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Aequationes Mathematicae

Quadratic functions fulfilling an additional condition along the hyperbola $xy = 1$

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Dedicated to Professors Maciej Sablik and László Székelyhidi on the occassion of their 70^{th} *birthdays.*

Abstract. In this paper we give necessary conditions for quadratic functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$.

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1. Preliminaries

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the set of all real numbers, rationals, and positive integers, respectively. We call a function $\varphi : \mathbb{R} \to \mathbb{R}$ *additive* if

$$\varphi(x + y) = \varphi(x) + \varphi(y) \tag{1}$$

holds for all $x, y \in \mathbb{R}$. The function φ is called $\mathbb{Q}$-*homogeneous* if the equation $\varphi(qx) = q\varphi(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is also well-known [10, Theorem 5.2.1], if $\varphi : \mathbb{R} \to \mathbb{R}$ is additive, then φ is $\mathbb{Q}$-homogeneous as well. An additive function is called a *linear function* if $\varphi(x) = x\varphi(1)$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *quadratic* if it satisfies the functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2}$$

for every $x, y \in \mathbb{R}$. As it is well known ([1], [2, Section 11.1]), we can associate with a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ the bi-additive and symmetric functional $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by the formula

$$F(x, y) = \frac{1}{2}\left[f(x + y) - f(x) - f(y)\right] \tag{3}$$

for all $x, y \in \mathbb{R}$. Then F is bi-additive (the mappings $t \mapsto F(t,x)$ and $t \mapsto F(x,t)$ $(t \in \mathbb{R})$ are additive for each $x \in \mathbb{R}$), and f is obtained as the diagonalization of F (i.e., $f(x) = F(x,x)$ for all $x \in \mathbb{R}$). Applying the $\mathbb{Q}$-homogeneity of additive functions, we have

$$F(rx, sy) = rsF(x,y) \quad \text{and} \quad f(rx) = F(rx,rx) = r^2F(x,x) = r^2f(x) \tag{4}$$

for every $r, s \in \mathbb{Q}$ and $x, y \in \mathbb{R}$. On the other hand, applying Eq. (3) and induction on n, one can easily prove the identity

$$f\left(\sum_{k=0}^{n} u_k\right) = \sum_{k=0}^{n} f(u_k) + 2 \sum_{0 \le i < j \le n} F(u_i,\, u_j) \tag{5}$$

for every $n \in \mathbb{N}$ and $u_0, u_1, \ldots, u_n \in \mathbb{R}$.

We say that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a *derivation* if φ satisfies (1) (i.e. φ is additive) and

$$\varphi(xy) = \varphi(x)y + x\varphi(y) \tag{6}$$

for all $x, y \in \mathbb{R}$. The family of derivations $\varphi\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{D}(\mathbb{R})$ in the sequel.

Equation (6) implies $\varphi(1) = 0$. Hence, any linear derivation is identically zero. On the other hand, it is also well known (and easy to prove) that the graph of any non-linear additive function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is dense in $\mathbb{R}^2$. In particular, the graph of any non-trivial (i.e., not identically zero) derivation $\varphi\colon \mathbb{R} \to \mathbb{R}$ has to be dense in $\mathbb{R}^2$. The existence of such functions is established, in a more general setting, for instance, in [13] (and in [10, Section 14.2]).

We need the notion of higher order derivation. The concept of derivations of higher order was introduced and characterized via functional equations by Unger and Reich [12]. The theory has been developed by Reich [11], Halter-Koch and Reich [9], Ebanks [5], and quite recently by Gselmann, Vincze and Kiss [8]. The recursive definition is based on the notion of bi-derivations. A functional $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *bi-derivation* if the mappings

$$t \mapsto B(t,x) \quad \text{and} \quad t \mapsto B(x,t) \quad (t \in \mathbb{R})$$

are derivations for each $x \in \mathbb{R}$.

Definition 1.1. The identically zero map is the only derivation of order zero. For each $n \in \mathbb{N}$, an additive mapping $\varphi\colon \mathbb{R} \to \mathbb{R}$ is called a derivation of order n, if there exists $B\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that B is a (symmetric) bi-derivation of order $n-1$ (that is, B is a derivation of order $n-1$ in each variable) and

$$\varphi(xy) - x\varphi(y) - \varphi(x)y = B(x,y) \qquad (x, y \in \mathbb{R}).$$

The set of derivations of order n will be denoted by $\mathcal{D}_n(\mathbb{R})$.

