Aequat. Math. 97 (2023), 981–993

0001-9054/23/5-60981-13
published online November 21, 2023
https://doi.org/10.1007/s00010-023-01017-1

Aequationes Mathematicae

Check for updates

Further remarks on local K-boundedness of K-subadditive set-valued maps

Eliza Jabłońska and Kazimierz Nikodem

Dedicated to Prof. Maciej Sablik and Prof. László Székelyhidi on their 70th birthday.

Abstract. Let X be an abelian metric group with an invariant metric, Y be a real normed space and K be a convex cone in Y. We prove that a K-subadditive (K-superadditive) compact- and convex-valued map $F\colon X \to \mathcal{CC}(Y)$, for which the functionals $f_{y^*}(x) = \inf y^*(F(x))$ are lower (upper, resp.) semicontinuous for any real continuous and non-negative on K functional y^*, has to be locally K-bounded on X. Our results refer to the papers Banakh and Jabłońska (Israel J Math 230:361–386, 2019), Jabłońska and Nikodem (Math Inequal Appl 22:1081–1089, 2019) and Nikodem (Aequationes Math 62:175–183, 2001).

Mathematics Subject Classification. Primary 39B62; Secondary 54C60, 26B25.

Keywords. K-subadditivity, K-superadditivity, Set-valued map, Upper semicontinuous function, Lower semicontinuous function, K-upper semicontinuity, K-lower semicontinuity, K-boundedness, Null-finite set, Haar-null set, Haar-meager set.

1. Introduction

Let X be an abelian metric group with an invariant metric and Y be a real topological vector space. Assume that K is a subsemigroup of Y (i.e. $K + K \subset K$). Denote by $n(Y)$ the family of all nonempty subsets of Y, and by $\mathcal{B}(Y)$ and $\mathcal{CC}(Y)$ its subfamilies of all bounded subsets of Y and all convex compact subsets of Y, respectively.

Definition 1. A set-valued map (s.v. map for short) $F\colon X \to n(Y)$ is called *K-subadditive* if

$$F(x_1) + F(x_2) \subset F(x_1 + x_2) + K \tag{1}$$

for all $x_1, x_2 \in X$. If F satisfies

$$F(x_1+x_2) \subset F(x_1)+F(x_2)+K \tag{2}$$

for all $x_1, x_2 \in X$ then it is called K-*superadditive*.

The concepts of K-subadditivity and K-superadditivity were introduced in [11], following the notions of K-midconvexity and K-midconcavity from [13]. Clearly, if F is a single-valued function, $Y=\mathbb{R}$ and $K=[0,\infty)$, then K-subadditivity means the classical subadditivity, i.e. $f\colon X \to \mathbb{R}$ satisfies

$$f(x_1+x_2) \le f(x_1)+f(x_2) \quad \text{for every } x_1, x_2 \in X,$$

as well as K-superadditivity means the classical superadditivity, i.e. $-f$ is subadditive.

Let us recall also the notion of K-lower (K-upper) semicontinuity from [13] which generalizes the classical notion of upper (lower, resp.) semicontinuity of a single-valued real function.

Definition 2. Let $x_0 \in X$. The s.v. map $F\colon X \to n(Y)$ is called:

- *K-lower semicontinuous at x_0*, if for every neighborhood W of 0 in Y there exists a neighborhood U of 0 in X such that
$$F(x_0) \subset F(x)+W+K \quad \text{for } x \in x_0+U,$$
- *K-upper semicontinuous at x_0*, if for every neighborhood W of 0 in Y there exists a neighborhood U of 0 in X such that
$$F(x) \subset F(x_0)+W+K \quad \text{for } x \in x_0+U.$$

Moreover, F is *K-lower* (*K-upper*) *semicontinuous on* X, if it is K-lower (K-upper, resp.) semicontinuous at each point $x \in X$.

