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The equation $f(xy) = f(x)h(y) + g(x)f(y)$ and representations on $\mathbb{C}^2$

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Abstract. Let G be a topological group, and let $C(G)$ denote the algebra of continuous, complex valued functions on G. We find the solutions $f, g, h \in C(G)$ of the Levi-Civita equation

$$f(xy) = f(x)h(y) + g(x)f(y),\ x, y \in G,$$

which is an extension of the sine addition law. Representations of G on $\mathbb{C}^2$ play an important role. As a corollary we get the solutions $f, g \in C(G)$ of the sine subtraction law $f(xy^*) = f(x)g(y) - g(x)f(y)$, $x, y \in G$, in which $x \mapsto x^*$ is a continuous involution, meaning that $(xy)^* = y^*x^*$ and $x^{**} = x$ for all $x, y \in G$.

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1. Introduction

Let S be a topological semigroup, $C(S)$ the algebra of continuous, complex valued functions on S. Our paper contributes to the theory of functional equations on groups and semigroups by a study of the Levi-Civita functional equation

$$f(xy) = f(x)h(y) + g(x)f(y) \text{ for } x, y \in S, \tag{1.1}$$

which is an extension of the well known and fundamental sine addition law

$$f(xy) = f(x)g(y) + g(x)f(y),\ x, y \in S. \tag{1.2}$$

The functions $f, g, h \in C(S)$ in (1.1) are the unknowns. We express that (1.1) holds by saying the triple (f, g, h) is a solution of (1.1) or that $f, g, h \in C(S)$ satisfies (1.1).

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We shall mainly consider topological groups G, where (1.1) is

$$f(xy) = f(x)h(y) + g(x)f(y), \; x, y \in G. \tag{1.3}$$

Our main goal is to describe the set of solutions $f, g, h \in C(G)$ of (1.3), and we attain it in Proposition 7.1 and Theorem 7.2 that provide explicit formulas for the solutions. We do not assume that G is abelian or that the solutions have commutativity properties like f being central, i.e., $f(xy) = f(yx)$ for all $x, y \in G$. This complicates of course the considerations.

An apparently unheeded occurrence of the functional Eq. (1.1) is in representation theory. Let $\rho : S \to M(2, \mathbb{C})$ (= complex 2×2 matrices) be a semigroup representation, i.e., satisfy $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in S$. Writing $\rho = \begin{pmatrix} g & f \\ k & h \end{pmatrix}$ where $f, g, h, k : S \to \mathbb{C}$, we find the triple (f, g, h) satisfies (1.1). Thus the map $\rho \mapsto (f, g, h)$ associates to any semigroup representation of S on $\mathbb{C}^2$ a solution of (1.1). Surprisingly a converse (Proposition 6.1) exists, which we shall exploit to solve (1.3).

Particular cases of (1.1) and (1.3) are present in the literature. As already mentioned the sine addition law (1.2) is (1.1) with $g = h$. Ebanks [6] considered the case of $g = 1$ in (1.3), i.e.,

$$f(xy) = f(x)h(y) + f(y), \quad x, y \in G, \tag{1.4}$$

on some semidirect products of groups, while Stetkær [14, Proposition 5.10 and Proposition 5.8(d)] solved it on nilpotent groups. A third particular case of (1.3) is Stetkær [15, Eq. (6)], i.e.,

$$f(xy) = f(x)\chi(y) + \mu(x)f(y), \; x, y \in G, \tag{1.5}$$

where χ and μ are given characters on G, and $f : G \to \mathbb{C}$ is the sole unknown function. Its solutions are known ([15, Theorem 11]), a fact that we shall use in the proof of our main result (Theorem 7.2).

The Eqs. (1.1), (1.2) and (1.3) and generalizations of them have recently been studied for semigroups and groups, but under the assumption that f is central. See for instance Ebanks [7,10].

Apart from the fact that (1.3) extends known functional equations, a motivation for studying it is that it is structurally similar to the sine addition law (1.2), so one might expect their solutions to be similar. This is indeed so on abelian groups, but in general (1.3) can possess non-abelian solutions (also for $f \neq 0$), which (1.2) can not.

Another justification for our study is that from our results for (1.3) we can derive the set of solution (f, g) of the sine subtraction law (8.1) on groups (Theorem 8.3). Ebanks [11, Theorem 3.2] solved (8.1) on semigroups, but only for f central.

Let us recapitulate our results. Theorem 7.2 provides for any topological group G formulas for the continuous, non-degenerate solutions (f, g, h) of (1.3) in terms of continuous irreducible representations of G on $\mathbb{C}^2$, continuous additive functions on G, continuous characters of G and continuous additive functions on the derived group $[G,G]$. The theorem divides the non-degenerate solutions into four disjoint classes. In two of the classes the component function f is central, and in the remaining two it is not. Proposition 7.1 lists the degenerate solutions of (1.3) such that $f \neq 0$.