Then $\mathcal{D}_1(\mathbb{R})$ is the set of derivations. Since the identically zero mapping from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ is a bi-derivation, we have the inclusion $\mathcal{D}_1(\mathbb{R}) \subseteq \mathcal{D}_2(\mathbb{R})$. Then an inductive argument yields the inclusion $\mathcal{D}_{n-1}(\mathbb{R}) \subseteq \mathcal{D}_n(\mathbb{R})$ for every

$n \in \mathbb{N}$. In the sequel we consider various characterizations of derivations of order 3.

Proposition 1.2. *(Unger and Reich [12] and Ebanks [5]). Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an additive function. Then $\varphi \in \mathcal{D}_3(\mathbb{R})$ if and only if*

$$\varphi\left(x^4\right) - 4x\varphi\left(x^3\right) + 6x^2\varphi\left(x^2\right) - 4x^3\varphi\left(x\right) = 0 \tag{7}$$

for all $x \in \mathbb{R}$.

We also need the following Lemma:

Lemma 1.3. (Amou [3, Lemma 2.3]). *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an additive function such that*

$$\varphi\left(x^8\right) - 14x^4\varphi\left(x^4\right) + 56x^6\varphi\left(x^2\right) - 64x^7\varphi\left(x\right) = 0 \tag{8}$$

for every $x \in \mathbb{R}$. Then $\varphi \in \mathcal{D}_3(\mathbb{R})$.

Though it is not explicitly mentioned by Amou [3], the converse implications is valid as well. Details are given in Proposition 1.5 below.

We shall also make use of the following observation.

Lemma 1.4. (Z. Boros and E. Garda-Mátyás [4]). *If $\mathbb{F}$ is a field, $n \in \mathbb{N}$, X is an arbitrary set, $V \subset \mathbb{F}$ contains at least $n+1$ elements, and the functions $G_k\colon X \to \mathbb{F}$ $(k = 0, 1, \ldots, n)$ satisfy the equation*

$$\sum_{k=0}^{n} G_k(x) r^k = 0 \tag{9}$$

for every $x \in X$ and $r \in V$, then $G_k(x) = 0$ for every $x \in X$ and $k \in \{0, 1, \ldots, n\}$.

In this paper, we shall apply Lemma 1.4 for $X = \mathbb{F} = \mathbb{R}$ and $V = \mathbb{Q}$.

Now we can establish a stronger version of Lemma 1.3.

Proposition 1.5. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an additive function. Then φ fulfills (8) if, and only if, $\varphi \in \mathcal{D}_3(\mathbb{R})$.*

Proof. In view of Proposition 1.2, we have to show that Eqs. (7) and (8) are equivalent for additive mappings $\varphi\colon \mathbb{R} \to \mathbb{R}$. Our argument is a refinement of Amou's proof [3, Lemma 2.3]. Namely, as it is explained in the cited argument, if the additive function φ fulfills (8), taking arbitrary $x \in \mathbb{R}$ and $r \in \mathbb{Q}$, substituting $x + r$ in place of x in (8), expanding the left hand side using the additivity and the rational homogeneity of φ, the coefficient of r^4 equals 56 times the left hand side of (7). Then our Lemma 1.4 yields the validity of (7).

Now let us assume that the additive function $\varphi\colon \mathbb{R} \to \mathbb{R}$ fulfills (7). Let us take $x \in \mathbb{R}$ and $r \in \mathbb{Q}$ arbitrarily. Replacing x with $x(x+r)$ we obtain

$$\begin{aligned} 0 &= \varphi\left(x^4(x+r)^4\right) - 4x(x+r)\varphi\left(x^3(x+r)^3\right) \\ &\quad +6x^2(x+r)^2\varphi\left(x^2(x+r)^2\right) - 4x^3(x+r)^3\varphi\left(x(x+r)\right) \\ &= \sum_{k=0}^{4} G_k(x) r^k, \end{aligned}$$

where

$$G_k(x) = \binom{4}{k} \sum_{j=0}^{8-k-1} A_{k,j} x^j \varphi(x^{8-k-j})$$

with

$$A_{4,j} = (-1)^j \binom{4}{j} \qquad (j = 0, 1, 2, 3, 4),$$

$A_{k,0} = A_{k,8-k} = 1 \quad (k = 0, 1, 2, 3)$ and

$$A_{k,j} = A_{k+1,j-1} + A_{k+1,j} \quad (j = 1, 2, \ldots, 8-k-1),\ (k = 0, 1, 2, 3).$$