Definition 3. A s.v. map $F\colon X \to \mathcal{B}(Y)$ is called:

- *weakly K-upper bounded* on a set $A \subset X$, if there is $B \in \mathcal{B}(Y)$ such that
$$F(x) \cap (B-K) \neq \emptyset \quad \text{for all } x \in A,$$
- *K-upper bounded* on a set $A \subset X$, if there is $B \in \mathcal{B}(Y)$ such that
$$F(x) \subset B-K \quad \text{for all } x \in A,$$
- *weakly K-lower bounded* on a set $A \subset X$, if there is $B \in \mathcal{B}(Y)$ such that
$$F(x) \cap (B+K) \neq \emptyset \quad \text{for all } x \in A,$$
- *K-lower bounded* on a set $A \subset X$, if there is $B \in \mathcal{B}(Y)$ such that
$$F(x) \subset B+K \quad \text{for all } x \in A.$$

The above idea of (weak) K-upper/K-lower boundedness was introduced in [13] as a generalization of the notion of boundedness from above/below of single-valued real functions. More precisely, weak $[0,\infty)$-upper boundedness, as well as $[0,\infty)$-upper boundedness of a single-valued real function means its classical boundedness from above. The same holds for (weak) $[0,\infty)$-lower boundedness of a single-valued real function.

Definition 4. A s.v. map $F\colon X \to \mathcal{B}(Y)$ is called:

- *locally (weakly) K-upper (K-lower) bounded at $x \in X$*, if it is (weakly) K-upper (K-lower, resp.) bounded on some neighborhood of x,
- *locally K-bounded at $x \in X$*, if it is both locally weakly K-upper bounded and locally K-lower bounded at x,
- *locally K-bounded on X*, if it is locally K-bounded at each point of X.

It is well known that for every subadditive real function defined on an abelian metric group boundedness from above on a "large" (in the sense of category or measure) set implies its local boundedness on the whole domain (see e.g. [12, Theorem 16.2.3]). Some recent results on subadditive functions can be found e.g. in [3]–[7].

In [2] the following generalization of the mentioned classical result was proved.

Theorem 1. [2, Theorem 2.2] *Let X be an abelian metric group with an invariant metric and $f\colon X \to \mathbb{R}$ be a subadditive function. If f is bounded from above on a non-null-finite set $A \subset X$ then f is locally bounded on X.*

The concept of null-finite sets was introduced in [1].[1]

Definition 5. A subset A of an abelian metric group X is called *null-finite* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ tending to zero in X such that the set $\{n \in \mathbb{N}\colon x + x_n \in A\}$ is finite for every $x \in X$.

In a complete abelian metric group X with an invariant metric the following sets are not null-finite: open sets, non-meager sets with the Baire property, sets of positive Haar measure provided X is locally compact, universally measurable sets which are not Haar-null, Borel sets which are not Haar-meager (see [1, Theorems 5.1 and 6.1]).

The notions of a Haar-null set and a Haar-meager set have been introduced by Christensen [8] and Darji [9], respectively. A subset B of an abelian Polish group X is called:

- *Haar-meager* if there exist a Borel set $A \supset B$, a compact metric space K and a continuous function $f\colon K \to X$ such that $f^{-1}(A + x)$ is meager in K for every $x \in X$;
- *Haar-null* if there exist a universally measurable set $A \supset B$ and a σ-additive probability Borel measure μ on X such that $\mu(A + x) = 0$ for every $x \in X$.

It was proved in [8] and [9] that every Haar-meager set is meager and, moreover, in every locally compact abelian Polish group the notions of a Haar-meager

[1] The notion of a null-finite set is directly equivalent to the notion of a non-shift-compact set from [5].

set and a Haar-null set are equivalent to the notions of a meager set and a set of Haar measure zero, respectively.

In [11] we proved the following generalization of Theorem 1.

Theorem 2. [11, Theorems 2 and 3] *Let X be an abelian metric group with an invariant metric and Y be a locally convex real topological vector space. Assume that $A \subset X$ is a non-null-finite set, K is a subsemigroup of Y, and $F\colon X \to \mathcal{B}(Y)$ is a s.v. map. If F satisfies one of the following conditions:*

(i) *F is K-subadditive and weakly K-upper bounded on A,*
(ii) *F is K-superadditive and K-lower bounded on A,*

then F is locally K-bounded on X.[2]

Here we use the above theorem to show some relationships between a K-subadditivie or K-superadditive s.v. map $F\colon X \to \mathcal{CC}(Y)$ and the functionals $f_{y^*}\colon X \to \mathbb{R}$ defined by

$$f_{y^*}(x) = \inf y^*(F(x)) \quad \text{for } x \in X, \tag{3}$$

where $y^* \in K^*$ and K^* means the set of all real continuous functionals on a real topological vector space Y which are non-negative on K, i.e.

$$K^* = \{y^* \in Y^*\colon y^*(y) \geq 0 \text{ for every } y \in K\}.$$

The results obtained refer to [10, Theorem 5] and [14, Theorem 1], where the continuity of a K-midconvex (K-midconcave) s.v. map was proved under the assumption that the functionals (3) are lower (upper, resp.) semicontinuous. This paper relates also to [1, Theorem 9.1].