Corollaries 7.3, 7.4 and Example 7.6 illustrate how Theorem 7.2 simplifies for connected, solvable groups like the $(ax+b)$-group and the Heisenberg group and for the semisimple Lie group $SL(2,\mathbb{R})$.

Theorem 8.3 derives the solutions $f, g \in C(G)$ such that $f \neq 0$ of the sine subtraction law (8.1).

2. Outline of the paper

Most of the terminology and the notation is settled in sections 3 and 4. Section 5 treats a certain transformation property for additive functions on $[G,G]$ which is needed in our description of the non-abelian solutions of (1.3) and (8.1). This finishes the preparations. Section 6 relates solutions of (1.3) to representations of G as alluded to in the introduction. Our main result (Theorem 7.2) is derived in section 7. It characterises the continuous, non-degenerate solutions of (1.3). Section 8 obtains in Theorem 8.3 the set of solutions of the sine subtraction law (8.1) by the help of Theorem 7.2.

3. Terminology, notation and definitions

In this section we collect the terminology, notation and definitions that we need.

A blanket notation that we use throughout the paper is: S **denotes a semigroup**, and G **a group with identity element** e.

Definition 3.1. If X is a set we let $\mathcal{F}(X,\mathbb{C})$ denote the complex algebra of complex valued functions on X. For a topological space X we let $C(X)$ denote the subalgebra of $\mathcal{F}(X,\mathbb{C})$ of the continuous functions in $\mathcal{F}(X,\mathbb{C})$.

Definition 3.2. $[G,G]$ denotes the commutator group of G (aka the derived group), i.e. the subgroup of G generated by $\{[x,y] := xyx^{-1}y^{-1} \mid x, y \in G\}$.

We include the Hausdorff property in the definition of a topological group.

$S^2 := \{x_1x_2 \in S \mid x_1, x_2 \in S\}$.

A function $\varphi : S \to \mathbb{C}$ is said to be additive if $\varphi(x_1x_2) = \varphi(x_1) + \varphi(x_2)$ for all $x_1, x_2 \in S$, multiplicative if $\varphi(x_1x_2) = \varphi(x_1)\varphi(x_2)$ for all $x_1, x_2 \in S$ and central if $\varphi(x_1x_2) = \varphi(x_2x_1)$ for all $x_1, x_2 \in S$.

$\mathbb{C}^*$ is the multiplicative group $(\mathbb{C}\backslash\{0\}, \cdot)$ of non-zero complex numbers endowed with its standard topology.

Definition 3.3. A character χ of G is a homomorphism $\chi : G \to \mathbb{C}^*$.

Definition 3.4. We define an involution of S to be a mapping $x \mapsto x^*$ of S into S such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Some authors prefer the terminology anti-homomorphic involution to highlight the switch of the order of x and y in the formula $(xy)^* = y^*x^*$.

A particular involution is the group inversion $x^* := x^{-1}$, $x \in G$.

Definition 3.5. Let S be endowed with an involution $x \mapsto x^*$. For $F \in \mathcal{F}(S, \mathbb{C})$ we define $F^* \in \mathcal{F}(S, \mathbb{C})$ by $F^*(x) := F(x^*)$, $x \in S$. A function $F \in \mathcal{F}(S, \mathbb{C})$ is said to be odd (with respect to $x \mapsto x^*$) if $F^* = -F$.

If the involution is the inversion of a group we write $\check{F}$ instead of F^*.

Definition 3.6. Let V be a complex vector space of dimension ≥ 1. We let $\mathcal{L}(V)$ denote the algebra of all linear maps of V into itself, $I \in \mathcal{L}(V)$ the identity operator and $GL(V)$ the group

$$GL(V) := \{T \in \mathcal{L}(V) \mid \exists T_1 \in \mathcal{L}(V) \text{ such that } TT_1 = T_1T = I\}.$$

If $V = \mathbb{C}^n$ for some $n \in \{1, 2, \dots\}$ then we interpret the elements of $\mathcal{L}(V)$ as matrices and view $\mathcal{L}(V)$ as $M(n, \mathbb{C})$:= the complex $n \times n$ matrices, and $GL(V)$ as $GL(n, \mathbb{C})$:= the invertible complex $n \times n$ matrices.

Theorem 3.7. *Let V be a finite-dimensional, complex vector space. There is exactly one topology on V that makes it a Hausdorff topological vector space. Any isomorphism of $\mathbb{C}^n$ onto V is a homeomorphism, when V is equipped with this topology.*

Proof. See Rudin [13, Theorem 1.21(a)]. □

We shall always endow finite-dimensional, complex vector spaces with the topology described in Theorem 3.7.

Definition 3.8. Let $V \neq \{0\}$ be a complex vector space.