Lemma 1.4 yields $G_k(x) = 0$ for every $x \in \mathbb{R}$ and $k \in \{0, 1, 2, 3, 4\}$. Now Eq. (8) follows from the observation that its left-hand side equals

$$G_0(x) + \frac{2}{3}x^2 G_2(x) + 2x^3 G_3(x) + 8x^4 G_4(x).$$

□

2. Motivation

In a recent paper, Z. Boros and E. Garda-Mátyás [4] investigated quadratic functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the additional equation

$$x^2 f(y) = y^2 f(x) \tag{10}$$

for the pairs $(x, y) \in \mathbb{R}^2$ that fulfill the condition $P(x, y) = 0$ for some fixed polynomial P of two variables. The authors [4, Problem 4.1] showed that there exist discontinuous quadratic solutions of Eq. (10) for the pairs $(x, y) \in \mathbb{R}^2$ that fulfill $xy = 1$, giving a counterexample, and formulated the following problem: Determine the general quadratic solution $f : \mathbb{R} \to \mathbb{R}$ of the equation

$$f(x) = x^4 f\left(\frac{1}{x}\right) \qquad (x \in \mathbb{R} \setminus \{0\}). \tag{11}$$

Though the continuity of f does not follow from this assumption, E. Garda-Mátyás [6] obtained some interesting results for the mappings $x \mapsto F(x, 1)$ and $x \mapsto F(x, 1/x)$. By Lemma 3.1 in [6] we have

$$F(x, 1) = x f(1) \tag{12}$$

for all $x \in \mathbb{R}$, and by Lemma 3.2 in [6] we have

$$f\left(x^2\right) = 2x^4 F\left(x, \frac{1}{x}\right) + 6x^2 f(x) - 7x^4 f(1) \tag{13}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

In this paper we give further necessary conditions for quadratic functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the additional Eq. (11).

3. Main results

Proposition 3.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a quadratic function, which satisfies the additional Eq.* (11)*. Let $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be given by* (3)*. Let us define a map $H\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by*

$$H(x,y) := F(x,y) - xyf(1) \tag{14}$$

and let $h(x) = H(x,x)$ $(x \in \mathbb{R})$. Then H is symmetric and bi-additive,

$$H(x,1) = 0, \tag{15}$$

$$h(x) = f(x) - x^2 f(1), \tag{16}$$

for every $x \in \mathbb{R}$, and we have

$$H\left(x, \frac{1}{x}\right) = F\left(x, \frac{1}{x}\right) - f(1), \tag{17}$$

$$h(x) = x^4 h\left(\frac{1}{x}\right) \tag{18}$$

for every $x \in \mathbb{R} \setminus \{0\}$.

Proof. From (12) and (14) we obtain (15) as

$$H(x,1) = F(x,1) - xf(1) = 0,$$

while (16) is obtained from (3) as

$$h(x) = H(x,x) = F(x,x) - x^2 f(1) = f(x) - x^2 f(1)$$

for every $x \in \mathbb{R}$. Replacing y with $\frac{1}{x}$ in Eq. (14) we obtain (17). From (16) we have $h(1) = 0$. The conditional Eq. (11) has the form

$$h(x) + x^2 f(1) = x^4 \left[h\left(\frac{1}{x}\right) + \frac{1}{x^2} f(1)\right],$$

which yields (18) for every real number $x \neq 0$. □

Lemma 3.2. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies* (11)*, then*

$$F\left(x^2, x\right) = 2xf(x) - x^3 f(1). \tag{19}$$

for all $x \in \mathbb{R}$, where $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by (3)*.*

Proof. We rearrange Eq. (13) in the following form

$$f\left(x^2\right) - x^4 f(1) = 2x^4 \left[F\left(x, \frac{1}{x}\right) - f(1)\right] + 6x^2\left[f(x) - x^2 f(1)\right].$$

From Eq. (16) we obtain $h\left(x^2\right) = f\left(x^2\right) - x^4 f(1)$. Then with (17), Eq. (13) has the form

$$h\left(x^2\right) = 2x^4 H\left(x, \frac{1}{x}\right) + 6x^2 h(x). \tag{20}$$

Using (20) for $x \in \mathbb{R} \setminus \{0, 1\}$, we write $h\left((x-1)^2\right)$ in two ways:

$$h\left((x-1)^2\right) = 2(x-1)^4 H\left(x-1, \frac{1}{x-1}\right) + 6(x-1)^2 h(x-1).$$