2. Main results

Theorem 3. *Let X be an abelian metric group with an invariant metric, and $A \subset X$ be a non-null-finite set. Assume that Y is a locally convex real topological vector space, K is a subsemigroup of Y, and $F\colon X \to \mathcal{B}(Y)$ is a s.v. map. If one of the following conditions holds:*

(i) *F is K-subadditive and weakly K-upper bounded on A,*
(ii) *F is K-superadditive and weakly K-lower bounded on A,*

then for every $y^* \in K^*$ the functional $f_{y^*}\colon X \to \mathbb{R}$ defined by (3) is locally bounded on X.

[2] In fact that theorem was formulated originally with the assumption that Y is an abelian metric group with an invariant metric, but the proof "works" also if Y is a locally convex real topological vector space.

Proof. Assume that F satisfies (i) (if F satisfies (ii), the proof runs in the same way). Then there exists a bounded set $B \subset Y$ such that

$$F(x) \cap (B - K) \neq \emptyset, \quad x \in A. \tag{4}$$

Fix any $y^* \in K^*$ and take the functional f_{y^*} defined by (3). Since F is K-subadditive and $y^* \in K^*$, we have

$$\begin{aligned} y^*(F(x_1)) + y^*(F(x_2)) &= y^*(F(x_1) + F(x_2)) \subset y^*(F(x_1 + x_2) + K) \\ &\subset y^*(F(x_1 + x_2)) + [0, \infty) \end{aligned}$$

for all $x_1, x_2 \in X$. Hence

$$\begin{aligned} f_{y^*}(x_1) + f_{y^*}(x_2) &= \inf y^*(F(x_1)) + \inf y^*(F(x_2)) \\ &= \inf \left(y^*(F(x_1)) + y^*(F(x_2))\right) \geq \inf y^*(F(x_1 + x_2)) \\ &= f_{y^*}(x_1 + x_2), \end{aligned}$$

which means that f_{y^*} is subadditive. By (4), for every $x \in A$ we have

$$y^*(F(x)) \cap y^*(B - K) \neq \emptyset.$$

Hence

$$y^*(F(x)) \cap \big(y^*(B) + (-\infty, 0]\big) \neq \emptyset. \tag{5}$$

But the set $y^*(B)$ is bounded, i.e. $y^*(B) \subset [m, M]$ for some $m < M$. Then, by (5),

$$y^*(F(x)) \cap (-\infty, M] \neq \emptyset,$$

which means that

$$f_{y^*}(x) \leq M, \quad x \in A.$$

Consequently, in view of Theorem 1, f_{y^*} is locally bounded on X. □

Now, which assumptions on the functionals f_{y^*} defined by (3) for $y^* \in K^*$ imply the local K-boundedness on X of a K-subadditive s.v. map F? The next theorem gives an answer.

Let us recall that in a real vector space by a *convex cone* we mean the set K satisfying $K + K \subset K$ and $tK \subset K$ for every $t \in [0, \infty)$.

Theorem 4. *Let X be a complete metric space with an invariant metric, Y be a real normed space, and K be a convex cone in Y. If a s.v. map $F\colon X \to \mathcal{CC}(Y)$ is K-subadditive and for every $y^* \in K^*$ the functional $f_{y^*}\colon X \to \mathbb{R}$ defined by* (3) *is lower semicontinuous on X, then F is locally K-bounded on X.*

Proof. Let B be the unit ball in Y and $B_n := nB$, $n \in \mathbb{N}$. Denote

$$A_n := \{x \in X\colon F(x) \cap \operatorname{cl}(B_n - K) \neq \emptyset\}, \quad n \in \mathbb{N}.$$

Clearly, $\bigcup_{n\in\mathbb{N}} A_n = X$. We will prove that the sets A_n are closed.

