1}^{1} t^\ell w_d(t)\,dt$. Choose arbitrary $1\le \ell\le n$ and $\mathbf{x}\in S^d$. Let u be the $(d+1)$-th coordinate of $\mathbf{x}$. If $u\in(-1,1)$, let $\widetilde{\mathbf{x}}\in\mathbb{R}^d$ be the vector obtained from $\mathbf{x}$ by deleting the $(d+1)$-th coordinate and normalizing the resulting vector to lie on S^{d-1}. If $|u|=1$, we let $\widetilde{\mathbf{x}}$ be any vector on S^{d-1}. Then

$$\begin{aligned} p_\ell(\mathbf{x},\omega_N) &= \sum_{i=1}^{\nu} \sum_{\mathbf{y}\in\pi_{t_i}(X_i)} (\mathbf{x}\cdot\mathbf{y})^\ell \\ &= \sum_{i=1}^{\nu}\sum_{\mathbf{z}\in X_i} \left((u\mathbf{e}_{d+1} + \sqrt{1-u^2}(\widetilde{\mathbf{x}},0))\cdot\left(t_i\mathbf{e}_{d+1} + \sqrt{1-t_i^2}(\mathbf{z},0)\right)\right)^\ell \\ &= \sum_{i=1}^{\nu}\sum_{\mathbf{z}\in X_i} \left(ut_i + \sqrt{1-u^2}\sqrt{1-t_i^2}(\widetilde{\mathbf{x}}\cdot\mathbf{z})\right)^\ell. \end{aligned}$$

Since quadrature (6.4) is exact for $h(t)=1$, we have $\sum_{i=1}^{\nu}\gamma_i = 1$. Then $\#X_i = \gamma_i N$, $i=1,\dots,\nu$. Since X_i is an n-design on S^{d-1}, by item (iii) of Theorem 2.1, for $u\in[-1,1]$ fixed, we have

$$p_\ell(\mathbf{x},\omega_N) = N\sum_{i=1}^{\nu}\gamma_i \int_{-1}^{1} \left(ut_i + t\sqrt{1-u^2}\sqrt{1-t_i^2}\right)^\ell w_{d-1}(t)\,dt.$$

For $(u,v)\in[-1,1]^2$, we have

$$\begin{aligned} q(u,v) &:= \int_{-1}^{1} \left(uv + t\sqrt{1-u^2}\sqrt{1-v^2}\right)^\ell w_{d-1}(t)\,dt \\ &= \int_{-1}^{1} \sum_{k=0}^{\ell} \binom{\ell}{k} u^{\ell-k}v^{\ell-k}(1-u^2)^{k/2}(1-v^2)^{k/2}t^k w_{d-1}(t)\,dt \qquad (6.6) \\ &= \sum_{\substack{k=0\\ k\text{ even}}}^{\ell} c_{k,d-1}\binom{\ell}{k} u^{\ell-k}v^{\ell-k}(1-u^2)^{k/2}(1-v^2)^{k/2}, \end{aligned}$$

where we omitted terms corresponding to k odd, since $c_{k,d-1}=0$ for odd k. Then $q(u,v)$ is a polynomial defined on $[-1,1]^2$. For every $u\in[-1,1]$ fixed, q is a polynomial in v of degree $\ell\le n$ and, in view of (6.4),

$$p_\ell(\mathbf{x},\omega_N) = N\sum_{i=1}^{\nu}\gamma_i q(u,t_i) = N\int_{-1}^{1} q(u,v)w_d(v)\,dv. \tag{6.7}$$

If ℓ is odd, the polynomial $q(u,v)$ is odd in v. Therefore,

$$p_\ell(\mathbf{x},\omega_N) = 0 = Nc_{\ell,d}.$$

Assume that ℓ is even. From (6.6), we have

$$\tau(u) := \int_{-1}^{1} q(u,v)w_d(v)\,dv = \sum_{\substack{k=0\\ k \text{ even}}}^{\ell} c_{k,d-1}\frac{\gamma_d}{\gamma_{d+k}}c_{\ell-k,d+k}\binom{\ell}{k}u^{\ell-k}(1-u^2)^{k/2}.$$