From (15) we have $h(x-1) = h(x)$ and $H\left(x-1, \frac{1}{x-1}\right) = H\left(x, \frac{1}{x-1}\right)$, so

$$h\left((x-1)^2\right) = 2(x-1)^4 H\left(x, \frac{1}{x-1}\right) + 6(x-1)^2 h(x),$$

while the addition rule (5) for the quadratic function h yields

$$h\left((x-1)^2\right) = h\left(x^2 - 2x + 1\right) = h\left(x^2 - 2x\right) = h\left(x^2\right) + 4h(x) - 4H\left(x^2, x\right).$$

From the equality of the left sides of the last two equations, it follows that

$$(x-1)^4 H\left(x, \frac{1}{x-1}\right) = \frac{1}{2} h\left(x^2\right) + \left(-3x^2 + 6x - 1\right) h(x) - 2H\left(x^2, x\right). \tag{21}$$

Using (18) for $x \in \mathbb{R} \setminus \{0, 1\}$, now we write $h\left(x^2 - x\right)$ in two ways:

$$\begin{aligned} h\left(x^2 - x\right) &= \left(x^2 - x\right)^4 h\left(\frac{1}{x^2 - x}\right) = x^4 (x-1)^4 h\left(\frac{1}{x-1} - \frac{1}{x}\right) \\ &= x^4 (x-1)^4 \left[h\left(\frac{1}{x-1}\right) + h\left(\frac{1}{x}\right) - 2H\left(\frac{1}{x}, \frac{1}{x-1}\right)\right] \\ &= x^4 (x-1)^4 \left[\frac{1}{(x-1)^4} h(x-1) + \frac{1}{x^4} h(x) - 2H\left(\frac{1}{x}, \frac{1}{x-1}\right)\right] \\ &= x^4 h(x) + (x-1)^4 h(x) - 2x^4 (x-1)^4 H\left(\frac{1}{x}, \frac{1}{x-1}\right). \end{aligned}$$

On the other hand, by the addition rule we have

$$h\left(x^2 - x\right) = h\left(x^2\right) + h(x) - 2H\left(x^2, x\right).$$

From the equality of the last two expressions we obtain

$$\begin{aligned} &x^4 (x-1)^4 H\left(\frac{1}{x}, \frac{1}{x-1}\right) = \\ &\quad -\frac{1}{2} h\left(x^2\right) + \left(x^4 - 2x^3 + 3x^2 - 2x\right) h(x) + H\left(x^2, x\right). \end{aligned} \tag{22}$$

Replacing x with $\frac{1}{x}$ in Eq. (22), taking also (15) and (18) into consideration, we have

$$-\frac{1}{x^4}\frac{(x-1)^4}{x^4}H\left(x,\frac{1}{x-1}\right)=$$
$$-\frac{1}{2x^8}h\left(x^2\right)+\frac{\left(1-2x+3x^2-2x^3\right)}{x^8}h(x)+H\left(\frac{1}{x^2},\frac{1}{x}\right).$$

Multiplying the latter equation by $-x^8$, we get

$$\begin{aligned}&(x-1)^4H\left(x,\frac{1}{x-1}\right)=\\&\frac{1}{2}h\left(x^2\right)+\left(2x^3-3x^2+2x-1\right)h(x)-x^8H\left(\frac{1}{x^2},\frac{1}{x}\right).\end{aligned}\tag{23}$$

From the equality of the left sides of (21) and (23) we obtain

$$\begin{aligned}&\left(-3x^2+6x-1\right)h(x)-2H\left(x^2,x\right)\\&=\left(2x^3-3x^2+2x-1\right)h(x)-x^8H\left(\frac{1}{x^2},\frac{1}{x}\right),\end{aligned}$$

therefore

$$2H\left(x^2,x\right)=\left(-2x^3+4x\right)h(x)+x^8H\left(\frac{1}{x^2},\frac{1}{x}\right).\tag{24}$$

Putting $\frac{1}{x}$ in place of x in this equality, we get

$$2H\left(\frac{1}{x^2},\frac{1}{x}\right)=\left(\frac{-2}{x^3}+\frac{4}{x}\right)\frac{1}{x^4}h(x)+\frac{1}{x^8}H\left(x^2,x\right).$$