If $A_n = X$, it is obvious. So, fix $n \in \mathbb{N}$ such that $A_n \neq X$ and fix $x_0 \in X \backslash A_n$. Then

$$F(x_0) \cap \mathrm{cl}\,(B_n - K) = \emptyset.$$

Since $F(x_0)$ is compact convex and B_n is convex, by the separation theorem (see [14, Lemma 1]) there exists a functional $y^* \in K^*$ such that

$$\inf y^*(F(x_0)) > \sup y^*(B_n).$$

Let

$$\varepsilon := \inf y^*(F(x_0)) - \sup y^*(B_n).$$

Since y^* is continuous at zero, we can find a neighborhood V of zero in Y with $y^*(V) \subset \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. By the lower semicontinuity of f_{y^*} at x_0 there exists a neighborhood U of x_0 such that

$$f_{y^*}(x) > f_{y^*}(x_0) - \frac{\varepsilon}{2} \quad \text{for every } x \in U.$$

Now, we prove that

$$F(x) \cap \mathrm{cl}\,(B_n - K) = \emptyset \quad \text{for every } x \in U. \tag{6}$$

Indeed, if for some $x \in U$ it was not true, there would exist a point

$$z \in F(x) \cap \mathrm{cl}\,(B_n - K) \subset F(x) \cap (B_n - K - V).$$

Let $z = b - k - v$ with some $b \in B_n$, $k \in K$ and $v \in V$. Then

$$\begin{aligned} y^*(b) &= y^*(z) + y^*(k) + y^*(v) > y^*(z) - \frac{\varepsilon}{2} \\ &\geq f_{y^*}(x) - \frac{\varepsilon}{2} > f_{y^*}(x_0) - \varepsilon = \sup y^*(B_n), \end{aligned}$$

a contradiction.

Thus (6) holds and, consequently, $U \subset X \backslash A_n$ which shows that $X \backslash A_n$ is open and hence A_n is closed.

Since X is a complete metric space and $X = \bigcup_{n \in \mathbb{N}} A_n$, by the Baire category theorem there exists $n_0 \in \mathbb{N}$ such that $\mathrm{int}\, A_{n_0} \neq \emptyset$. By the definition of A_{n_0} we have

$$F(x) \cap \big((B_{n_0} + B) - K\big) \neq \emptyset \quad \text{for } x \in A_{n_0}.$$

Since the set $B_{n_0} + B$ is bounded, this means that F is weakly K-upper bounded on the set A_{n_0} with non-empty interior. Consequently, by Theorem 2, F is locally K-bounded on X. This finishes the proof. □

The next example shows that it is not possible to get the same result in Theorem 4 if we weaken the assumption on the functionals f_{y^*}, i.e. replace lower semicontinuity on X of f_{y^*} by local boundedness from below on X.

Example 1. Let $K = [0, \infty)$ and $F\colon \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be defined by

$$F(x) = \big[|a(x)|, |a(x)| + 1\big] \quad \text{for every } x \in \mathbb{R},$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function. Clearly, then F is K-subadditive, even locally K-lower bounded on X, and every $y^* \in K^*$ is given by $y^*(x) = cx$ with some $c \geq 0$. Hence

$$f_{y^*}(x) = \inf y^*\big(F(x)\big) = \inf c\big[|a(x)|, |a(x)| + 1\big] = c|a(x)| \geq 0$$

which means that f_{y^*} is bounded from below on the whole $\mathbb{R}$, but F is not locally weakly K-upper bounded at any point.

Indeed, if for some open set $U \subset \mathbb{R}$ and some $[m, M] \subset \mathbb{R}$ we had $F(x) \cap \big([m, M] - K\big) \neq \emptyset$ for $x \in U$, then $a(x) \leq |a(x)| \leq M$ for $x \in U$, which would be impossible because of the discontinuity of a.

Problem 1. Let X, Y, K and F be as in Theorem 4. Can we obtain the same result replacing the lower semicontinuity of f_{y^*} on X by upper semicontinuity on X for every $y^* \in K^*$, or by local boundedness on X for every $y^* \in K^*$?

We know that the answer to Problem 1 is positive in the case when $Y = \mathbb{R}$ and $K = [0, \infty)$ (even under the weaker assumptions that for some nonzero $y^* \in K^*$ the functional f_{y^*} is upper semicontinuous at a point of X or it is locally bounded at a point of X).