(a) A semigroup representation of S on V is a map $\pi : S \to \mathcal{L}(V)$ such that $\pi(xy) = \pi(x)\pi(y)$ for all $x, y \in S$. We define $\dim \pi := \dim V$.
(b) Let π be a semigroup representation of S on V. We say that π is irreducible if the only π-invariant subspaces of V are the trivial ones (that is $\{0\}$) and V).
(c) A representation of G on V is a semigroup representation π of G on V such that $\pi(e) = I$. A representation of G on V is the same as a homomorphism of the group G into the group $GL(V)$.
(d) Let S be a topological semigroup and π be a semigroup representation of S on a finite dimensional vector space V. We say that π is continuous, if the map $\pi : S \to \mathcal{L}(V)$ is continuous.

4. Lie's theorem etc.

Lie's theorem is a basic result in the theory of solvable, topological groups. It can be found in Hewitt and Ross [12, Corollary 29.43].

Theorem 4.1. (S. Lie) *Let π be a continuous representation of a connected, solvable, topological group G on a finite-dimensional, complex vector space $V \neq \{0\}$.*

Then there exist a non-zero vector $v_0 \in V$ and a continuous character χ of G such that $\pi(x)v_0 = \chi(x)v_0$ for all $x \in G$.

Remark 4.2. [12] defines solvability of a topological group using closures of commutator subgroups instead of the usual algebraic definition that uses commutator subgroups (see [12, Definition 29.39]). However, the definitions are equivalent as shown in Corollaire 1 in Chapter III, §9 of Bourbaki [5].

If G is abelian, an elementary version of Theorem 4.1 is available.

Proposition 4.3. *Let π be a semigroup representation of an abelian semigroup S on a finite-dimensional, complex vector space $V \neq \{0\}$. Then there exist a non-zero vector $v_0 \in V$ and a multiplicative function χ on S such that $\pi(x)v_0 = \chi(x)v_0$ for all $x \in S$.*

Proof. We skip the proof. □

Proposition 4.4. *Let π be a representation of a group G on $\mathbb{C}^2$. If π is not irreducible, then π is equivalent to a representation of G on $\mathbb{C}^2$ of the form $\begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix}$, where $\chi, \mu \in \mathcal{F}(G, \mathbb{C})$ are characters and where $\varphi \in \mathcal{F}(G, \mathbb{C})$ satisfies*

$$\varphi(xy) = \varphi(x)\chi(y) + \mu(x)\varphi(y) \text{ for all } x, y \in G. \tag{4.1}$$

If furthermore G is a topological group and π is continuous, then $\chi, \mu, \varphi \in C(G)$.

Proof. Since π is not irreducible there exists a non-trivial, π-invariant subspace of $\mathbb{C}^2$. This subspace must be 1-dimensional, since π is 2-dimensional, so it has the form $\mathbb{C}v_1$ for some nonzero vector $v_1 \in \mathbb{C}^2$. Let $v_2 \in \mathbb{C}^2 \setminus \mathbb{C}v_1$. Now π has with respect to the basis $\{v_1, v_2\}$ of $\mathbb{C}^2$ the form $\begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix}$ for some functions $\chi, \mu, \varphi \in \mathcal{F}(G, \mathbb{C})$. It follows that $\widetilde{\pi} := \begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix}$ is a representation of G on $\mathbb{C}^2$, and that it is equivalent to π. That χ and μ are characters of G and that φ satisfies (4.1) follow from $\widetilde{\pi}$ being a representation.

Since $\widetilde{\pi}$ and π equivalent there exists $A \in GL(2, \mathbb{C})$ such that $\begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix} = A\pi A^{-1}$. This formula implies the continuity statements. □

5. On a certain transformation law

The commutator subgroup $[G,G]$ is a normal subgroup of G, and so it is invariant under inner automorphisms of G. The transformation law (5.1) is a condition on functions on $[G,G]$ explicating how we want them to behave under inner automorphisms. It occurs in the literature ([15, Theorem 11(a)]). Our main result (Theorem 7.2) uses it to describe some of the solutions of the functional Eq. (1.3).

In this section we shall derive some properties that additive functions $A : [G,G] \to \mathbb{C}$ satisfying (5.1), have.

Definition 5.1. Let $A : [G,G] \to \mathbb{C}$ be an additive function and χ and μ characters of G. If

$$A(xcx^{-1}) = \frac{\mu(x)}{\chi(x)}A(c) \text{ for all } x \in G \text{ and } c \in [G,G], \tag{5.1}$$

we say A satisfies the transformation law (5.1) with respect to χ and μ.

Let $A : [G,G] \to \mathbb{C}$ be additive. For some groups (5.1) is trivially true, because $A = 0$ on them: If G is abelian so that $[G,G] = \{e\}$, then $A = 0$, being additive. If $[G,G] = G$ we also have $A = 0$: Since A is additive on $[G,G]$, it is additive on $G = [G,G]$, and being additive on G it vanishes on the commutator subgroup $[G,G]$.