Substituting this expansion of $H\left(\frac{1}{x^2},\frac{1}{x}\right)$ into Eq. (24), we obtain

$$2H\left(x^2,x\right)=\left(-2x^3+4x\right)h(x)+(-x+2x^3)h(x)+\frac{1}{2}H\left(x^2,x\right),$$

therefore we have

$$H\left(x^2,x\right)=2xh(x),\tag{25}$$

i.e., $F\left(x^2,x\right)=2xf(x)-x^3f(1)$. The validity of (25) for $x\in\{0,1\}$ is obvious (cf. (15)). □

Lemma 3.3. *If a quadratic function* $f\colon\mathbb{R}\to\mathbb{R}$ *satisfies* (11)*, then*

$$f\left(x^4\right)=20x^4f\left(x^2\right)-64x^6f(x)+45x^8f(1)\tag{26}$$

for every $x\in\mathbb{R}$.

Proof. Let us consider the functions H and h introduced in Proposition 3.1. Replacing x with $x-\frac{1}{x}$ in Eq. (20), we obtain

$$h\left(\left(x-\frac{1}{x}\right)^2\right)=2\left(x-\frac{1}{x}\right)^4H\left(x-\frac{1}{x},\frac{1}{x-\frac{1}{x}}\right)+6\left(x-\frac{1}{x}\right)^2h\left(x-\frac{1}{x}\right).$$

On the other hand,

$$\begin{aligned} h\left(\left(x-\frac{1}{x}\right)^2\right) &= h\left(x^2+\frac{1}{x^2}-2\right) = h\left(x^2+\frac{1}{x^2}\right) \\ &= h\left(x^2\right)+h\left(\frac{1}{x^2}\right)+2H\left(x^2,\,\frac{1}{x^2}\right) \\ &= \left(1+\frac{1}{x^8}\right)h\left(x^2\right)+2H\left(x^2,\,\frac{1}{x^2}\right). \end{aligned}$$

From the equality of the left sides of the last two equations, it follows that

$$\begin{aligned} 2H\left(x^2,\,\frac{1}{x^2}\right) = {} & 2\left(x-\frac{1}{x}\right)^4 H\left(x-\frac{1}{x},\,\frac{1}{x-\frac{1}{x}}\right) \\ & + 6\left(x-\frac{1}{x}\right)^2 h\left(x-\frac{1}{x}\right)-\left(1+\frac{1}{x^8}\right)h\left(x^2\right). \end{aligned} \tag{27}$$

Using (20) for $x \in \mathbb{R}\setminus\{-1,\,0\}$, we write $h\left((x+1)^2\right)$ in two ways:

$$h\left((x+1)^2\right) = 2(x+1)^4 H\left(x+1,\,\frac{1}{x+1}\right)+6(x+1)^2 h(x+1).$$

From (15) we have $h(x+1)=h(x)$ and $H\left(x+1\,,\,\frac{1}{x+1}\right)=H\left(x\,,\,\frac{1}{x+1}\right)$, so

$$h\left((x+1)^2\right) = 2(x+1)^4 H\left(x,\,\frac{1}{x+1}\right)+6(x+1)^2 h(x),$$

while using (25), we have

$$\begin{aligned} h\left((x+1)^2\right) &= h\left(x^2+2x+1\right) = h\left(x^2+2x\right) \\ &= h\left(x^2\right)+4h(x)+4H\left(x^2,x\right) = h\left(x^2\right)+(8x+4)h(x). \end{aligned}$$

From the equality of the left sides of the last two equations we obtain

$$(x+1)^4 H\left(x,\,\frac{1}{x+1}\right) = \frac{1}{2}h\left(x^2\right)-\left(3x^2+2x+1\right)h(x). \tag{28}$$

Using (18) for $x \in \mathbb{R}\setminus\{-1,\,0\}$, now we write $h\left(x^2+x\right)$ in two ways:

$$\begin{aligned} h\left(x^2+x\right) &= \left(x^2+x\right)^4 h\left(\frac{1}{x^2+x}\right) = x^4(x+1)^4 h\left(\frac{1}{x}-\frac{1}{x+1}\right) \\ &= x^4(x+1)^4\left[h\left(\frac{1}{x}\right)+h\left(\frac{1}{x+1}\right)-2H\left(\frac{1}{x}\,,\,\frac{1}{x+1}\right)\right] \\ &= x^4(x+1)^4\left[\frac{1}{x^4}h\left(x\right)+\frac{1}{(x+1)^4}h\left(x+1\right)-2H\left(\frac{1}{x}\,,\,\frac{1}{x+1}\right)\right] \\ &= (x+1)^4 h(x)+x^4 h(x)-2x^4(x+1)^4 H\left(\frac{1}{x}\,,\,\frac{1}{x+1}\right). \end{aligned}$$