Indeed, let $F(x) = [m(x), M(x)]$ for $x \in X$. Then the K-subadditivity of F means the subadditivity of m. Moreover, $f_{y*}(x) = c\, m(x)$, $x \in X$, with $c > 0$ and the upper semicontinuity (local boundedness) of f_{y*} at some $x_0 \in X$ implies the upper semicontinuity (local boundedness, resp.) of m at x_0, and hence m is locally bounded on X in view of Theorem 1. Thus, for every $x \in X$ we can find a neighborhood U of x such that $m(U) \subset [a, A]$ with some $a < A$. For every $t \in U$ we get

$$\begin{aligned} F(t) \cap ([a, A] - K) &= [m(t), M(t)] \cap (-\infty, A] \neq \emptyset; \\ F(t) &= [m(t), M(t)] \subset [a, \infty) = [a, A] + K, \end{aligned}$$

which means that F is locally K-bounded on X.

Let us recall that the subadditivity of a single-valued real function $f\colon X \to \mathbb{R}$ means the superadditivity of the function $-f$. Unfortunately there is no analogous property for K-subadditive s.v. maps, i.e. the K-superadditivity of F does not mean the $\pm K$-subadditivity of $-F$ (see [11, Examples 2 and 3]). That is why we have to prove an analogous result for K-superadditive s.v. maps independently.

Theorem 5. *Let X be a complete metric space with an invariant metric, Y be a real normed space and K be a convex cone in Y. If a s.v. map $F\colon X \to \mathcal{CC}(Y)$ is K-superadditive and for every $y^* \in K^*$ the functional $f_{y^*}\colon X \to \mathbb{R}$ defined by (3) is upper semicontinuous on X, then F is locally K-bounded on X.*

Proof. Let B be the unit ball in Y and $B_n := nB$ for $n \in \mathbb{N}$. Define

$$A_n := \{x \in X : F(x) \subset \mathrm{cl}\,(B_n + K)\}, \quad n \in \mathbb{N}. \tag{7}$$

Then $\bigcup_{n\in\mathbb{N}} A_n = X$. We will show that $F(x) \subset \mathrm{cl}\,(B_n + K)$ for every $x \in \mathrm{cl}\,A_n$ and $n \in \mathbb{N}$.

For a proof by contradiction suppose that there are $n_0 \in \mathbb{N}$, $x_0 \in \mathrm{cl}\,A_{n_0}$ and $z \in F(x_0)\backslash \mathrm{cl}\,(B_{n_0} + K)$. Since the set $\mathrm{cl}\,(B_{n_0} + K)$ is convex and closed, by the separation theorem (see e.g [15, Theorem 3.4]) there exists a continuous linear functional $y^* \in Y^*$ such that

$$y^*(z) < \inf y^*\big(\mathrm{cl}\,(B_{n_0} + K)\big). \tag{8}$$

Since

$$y^*(k) \geq y^*(z) - y^*(b_0) =: M \quad \text{for all } k \in K$$

with arbitrarily fixed $b_0 \in B_{n_0}$, and, moreover,

$$y^*(k) = \frac{1}{m} y^*(mk) \geq \frac{1}{m} M \quad \text{for every } m \in \mathbb{N},$$

for $m \to \infty$ we obtain $y^*(k) \geq 0$ for all $k \in K$ which means that $y^* \in K^*$. Now, put

$$\varepsilon := \inf y^*\big(\mathrm{cl}\,(B_{n_0} + K)\big) - y^*(z).$$

By the upper semicontinuity of f_{y^*} at x_0 there exists a neighborhood U of x_0 such that

$$f_{y^*}(x) < f_{y^*}(x_0) + \varepsilon \quad \text{for every } x \in U. \tag{9}$$

Since $x_0 \in \mathrm{cl}\,A_{n_0}$, there exists an $x_1 \in A_{n_0} \cap U$. Then, according to (9) and the definition of ε, we obtain

$$\begin{aligned} f_{y^*}(x_1) &< f_{y^*}(x_0) + \varepsilon \leq y^*(z) + \varepsilon = \inf y^*\big(\mathrm{cl}\,(B_{n_0} + K)\big) \\ &\leq \inf y^*\big(F(x_1)\big) = f_{y^*}(x_1). \end{aligned}$$