Proposition 5.2 shows that the transformation law (5.1) forces A to vanish on nilpotent groups like the Heisenberg group when $\chi \neq \mu$.

Proposition 5.2. *Let G be nilpotent, and let χ and μ be distinct characters of G. If $A : [G,G] \to \mathbb{C}$ is additive and satisfies (5.1), then $A = 0$.*

Proof. We shall show that $A(c) = 0$ for all $c \in [G,G]$. For $y \in G$ we define $\mathrm{Ad}(y) : G \to G$ by $\mathrm{Ad}(y)(x) := [y,x]$ for $x \in G$. Due to the nilpotency there exists $N \in \mathbb{N}$ such that $\mathrm{Ad}(y)^N x = e$ for all $x,y \in G$. Note that $\mathrm{Ad}(y)([G,G]) \subseteq [G,G]$. Choose $y_0 \in G$ such that $\mu(y_0)/\chi(y_0) \neq 1$. We get from the additivity of A and (5.1) for any $c \in [G,G]$ that

$$\begin{aligned} A(\mathrm{Ad}(y_0)c) &= A([y_0,c]) = A(y_0cy_0^{-1}c^{-1}) = A(y_0cy_0^{-1}) + A(c^{-1}) \\ &= \frac{\mu(y_0)}{\chi(y_0)}A(c) - A(c) = \left(\frac{\mu(y_0)}{\chi(y_0)} - 1\right)A(c), \end{aligned}$$

and continuing by induction we find that

$$A(\mathrm{Ad}(y_0)^n c) = \left(\frac{\mu(y_0)}{\chi(y_0)} - 1\right)^n A(c) \quad \text{for } n = 1,2,\ldots$$

Taking $n = N$ and using that $A(e) = 0$ (A is additive) we infer that

$$0 = \left(\frac{\mu(y_0)}{\chi(y_0)} - 1\right)^N A(c),$$

which implies that $A(c) = 0$ as desired. $\square$

For good measure: There are non-zero, continuous, additive functions $A : [G, G] \to \mathbb{C}$ satisfying (5.1). The $(ax + b)$-group harbours several of them (see [15, Section 7.2]).

We return now to general groups G. We discuss in Proposition 5.3 properties of certain functions $\mathcal{A} \in \mathcal{F}([G, G], \mathbb{C})$ that play a prominent role in our formulas for the non-abelian solutions of (1.3) and (8.1) (Theorems 7.2(d) and 8.3(d)). The functions occur in the literature ([15, Theorem 11(a)]).

Proposition 5.3. *Suppose $\mathcal{A} \in \mathcal{F}([G, G], \mathbb{C})$ is additive and satisfies the transformation law (5.1) with distinct characters χ and μ of G.*

Choose $y_0 \in G$ such that $\chi(y_0) \neq \mu(y_0)$, and define $\mathcal{A} \in \mathcal{F}(G, \mathbb{C})$ by $\mathcal{A}(x) := A([y_0, x])\chi(x)$, $x \in G$. Then

(a) $\mathcal{A}(xy) = \mathcal{A}(x)\chi(y) + \mu(x)\mathcal{A}(y)$ for all $x, y \in G$, so $(\mathcal{A}, \mu, \chi)$ is a solution of (1.5) and hence of (1.3).
(b) If $\mathcal{A} \neq 0$, then $\mathcal{A}$ is not central.
(c) If $\mathcal{A} \neq 0$, then $\mathcal{A}$, χ and μ are linearly independent.
(d) $\check{\mathcal{A}} = -\mathcal{A}/(\chi\mu)$.
(e) Let $x \mapsto x^*$ be an involution of G. Then

$$\mathcal{A}^* = -\mathcal{A} \iff A([y_0, x^*])\chi^*(x) = A([y_0, x^{-1}])\mu(x) \text{ for all } x \in G.$$

Proof. (a) Using the formula $[y_0, xy] = [y_0, x]x[y_0, y]x^{-1}$ and (5.1) we find for $x, y \in G$ that

$$\begin{aligned}\mathcal{A}(xy) &= A([y_0, xy])\chi(xy) = A([y_0, x]x[y_0, y]x^{-1})\chi(x)\chi(y) \\ &= (A([y_0, x]) + A(x[y_0, y]x^{-1}))\chi(x)\chi(y) \\ &= \mathcal{A}(x)\chi(y) + \frac{\mu(x)}{\chi(x)}A([y_0, y])\chi(x)\chi(y) = \mathcal{A}(x)\chi(y) + \mu(x)\mathcal{A}(y),\end{aligned}$$

which is (a).