On the other hand, using (25), we have

$$h\left(x^2+x\right)=h\left(x^2\right)+h(x)+2H\left(x^2,x\right)=h\left(x^2\right)+h(x)+4xh(x).$$

From the equality of the last two expressions, it follows that

$$x^4(x+1)^4H\left(\frac{1}{x},\frac{1}{x+1}\right)=-\frac{1}{2}h\left(x^2\right)+\left(x^4+2x^3+3x^2\right)h(x). \tag{29}$$

Using (25) in Eq. (21), we obtain

$$(x-1)^4H\left(x,\frac{1}{x-1}\right)=\frac{1}{2}h\left(x^2\right)+\left(-3x^2+2x-1\right)h(x). \tag{30}$$

Using (25) in Eq. (22), we have

$$x^4(x-1)^4H\left(\frac{1}{x},\frac{1}{x-1}\right)=-\frac{1}{2}h\left(x^2\right)+\left(x^4-2x^3+3x^2\right)h(x). \tag{31}$$

Now we write

$$\begin{aligned}
2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{x-\frac{1}{x}}\right)&=2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{x}{x^2-1}\right)\\
&=2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)\right)\\
&=\left(x^2-1\right)^4 H\left(x,\frac{1}{x-1}\right)+\left(x^2-1\right)^4 H\left(x,\frac{1}{x+1}\right)\\
&\quad-\left(x^2-1\right)^4 H\left(\frac{1}{x},\frac{1}{x-1}\right)-\left(x^2-1\right)^4 H\left(\frac{1}{x},\frac{1}{x+1}\right).
\end{aligned}$$

Substituting (28),(29),(30) and (31) into the latter equation, after some computation we get

$$\begin{aligned}
&2\left(x^2-1\right)^4 H\left(x-\frac{1}{x},\frac{1}{x-\frac{1}{x}}\right)=\\
&\frac{\left(x^4+6x^2+1\right)\left(x^4+1\right)}{x^4}h\left(x^2\right)-\frac{2\left(3x^8+12x^6+2x^4+12x^2+3\right)}{x^2}h(x).
\end{aligned} \tag{32}$$

Substituting (32) into Eq. (27), we have

$$\begin{aligned}
&2H\left(x^2,\frac{1}{x^2}\right)\\
&=\frac{\left(x^4+6x^2+1\right)\left(x^4+1\right)}{x^8}h\left(x^2\right)-\frac{2\left(3x^8+12x^6+2x^4+12x^2+3\right)}{x^6}h(x)\\
&\quad+6\left(x-\frac{1}{x}\right)^2h\left(x-\frac{1}{x}\right)-\left(1+\frac{1}{x^8}\right)h\left(x^2\right).
\end{aligned} \tag{33}$$

Expressing $H\left(x,\,\frac{1}{x}\right)$ from Eq. (20), we get

$$\begin{aligned} h\left(x-\frac{1}{x}\right) &= h(x)+h\left(\frac{1}{x}\right)-2H\left(x,\,\frac{1}{x}\right)\\ &= h(x)+\frac{1}{x^4}h(x)-\frac{1}{x^4}h\left(x^2\right)+\frac{6}{x^2}h(x)\\ &= \frac{x^4+6x^2+1}{x^4}h(x)-\frac{1}{x^4}h\left(x^2\right). \end{aligned}$$

Substituting this into Eq. (33), after some computation we obtain

$$H\left(x^2,\,\frac{1}{x^2}\right)=\frac{7}{x^4}h\left(x^2\right)-\frac{32}{x^2}h(x). \tag{34}$$

Replacing x with x^2 in Eq. (20), we have

$$h\left(x^4\right)=2x^8H\left(x^2,\,\frac{1}{x^2}\right)+6x^4h\left(x^2\right).$$

Finally we substitute (34) into the latter equation to obtain

$$h\left(x^4\right)=20x^4h\left(x^2\right)-64x^6h(x). \tag{35}$$

The statement of the Lemma follows from Eqs. (35) and (16). □

Theorem 3.4. *If a quadratic function $f\colon\mathbb{R}\to\mathbb{R}$ satisfies (11), then there exists a symmetric bi-derivation H of order 3 for which*

$$f(x)=H(x,x)+x^2f(1)\quad \textit{for every}\quad x\in\mathbb{R}.$$

Proof. As well as in the previous arguments, we consider the functions H and h introduced in Proposition 3.1.