This contradiction proves that F is K-lower bounded on $\mathrm{cl}\,A_n$ for every $n \in \mathbb{N}$, i.e.

$$F(x) \subset \mathrm{cl}\,(B_n + K) \quad \text{for every } x \in \mathrm{cl}\,A_n.$$

Since $X = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\,A_n$ and X is complete, in view of the Baire category theorem there exists $n_0 \in \mathbb{N}$ such that $\mathrm{int}\,\mathrm{cl}\,A_{n_0} \neq \emptyset$. Moreover,

$$F(x) \subset \mathrm{cl}\,(B_{n_0} + K) \subset B_{n_0} + B + K \quad \text{for every } x \in \mathrm{cl}\,A_{n_0},$$

so F is K-lower bounded on an open set. Now, to complete the proof it is enough to apply Theorem 2. □

Modifying Example 1, it is easy to observe that it is impossible to get the same result in Theorem 5 assuming the local boundedness from above on X (instead of upper semicontinuity on X) of functionals f_{y^*}.

Example 2. Let $K = (-\infty, 0]$ and $F\colon \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be defined by

$$F(x) = \big[|a(x)|, |a(x)| + 1\big] \quad \text{for every } x \in \mathbb{R},$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function. Clearly, then F is K-superadditive, even locally K-upper bounded on X, and every $y^* \in K^*$ is given by $y^*(x) = cx$ with some $c \le 0$. Hence

$$f_{y^*}(x) = \inf y^*\big(F(x)\big) = \inf c\big[|a(x)|, |a(x)| + 1\big] = c\big(|a(x)| + 1\big) \le 0$$

which means that f_{y^*} is bounded from above on the whole $\mathbb{R}$, but F is not locally K-lower bounded at any point.

The following question arises which is analogous to Problem 1.

Problem 2. Let X, Y, K and F be as in Theorem 5. Can we obtain the same result replacing the upper semicontinuity of f_{y^*} on X by lower semicontinuity on X for every $y^* \in K^*$, or by local boundedness on X for every $y^* \in K^*$?

For now we can give a positive answer only in the case $Y = \mathbb{R}$ and $K = [0, \infty)$ (the solution runs in the same way as the solution of Problem 1).

Remark 1. Notice that in the conclusion of Theorems 4 and 5 we are not able to get the local K-upper boundedness (instead of the local weak K-upper boundedness) of F on X.

For example, let $K = [0, \infty)$ and $F\colon \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be given by

$$F(x) = \begin{cases} \left[0, \frac{1}{|x|}\right], & x \neq 0, \\ \{0\}, & x = 0. \end{cases}$$

Clearly, F is K-subadditive and K-superadditive. Moreover, for every $y^* \in K^*$ (i.e. $y^*(x) = cx$ with $c \ge 0$) the functional $f_{y^*} = 0$, so it is continuous on $\mathbb{R}$. But F is not K-upper bounded at 0.

3. Applications and final remarks

One can easily observe that if $F\colon X \to \mathcal{B}(Y)$ is a s.v. map, where X is a metric space and Y is a real normed space, then

(i) the K-upper semicontinuity of F at a point $x_0 \in X$ implies local K-lower boundedness at this point,

(ii) the K-lower semicontinuity of F at a point $x_0 \in X$ implies local weak K-upper boundedness at this point.

Indeed, to obtain (i) it is enough to put the bounded set $B := F(x_0) + W$ with a fixed neighbourhood $W \subset Y$ of 0. The proof of (ii) runs by contradiction. Fix a neighborhood $W \subset Y$ of 0. Then, by the K-lower semicontinuity of F, there is a neighbourhood U of x_0 such that

$$F(x_0) \subset F(x) + W + K \quad \text{for every } \; x \in U.$$

If for the bounded set $B := F(x_0) - W$ we could find $x_1 \in U$ such that $F(x_1) \cap (B - K) = \emptyset$, then

$$(F(x_1) + W + K) \cap F(x_0) = \emptyset.$$

This contradiction proves that F is weakly K-upper bounded at x_0.

Moreover, in view of Examples 1 and 2, a K-subadditive s.v. map which is locally K-lower bounded on X, as well as K-superadditive s.v. map which is locally K-upper bounded on X, needn't be locally K-bounded on X.

However, as an immediate consequence of Theorems 4 and 5, we obtain the following result.