(b) We prove the equivalent statement $\mathcal{A}$ central $\Rightarrow \mathcal{A} = 0$ as follows. If $\mathcal{A}$ is central then $\mathcal{A}(xy_0) = \mathcal{A}(y_0x)$. Using (a) and that $\mathcal{A}(y_0) = A([y_0, y_0])\chi(y_0) = A(e)\chi(y_0) = 0 \cdot \chi(y_0) = 0$ we get $\mathcal{A}(x)\chi(y_0) = \mu(y_0)\mathcal{A}(x)$. However, $\chi(y_0) \neq \mu(y_0)$, so $\mathcal{A}(x) = 0$.

(c) Suppose for contradiction that $\mathcal{A}$, μ and χ are linearly dependent. Since μ and χ are linearly independent as distinct characters (by Artin's theorem [14, Corollary 3.20]), it follows that $\mathcal{A} \in \operatorname{span}\{\chi, \mu\}$. But then $\mathcal{A}$ is central, and so $\mathcal{A} = 0$ by (b).

(d) Put $y = x^{-1}$ in (a).

(e) From (d) we find that

$$-\mathcal{A}(x) = \check{\mathcal{A}}(x)\chi(x)\mu(x) = A([y_0, x^{-1}])\chi(x^{-1})\chi(x)\mu(x) = A([y_0, x^{-1}])\mu(x).$$

Thus $-\mathcal{A} = \mathcal{A}^* \iff A([y_0, x^{-1}])\mu(x) = A([y_0, x^*])\chi(x^*)$, which is (e). $\square$

6. Solutions, representations and non-degeneracy

Proposition 6.1 associates the solutions with $f \neq 0$ of (1.1) to semigroup representations. This works even for degenerate solutions. Proposition 6.4 shows that the non-degenerate solutions of (1.3) are associated to representations and not just to semigroup representations.

Proposition 6.1. *Let S be a topological semigroup, and let $f, g, h \in C(S)$, where $f \neq 0$, satisfy the functional Eq. (1.1) on S. Then there exists exactly one function $k \in \mathcal{F}(S, \mathbb{C})$ such that the matrix valued function*

$$\rho := \begin{pmatrix} g & f \\ k & h \end{pmatrix} \tag{6.1}$$

is a semigroup representation of S on $\mathbb{C}^2$; the function k is

$$k(y) = \frac{1}{f(x_0)}[g(x_0 y) - g(x_0)g(y)] \text{ for } y \in S, \tag{6.2}$$

$$= \frac{1}{f(x_0)}[h(y x_0) - h(x_0)h(y)] \text{ for } y \in S, \tag{6.3}$$

where $x_0 \in S$ is any element such that $f(x_0) \neq 0$. Furthermore k and ρ are continuous.

The statement of Proposition 6.1 that ρ given by (6.1) is a semigroup representation, is a condensed way of expressing that the formulas of the following system (6.4) - (6.7) hold for all $x, y \in S$.

$$g(xy) = g(x)g(y) + f(x)k(y), \tag{6.4}$$

$$f(xy) = g(x)f(y) + f(x)h(y), \tag{6.5}$$

$$k(xy) = k(x)g(y) + h(x)k(y), \tag{6.6}$$

$$h(xy) = k(x)f(y) + h(x)h(y). \tag{6.7}$$

Proof (of Proposition 6.1). Let $f, g, h \in C(S)$ where $f \neq 0$, satisfy (1.1) on S, and fix $x_0 \in S$ such that $f(x_0) \neq 0$.

The uniqueness of k: If ρ is a semigroup representation then (6.4) holds. Taking $x = x_0$ we get that $k(x) = f(x_0)^{-1}[g(x_0 x) - g(x_0)g(x)]$ for $x \in S$. This proves the uniqueness of k.

The proof of [15, Proposition 3] contains the formula

$$f(x)[h(yz) - h(y)h(z)] = [g(xy) - g(x)g(y)]f(z), \ \forall x, y, z \in S, \tag{6.8}$$

(derived as usual from $f(x(yz)) = f((xy)z)$) which implies that there exists $k \in C(S)$ such that for all $x, y \in S$ we have that

$$g(xy) - g(x)g(y) = f(x)k(y) \quad \text{and} \quad h(xy) - h(x)h(y) = k(x)f(y). \tag{6.9}$$

Indeed, to show the existence of $k \in C(X)$ such that (6.9) holds we choose $x_0 \in S$ such that $f(x_0) \neq 0$ and define $k \in C(X)$ by

$$k(y) := \frac{1}{f(x_0)}[g(x_0y) - g(x_0)g(y)] \text{ for } y \in S. \tag{6.10}$$

Multiplying (6.10) by $f(x)$ we get $f(x)k(y) = \frac{1}{f(x_0)}[g(x_0y) - g(x_0)g(y)]f(x)$ for $x, y \in S$. Applying (6.8) to this we find that

$$f(x)k(y) = \frac{1}{f(x_0)}f(x_0)[h(yx) - h(y)h(x)] = h(yx) - h(y)h(x),$$

which is the second identity of (6.9). To obtain the first identity of (6.9) we apply first (6.8) and then the second identity of (6.9) to get that

$$[g(xy) - g(x)g(y)]f(x_0) = f(x)[h(yx_0) - h(y)h(x_0)] = f(x)k(y)f(x_0),$$

which is the first identity of (6.9), since $f(x_0) \neq 0$.