Let $x,y\in\mathbb{R}$ and $r\in\mathbb{Q}$. Substituting $x+ry$ in place of x in Eq. (25), we get

$$H\left(x^2+2rxy+r^2y^2,x+ry\right)=2(x+ry)h(x+ry).$$

Rearranging the latter equation and using (25) we obtain

$$\begin{aligned} 0 &= 2rH(xy,x)+r^2H\left(y^2,x\right)+rH\left(x^2,y\right)+2r^2H(xy,y)\\ &\quad -2r^2xh(y)-4rxH(x,y)-2ryh(x)-4r^2yH(x,y). \end{aligned}$$

Thus we get a polynomial in r. The coefficient of r^1 equals zero (by Lemma 1.4), hence we obtain

$$2H(xy,x)+H\left(x^2,y\right)=4xH(x,y)+2yh(x). \tag{36}$$

Let $x,y\in\mathbb{R}$ and $r\in\mathbb{Q}$. Replacing x with $x+ry$ in Eq. (35) (derived in the proof of Lemma 3.3) we obtain

$$h\left((x+ry)^4\right)=20(x+ry)^4h\left((x+ry)^2\right)-64(x+ry)^6h(x+ry). \tag{37}$$

Expanding the powers of sums on both sides, Eq. (37) can be written as

$$h\left(\sum_{k=0}^{4}\binom{4}{k}x^k r^{n-k}y^{n-k}\right) = 20\left(\sum_{l=0}^{4}\binom{4}{l}x^l r^{4-l}y^{4-l}\right)h\left(x^2+2rxy+r^2y^2\right)$$
$$-64\sum_{q=0}^{6}\binom{6}{q}x^q r^{n-q}y^{n-q}h(x+ry).$$

Applying the identity (5), the rational homogeneity properties of H and h, Eq. (15), and using $h(1)=0$, we obtain

$$\begin{aligned}
0 &= \sum_{k=0}^{4}\binom{4}{k}^2 r^{8-2k}h\left(x^k y^{4-k}\right) \\
&+ 2\sum_{0\le i<j\le 4}\binom{4}{i}\binom{4}{j}r^{8-(i+j)}H\left(x^i y^{4-i},\, x^j y^{4-j}\right) \\
&- 20\left(\sum_{l=0}^{4}\binom{4}{l}x^l r^{4-l}y^{4-l}\right) \\
&\cdot\left[h\left(x^2\right)+4r^2h(xy)+r^4h\left(y^2\right)+4rH\left(x^2,xy\right)\right. \\
&\left.+4r^3H\left(y^2,xy\right)+2r^2H\left(x^2,y^2\right)\right] \\
&+ 64\sum_{q=0}^{6}\binom{6}{q}x^q r^{6-q}y^{6-q}\left[h(x)+2rH(x,y)+r^2h(y)\right].
\end{aligned} \tag{38}$$

The coefficient of r^1 equals zero, hence we get

$$\begin{aligned}
0 &= 2\binom{4}{3}\binom{4}{4}H\left(x^3y,\, x^4\right) - 20\left[\binom{4}{3}x^3yh\left(x^2\right)+\binom{4}{4}x^4 4H\left(x^2,xy\right)\right] \\
&\quad + 64\binom{6}{5}x^5yh(x) + 64\binom{6}{6}x^6 2H(x,y) \\
&= 8H\left(x^3y,\, x^4\right) - 80x^3yh\left(x^2\right) - 80x^4H\left(x^2,xy\right) \\
&\quad + 384x^5yh(x) + 128x^6H(x,y).
\end{aligned}$$

Thus

$$H\left(x^3y,\, x^4\right) = 10x^3yh\left(x^2\right) + 10x^4H\left(x^2,xy\right) - 48x^5yh(x) - 16x^6H(x,y). \tag{39}$$

Replacing y with xy in Eq. (39), we get

$$H\left(x^4y,\, x^4\right) = 10x^4yh\left(x^2\right) + 10x^4H\left(x^2,x^2y\right) - 48x^6yh(x) - 16x^6H(x,xy). \tag{40}$$

Putting x^2 in place of x in Eq. (36) we have

$$2H\left(x^2y,\, x^2\right) = -H\left(x^4,y\right) + 4x^2H\left(x^2,y\right) + 2yh\left(x^2\right). \tag{41}$$