Corollary 6. *Let X be a complete metric space with an invariant metric, Y be a real normed space, and K be a convex cone in Y. If a s.v. map $F\colon X \to \mathcal{CC}(Y)$ satisfies one of the following two conditions:*

(i) *F is K-subadditive and K-upper semicontinuous on X,*
(ii) *F is K-superadditive and K-lower semicontinuous on X,*

then F is locally K-bounded on X.

Proof. Assume that F satisfies (i) (if F satisfies (ii) the proof runs in the same way). Let B be the unit ball in Y. Fix arbitrary $x_0 \in X$, $y^* \in K^*$ and $\varepsilon > 0$. Since F is K-upper semicontinuous at x_0, there exists a neighborhood U of 0 in X such that

$$F(x) \subset F(x_0) + \frac{\varepsilon}{\|y^*\|}B + K \quad \text{for every } x \in x_0 + U.$$

Hence

$$y^*\big(F(x)\big) \subset y^*\big(F(x_0)\big) + \frac{\varepsilon}{\|y^*\|}[-\|y^*\|, \|y^*\|] + [0, \infty), \quad x \in x_0 + U.$$

Consequently,

$$f_{y^*}(x) \geq f_{y^*}(x_0) - \varepsilon, \quad x \in x_0 + U,$$

which shows that f_{y^*} is lower semicontinuous at x_0. To complete the proof it is enough to apply Theorem 4. □

In Theorems 4 and 5 we assume that K is a convex cone in a real normed space, but considerations on K-subadditivity/K-superadditivity seem to be much more natural for a subsemigroup K.

It is easy to check that if K is a subsemigroup with 0, then $\operatorname{conv} K$ is a convex cone. Clearly then

- K-subadditivity implies $\operatorname{conv} K$-subadditivity,
- K-superadditivity implies $\operatorname{conv} K$-superadditivity,
- local K-boundedness at a point implies local $\operatorname{conv} K$-boundedness at the same point,
- $y^* \in K^*$ implies $y^* \in (\operatorname{conv} K)^*$.

Consequently, by Theorems 4 and 5, we get the following corollary.

Corollary 7. *Let X be a complete metric space with an invariant metric, Y be a real normed space, and K be a subsemigroup of Y with 0. Assume that $F\colon X \to \mathcal{CC}(Y)$ is a s.v. map and $f_{y^*}\colon X \to \mathbb{R}$ is the functional defined by (3) for every $y^* \in K^*$. If one of the following conditions holds:*

- (i) *F is K-subadditive and f_{y^*} is lower semicontinuous on X for every $y^* \in K^*$,*
- (ii) *F is K-superadditive and f_{y^*} is upper semicontinuous on X for every $y^* \in K^*$,*

then F is locally conv *K-bounded on X.*

Problem 3. Is it possible to get the stronger result that F is locally K-bounded on X in Corollary 7?

The next corollary gives an answer in the case $Y = \mathbb{R}$.

Corollary 8. *Let X be a complete metric space with an invariant metric, and K be a subsemigroup of $\mathbb{R}$ with 0. Assume that $F\colon X \to \mathcal{CC}(\mathbb{R})$ is a s.v. map and $f_{y^*}\colon X \to \mathbb{R}$ is the functional defined by (3) for every $y^* \in K^*$. If one of the following conditions holds:*

- (i) *F is K-subadditive and f_{y^*} is lower semicontinuous on X for every $y^* \in K^*$,*
- (ii) *F is K-superadditive and f_{y^*} is upper semicontinuous on X for every $y^* \in K^*$,*

then F is locally K-bounded on X.

Proof. Assume that F satisfies (i) (the proof for (ii) is similar). Let $F(x) = [m(x), M(x)]$ for $x \in X$.

The case when $\operatorname{conv} K = \{0\}$ is trivial because then $K = \{0\}$, so it is enough to use Corollary 7.