We next derive the identities (6.4)–(6.7). Now (6.5) is the functional Eqs. (1.1), and (6.4) and (6.7) are contained in (6.9). It remains to derive (6.6), which we proceed to do. Using the defining formula (6.10) we find by the help of (6.4) and (6.5) that

$$\begin{aligned}
f(x_0)k(xy) &= g(x_0(xy)) - g(x_0)g(xy) = g((x_0x)y) - g(x_0)g(xy) \\
&= g(x_0x)g(y) + f(x_0x)k(y) - g(x_0)g(x)g(y) - g(x_0)f(x)k(y) \\
&= g(x_0)g(x)g(y) + f(x_0)k(x)g(y) + g(x_0)f(x)k(y) \\
&\quad + f(x_0)h(x)k(y) - g(x_0)g(x)g(y) - g(x_0)f(x)k(y) \\
&= f(x_0)[k(x)g(y) + h(x)k(y)] \text{ for all } x, y \in S.
\end{aligned}$$

Dividing this by $f(x_0)$ we get (6.6).

The formula (6.3) follows from (6.7).

We have already seen that $k \in C(S)$ (by (6.10)). The continuity of ρ is now a triviality, because all its matrix elements are continuous. $\square$

As regards topological groups Proposition 6.4 presents three criteria for ρ from Proposition 6.1 to be a representation and not just a semigroup representation. To formulate the criteria we write down what the notion of non-degeneracy of solutions of general Levi-Civita functional equations becomes in the present situation of the functional Eq. (1.1).

Definition 6.2. Let $f, g, h \in \mathcal{F}(S, \mathbb{C})$ be a solution of (1.1).

The solution is non-degenerate, if both sets $\{f, g\}$ and $\{f, h\}$ are linearly independent in $\mathcal{F}(S, \mathbb{C})$. If they are not, we say (f, g, h) is degenerate.

Some of our results, for example the existence of ρ in Proposition 6.1, have $f \neq 0$ among their hypotheses, so it is relevant to note that $f \neq 0$ for each non-degenerate solution (f, g, h) of (1.1).

Remark 6.3. If $f, g, h \in \mathcal{F}(S, \mathbb{C})$, where $f \neq 0$, is a degenerate solution of (1.1), then the matrix $\rho(x)$ from Proposition 6.1 has determinant 0 for each $x \in S$. In particular ρ is not a representation.

This observation about degenerate solutions should be contrasted with (b) of Proposition 6.4 that discusses non-degenerate solutions. Remark 6.3 is not needed later.

Proof. We prove this only when $g = cf$ for some $c \in \mathbb{C}$, because the possibility of h being proportional to f can be handled similarly. For any $x, y \in S$ we find by the help of (6.4), $g = cf$ and (1.1) that $k = ch$. Indeed,

$$\begin{aligned} &f(x)k(y) = g(xy) - g(x)g(y) = c[f(xy) - cf(x)f(y)] \\ &= c[f(x)h(y) + g(x)f(y) - cf(x)f(y)] = c[f(x)h(y) + (g(x) - cf(x))f(y)] \\ &= c[f(x)h(y) + 0 \cdot f(y)] = cf(x)h(y), \text{ so } k = ch. \end{aligned}$$

Now $\rho(x) = \begin{pmatrix} cf(x) & f(x) \\ ch(x) & h(x) \end{pmatrix}$ gives that $\det \rho(x) = 0$. □

Proposition 6.4 contains three characterisations of non-degeneracy on groups.

Proposition 6.4. *Let the triple $f, g, h \in \mathcal{F}(G, \mathbb{C})$ where $f \neq 0$ satisfy (1.3), and let ρ be the corresponding semigroup representation of G on $\mathbb{C}^2$ from Proposition 6.1. Then the following statements are equivalent.*

- *(a) The solution (f, g, h) of (1.3) is non-degenerate.*
- *(b) ρ is a representation of G on $\mathbb{C}^2$.*
- *(c) $f(e) = 0$.*
- *(d) f is not proportional to a character of G.*

Proof. (a) $\Rightarrow$ (b). Suppose (a) holds. Putting first $x = e$, and then $y = e$ in (1.3), we find from the non-degeneracy that $g(e) = h(e) = 1$ and $f(e) = 0$. Taking $y = e$ in (6.4) we find that $k(e) = 0$. Now $\rho(e) = I$ by (6.1).

(b) $\Rightarrow$ (c). Suppose (b). Clearly $\rho(e) = I$ implies $f(e) = 0$ by (6.1).