It remains to show that $\tau(u) = c_{\ell,d}$, $u \in [-1,1]$. For $u \in (0,1]$, making the substitution $u = (w^2+1)^{-1/2}$, $w \in [0,\infty)$, we obtain

$$\tau(u) = \frac{1}{(w^2+1)^{\ell/2}}\sum_{\substack{k=0\\ k \text{ even}}}^{\ell}\theta^\ell_{k,d}\binom{\ell}{k}w^k = \frac{1}{(w^2+1)^{\ell/2}}\sum_{j=0}^{\ell/2}\theta^\ell_{2j,d}\binom{\ell}{2j}w^{2j}, \tag{6.8}$$

where $\theta^\ell_{k,d} := c_{k,d-1}\frac{\gamma_d}{\gamma_{d+k}}c_{\ell-k,d+k}$.

We will show that the coefficient of w^{2j} in (6.8) is proportional to the binomial coefficient $\binom{\ell/2}{j}$. Then the sum in (6.8) will be a constant multiple of $(w^2+1)^{\ell/2}$ thus showing that $\tau(u)$ is constant. It is not difficult to verify that $c_{2j,d} = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(j+\frac{d+1}{2}\right)}$, $j \geq 0$, $\gamma_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}$, and $\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(2k)!} = \frac{1}{k!\cdot 2^{2k}}$, $k \geq 0$. Then

$$\begin{aligned}
\theta^\ell_{2j,d}\binom{\ell}{2j} &= \frac{\ell!\cdot\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{\ell+d+1}{2}\right)}\cdot\frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(2j)!}\cdot\frac{\Gamma\left(\frac{\ell-2j}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(\ell-2j)!}\\
&= \frac{\ell!\cdot\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{\ell+d+1}{2}\right)}\cdot\frac{1}{j!\cdot 2^{2j}}\cdot\frac{1}{2^{\ell-2j}(\ell/2-j)!}\\
&= \frac{\cdot\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{\ell+d+1}{2}\right)}\cdot\frac{\ell!}{(\ell/2)!\cdot 2^\ell}\cdot\binom{\ell/2}{j} = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+d+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\cdot\binom{\ell/2}{j}\\
&= c_{\ell,d}\binom{\ell/2}{j}
\end{aligned}$$

holds for every $j = 0,1,\ldots,\ell/2$. From (6.8), we now have

$$\tau(u) = \frac{c_{\ell,d}}{(w^2+1)^{\ell/2}}\sum_{j=0}^{\ell/2}\binom{\ell/2}{j}w^{2j} = c_{\ell,d}, \quad u \in (0,1].$$

Since $\tau(u)$ is an even polynomial, we have $\tau(u) = c_{\ell,d}$, $u \in [-1,1]$. From (6.7) we have

$$p_\ell(\mathbf{x},\omega_N) = N\tau(u) = Nc_{\ell,d}, \quad u \in [-1,1].$$

Thus, (6.5) holds for every $\ell = 1,\ldots,n$ and, trivially, for $\ell = 0$. Then item (iii) of Theorem 2.1 implies that ω_N is an N-point n-design on S^d. It is contained in the union of ν parallel hyperplanes. □

We are now ready to prove the remaining case of Theorem 6.8.

Case III: $m, d \geq 2$. In view of Lemma 6.10, there is a cardinality N_0 such that for every $N \geq N_0$, there exists a spherical $(2m-1)$-design on S^{d-1}. Denote $\rho := \min_{i=\overline{1,m}} a_0(\varphi_i)$ and let μ be the least common denominator of positive rationals $\frac{a_0(\varphi_i)}{\rho} N_0$, $i = 1, \ldots, m$. Let X_i be a $(2m-1)$-design on S^{d-1} of cardinality $N_i := \frac{\mu a_0(\varphi_i)}{\rho} N_0$, $i = 1, \ldots, m$. Applying Lemma 6.11 with $\nu = m$, $\gamma_i = a_0(\varphi_i)$, $i = 1, \ldots, m$, and t_i's being the zeros of $P_m^{(d)}$ and taking (6.3) into account, we obtain a spherical $(2m-1)$-design ω_N on S^d of cardinality $N = \frac{\mu}{\rho} N_0$ contained in m parallel hyperplanes. Then ω_N is m-stiff. $\square$

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Declarations

Conflict of interest The authors declare no competing interests.

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