Now, consider the case when $\operatorname{conv} K = [0, \infty)$. By Corollary 7 F is locally $\operatorname{conv} K$-bounded on X; i.e., for every $x \in X$ there are a neighborhood U of x and intervals $[m_1, M_1], [m_2, M_2]$ with $m_i < M_i$ for $i = 1, 2$ such that for any $t \in U$

$$[m(t), M(t)] \cap (-\infty, M_1] = F(t) \cap ([m_1, M_1] - \operatorname{conv} K) \neq \emptyset,$$
$$[m(t), M(t)] = F(t) \subset [m_2, M_2] + \operatorname{conv} K = [m_2, \infty).$$

Hence $m(U) \subset [m_2, M_1]$. Since $0 \in K$, for $B = [m_2, M_1]$ and every $t \in U$

$$F(t) \cap (B - K) \supset [m(t), M(t)] \cap B = [m(t), M(t)] \cap [m_2, M_1] \neq \emptyset,$$

which means that F is weakly K-upper bounded at $x \in X$. Hence, in view of Theorem 2, F is locally K-bounded on X.

If $\operatorname{conv} K = (-\infty, 0]$, then the proof runs in a similar way (then M is bounded at x).

Finally, let $\operatorname{conv} K = \mathbb{R}$. Then there are $k_1, k_2 \in K$ with $k_1 < 0$ and $k_2 > 0$. Moreover, for every $x \in X$

$$F(x) \cap \big([k_1, k_2] - K\big) = F(x) \cap \mathbb{R} \neq \emptyset,$$
$$F(x) \subset \mathbb{R} = [k_1, k_2] + K,$$

which means that F is locally K-bounded on X. □

Acknowledgements

The research of E. Jabłońska was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks and dean grant within subsidy of Ministry of Science and Higher Education.

Author contributions E.J. and K.N. wrote the main manuscript text and both reviewed the manuscript.

Funding Not applicable.

Availability of data and materials Not applicable.

Declarations
Conflict of interest Not applicable.

Ethical approval Not applicable.

References

[1] Banakh, T., Jabłońska, E.: Null-finite sets in metric groups and their applications. Israel J. Math. **230**, 361–386 (2019)
[2] Bingham, N.H., Jabłońska, E., Jabłoński, W., Ostaszewski, A.J.: On subadditive functions bounded above on a "large" set. Results Math. **75**, 58 (2020)

[3] Bingham, N.H., Ostaszewski, A.J.: Generic subadditive functions. Proc. Am. Math. Soc. **136**, 4257–4266 (2008)
[4] Bingham, N.H., Ostaszewski, A.J.: Automatic continuity: subadditivity, convexity, uniformity. Aequationes Math. **78**, 257–270 (2009)
[5] Bingham, N.H., Ostaszewski, A.J.: Normed groups: dichotomy and duality. Dissertationes Math. **472**, 138p (2010)
[6] Bingham, N.H., Ostaszewski, A.J.: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening. Indag. Math. (N.S.) **29**, 687–713 (2018)
[7] Bingham, N.H., Ostaszewski, A.J.: Variants of the Berz sublinearity theorem. Aequationes Math. **93**, 351–369 (2019)
[8] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. Israel J. Math. **13**, 255–260 (1972)
[9] Darji, U.B.: On Haar meager sets. Topology Appl. **160**, 2396–2400 (2013)
[10] Jabłońska, E., Nikodem, K.: Continuity properties of K-midconvex and K-midconcave set-valued maps. Math. Inequal. Appl. **22**, 1081–1089 (2019)
[11] Jabłońska, E., Nikodem, K.: K-subadditive and K-superadditive set-valued functions bounded on large sets. Aequationes Math. **95**, 1221–1231 (2021)
[12] Kuczma, M.: An introduction to the theory of functional equations and inequalities. In: Cauchy's Equation and Jensen's Inequality. PWN-Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985, 2nd edn. Birkhäuser, Basel (2009)
[13] Nikodem, K.: K-convex and K-concave set-valued functions. Zeszyty Nauk. Politechniki Łódzkiej Mat. **559**; Rozprawy Mat. 114, Łódź (1989)
[14] Nikodem, K.: Continuity properties of midconvex set-valued maps. Aequationes Math. **62**, 175–183 (2001)
[15] Rudin, W.: Functional Analysis. McGraw-Hill Inc. (1973)

Eliza Jabłońska
Faculty of Applied Mathematics
AGH University of Krakow
Mickiewicza 30
30-059 Kraków
Poland
e-mail: elizajab@agh.edu.pl

Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: knikodem@ath.bielsko.pl

Received: December 1, 2022
Revised: November 7, 2023
Accepted: November 9, 2023