(c) $\Rightarrow$ (d) is obvious, since $f \neq 0$.

(d) $\Rightarrow$ (a) Assume (d). Suppose for a contradiction that $g = \alpha f$ for some $\alpha \in \mathbb{C}$. It suffices to consider this case, since similar arguments work if h is assumed proportional to f. From (1.3) we derive that $f(xy) = f(x)\chi(y)$, where $\chi := h + \alpha f$. This implies, since $f \neq 0$, that χ is a character of G. Taking $x = e$ we get $f = f(e)\chi$, which contradicts the assumption (d). □

7. The solutions of (1.3) on groups

Section 7 consists of three subsections. Subsection 7.1 lists the degenerate solutions of (1.3) with $f \neq 0$ (Proposition 7.1). Subsection 7.2 contains preliminary considerations for subsection 7.3 in which we find the continuous, non-degenerate solutions of (1.3). We set them down in Theorem 7.2 which is the main result of our paper. Throughout the section the underlying space is a group G.

7.1. The degenerate solutions of (1.3)

Proposition 7.1. *If the triple $f, g, h \in \mathcal{F}(G, \mathbb{C})$ is a degenerate solution of (1.3) with $f \neq 0$, then there exist constants $c \in \mathbb{C}^*$, $c_1 \in \mathbb{C}$ and a character χ of G such that $f = c\chi$, $g = c_1\chi$ and $h = (1 - c_1)\chi$.*

Conversely, if $c, c_1 \in \mathbb{C}$ are constants and χ is a character of G then the triple $f := c\chi$, $g := c_1\chi$, $h := (1 - c_1)\chi$ is a degenerate solution of (1.3).

Proof. We will only derive the direct statement, since the converse is trivial to verify. So suppose the triple $f, g, h \in \mathcal{F}(G, \mathbb{C})$ is a degenerate solution of (1.3) with $f \neq 0$. We read from Proposition 6.4 that $f = c\chi$ for some $c \in \mathbb{C}^*$ and some character χ of G. Then we get from (1.3) the formula

$$1 = \frac{h}{\chi}(y) + \frac{g}{\chi}(x) \quad \text{for all } x, y \in G,$$

and this yields the desired formulas for g and h. □

7.2. Preliminaries

Let (f, g, h) be a continuous, non-degenerate solution of (1.3) on the topological group G, and $\rho := \begin{pmatrix} g & f \\ k & h \end{pmatrix}$ the corresponding semigroup representation of G on $\mathbb{C}^2$ introduced in Proposition 6.1. ρ is continuous, and by Proposition 6.4 it is a representation of G on $\mathbb{C}^2$.

Case 1. ρ irreducible. By the theory of group representations (see for instance [14, Corollary E.12]) the space of matrix coefficients of ρ is 4-dimensional. In particular f, g and h are linearly independent.

Case 2. ρ not irreducible. We read from Proposition 4.4 that

$$\rho = \begin{pmatrix} g & f \\ k & h \end{pmatrix} = A \begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix} A^{-1} \tag{7.1}$$

where $A \in GL(2, \mathbb{C})$, $\mu, \chi \in C(G))$ are characters, and $\varphi \in C(G))$ satisfies

$$\varphi(xy) = \varphi(x)\chi(y) + \mu(x)\varphi(y) \text{ for all } x, y \in G. \tag{7.2}$$

When we apply the trace function to (7.1) we get the formula

$$g + h = \mu + \chi. \tag{7.3}$$

Using the notation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta := \det A$ we get from (7.1) by elementary computations the formula

$$\begin{pmatrix} g & f \\ k & h \end{pmatrix} = A \begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix} A^{-1} = \frac{1}{\Delta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu & \varphi \\ 0 & \chi \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$= \frac{1}{\Delta} \begin{pmatrix} ad\mu - ac\varphi - bc\chi & -ab\mu + a^2\varphi + ab\chi \\ cd\mu - c^2\varphi - cd\chi & -bc\mu + ac\varphi + ad\chi \end{pmatrix},$$

from which we read that

$$f = \frac{a^2}{\Delta}\varphi + \frac{ab}{\Delta}(\chi - \mu), \quad g = \frac{1}{\Delta}(ad\mu - ac\varphi - bc\chi). \tag{7.4}$$

Furthermore $f \neq 0$, so $a \neq 0$ by (7.4). Combining (7.3) and (7.4) gives

$$g = -\frac{c}{a}f + \mu \quad \text{and} \quad h = \frac{c}{a}f + \chi. \tag{7.5}$$

7.3. The non-degenerate solutions of (1.3)

Subsection 7.3 states and proves the main result of the present paper, Theorem 7.2. The theorem describes the set of continuous, non-degenerate solutions (f, g, h) of (1.3) on topological groups, dividing it into four mutually disjoint classes (a)–(d). The main news about the functional Eq. (1.3) is the emergence of the solutions in (a) and (d). These two classes are void in the classic case of an abelian group, as we shall see from Corollary 7.5. Theorem 7.2 simplifies on other important types of groups. Corollaries 7.3 and 7.4 and Example 7.6 illustrate this.