Substituting this expansion of $H\left(x^2y,\, x^2\right)$ into Eq. (40), we obtain

$$\begin{aligned} H\left(x^4y,\, x^4\right) =& 20x^4yh\left(x^2\right) + 20x^6H\left(x^2,y\right) - 5x^4H\left(x^4,y\right) \\ & - 96x^6yh(x) - 32x^6H(x,xy). \end{aligned} \tag{42}$$

Expressing $H(x,xy)$ from Eq. (36), then substituting it into Eq. (42), we get

$$\begin{aligned} H\left(x^4y,\, x^4\right) =& 20x^4yh\left(x^2\right) + 28x^6H\left(x^2,y\right) - 5x^4H\left(x^4,y\right) \\ & - 64x^6yh(x) - 32x^7H(x,y). \end{aligned} \tag{43}$$

Now, replacing x with x^4 in Eq. (36) we have

$$2H\left(x^4y,\, x^4\right) = -H\left(x^8,y\right) + 4x^4H\left(x^4,y\right) + 2yh\left(x^4\right). \tag{44}$$

And finally, from the equality of the left sides of (43) and (44), with (35), we obtain

$$H\left(x^8,y\right) - 14x^4H\left(x^4,y\right) + 56x^6H\left(x^2,y\right) - 64x^7H(x,y) = 0. \tag{45}$$

The latter equation holds for an arbitrary fixed $y \in \mathbb{R}$ and for every $x \in \mathbb{R}$. By Lemma 1.3, H is a derivation of order 3 in its first variable. Since H is a symmetric, bi-additive function, it follows that H is a derivation of order 3 in each variable, so H is a symmetric bi-derivation of order 3. □

Theorem 3.5. *If a quadratic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies (11), the bi-additive and symmetric functionals $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $H\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are given by the formulas (3) and (14), respectively, and*

$$T\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

is defined by the formula

$$T(x,y,z) = H(xy,z) - xH(y,z) - yH(x,z) \quad (x,y,z \in \mathbb{R}), \tag{46}$$

then

$$T(x,y,z) + T(z,x,y) + T(y,z,x) = 0 \tag{47}$$

for every $x,y,z \in \mathbb{R}$.

Proof. Clearly, Eq. (36) yields

$$0 = H\left(x^2,y\right) + 2H(xy,x) - 2yH(x,x) - 4xH(x,y) \tag{48}$$

for every $x,y \in \mathbb{R}$. Taking arbitrary $x,z \in \mathbb{R}$, $r \in \mathbb{Q}$, replacing x with $x+rz$ in Eq. (48), expanding it using the symmetry and the bi-additivity of H, applying Lemma 1.4 for the coefficient of r, and dividing the obtained equation by 2, we have

$$\begin{aligned} 0 = & H(xy,z) + H(yz,x) + H(zx,y) \\ & -2xH(y,z) - 2yH(z,x) - 2zH(x,y), \end{aligned}$$

which can be reformulated as Eq. (47). □

Remark 3.6. If $H\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a symmetric bi-derivation, $c\in\mathbb{R}$, and $f\colon \mathbb{R}\to\mathbb{R}$ is given by the formula

$$f(x) = cx^2 + H(x,x) \quad (x\in\mathbb{R}), \tag{49}$$

then f is a quadratic function fulfilling the additional Eq. (11) (i.e., f satisfies the additional equation $y^2 f(x) = x^2 f(y)$ under the condition $xy = 1$). This sufficient condition for the mapping H implies $T(x,y,z) = 0$ identically for the tri-additive mapping T given by the formula (46). Our necessary conditions established in Theorems 3.4 and 3.5 are similar but, in fact, weaker then this sufficient condition for the mapping H. Therefore, this paper provides only a partial solution to [4, Problem 1]. However, recent investigations by Masaaki Amou (presented at the 58th International Symposium on Functional Equations, Innsbruck, Austria, June 19–26, 2022) suggest that this sufficient condition need not be necessary.

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Author contributions Edit Garda-Mátyás elaborated most of the calculations and established Theorem 3.4 together with the preceding lemmas. She prepared the preliminary version of the manuscript and included these results in her PhD dissertation [7], which was written under the supervision of her present co-author. Zoltán Boros called his co-author's attention to the most relevant reference citations, suggested the research project, assisted the proper formulation of the results, improved the presentation and added a new result (Theorem 3.5) together with the concluding remark.

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Declarations

Conflict of interest This is to declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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