Theorem 7.2. *Let G be a topological group.*

The continuous, non-degenerate solutions of (1.3) are the following triples $f, g, h \in C(G)$, where $c, c_0 \in \mathbb{C}$ and $c_1 \in \mathbb{C}^*$, and $\chi, \mu \in C(G)$ are characters of G. The classes (a), (b), (c) and (d) are mutually disjoint.

(a) There exists a function $k \in C(G)$ such that the matrix valued function $\begin{pmatrix} g & f \\ k & h \end{pmatrix}$ is a continuous, irreducible representation of G on $\mathbb{C}^2$.

(b) $f = A\chi$, $g = cf + \chi$, $h = -cf + \chi$, where $A \in C(G)$ is a non-zero, additive function on G.

(c) $f = c_1(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$, where $\chi \neq \mu$.

(d) $f = \mathcal{A} + c_0(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$.
Here $\chi \neq \mu$, $y_0 \in G$ is chosen such that $\chi(y_0) \neq \mu(y_0)$, $A \in C([G, G])$ is a non-zero, additive function satisfying the transformation law (5.1), and $\mathcal{A} := A([y_0, \cdot])\chi$.

In (b) and (c) the functions f, g and h are abelian, while the f's in (a) and (d) are not central (and so in particular are not abelian).

In (a) the set $\{f, g, h, k\}$ is linearly independent.
In (d) both $\{\mathcal{A}, \chi, \mu\}$ and $\{f, g, h\}$ are linearly independent.

Proof (of Theorem 7.2). We divide the proof into 4 parts.

Part I: We prove that any continuous, non-degenerate solution (f, g, h) of (1.3) falls into one of the four classes (a)–(d). We let ρ denote the continuous representation of G on $\mathbb{C}^2$ given by (6.1) (Cf. Proposition 6.4).

If ρ is irreducible then (f, g, h) falls in class (a). That aside we may by case 2 of subsection 7.2 suppose that ρ is not irreducible and that it has the form (7.1). There are 3 possibilities for χ, μ and φ which we treat one by one.

The possibility $\chi = \mu$. Here (7.4) gives that $f = a^2\varphi/\Delta$, where φ is a continuous solution of (7.2). Now $\varphi = A\chi$ where $A \in C(G)$ is additive, by the theory for the sine addition law (7.2). Thus $f = a^2 A\chi/\Delta$, where $A \neq 0$ because $f \neq 0$. Incorporating a^2/Δ into A, and taking (7.5) into account we see that the solution (f, g, h) falls into class (b).

The possibility $\chi \neq \mu$ and φ, χ and μ linearly dependent. As distinct characters χ and μ are linearly independent (by Artin's theorem [14, Corollary 3.20]), so φ is a linear combination of them and hence central. From [15, Proposition 5] we read that $\varphi = \alpha(\chi - \mu)$ for some $\alpha \in \mathbb{C}$, and so from (7.4) that $f = c_0(\chi - \mu)$ for some $c_0 \in \mathbb{C}$. Actually $c_0 \in \mathbb{C}^*$, because $f \neq 0$. Together with (7.5) this reveals that the solution (f, g, h) falls into class (c).

The possibility $\chi \neq \mu$ and φ, χ and μ linearly independent. Applying [15, Theorem 11] to (7.2) we get that $\varphi = \mathcal{A} + \alpha(\chi - \mu)$, where $\mathcal{A}$ is as described in (d) and $\alpha \in \mathbb{C}$. We note that $\mathcal{A} \neq 0$, because $\{\varphi, \chi, \mu\}$ is assumed linearly independent. From (7.4) we find that

$$f = \frac{a^2}{\Delta}\varphi + \frac{ab}{\Delta}(\chi - \mu) = \frac{a^2}{\Delta}\mathcal{A} + \left(\frac{a^2}{\Delta}\alpha + \frac{ab}{\Delta}\right)(\chi - \mu),$$

which shows that f has the desired form, so that (f, g, h) falls into class (d).

This finishes the proof of part I.

Part II. We prove the converse of part I, i.e., that any triple (f, g, h) from (a), (b), (c) and (d) is a continuous, non-degenerate solution of (1.3).

The continuity is obvious from the explicit formulas for f, g and h.

Class (a) We get from $\rho(xy) = \rho(x)\rho(y)$ that (f, g, h) is a solution of (1.3). The irreducibility of ρ ensures the space of its matrix coefficients is 4-dimensional ([14, Corollary E.12]), so that $\{f, g, h, k\}$ is a linearly independent set. This implies the non-degeneracy of the solution (f, g, h).

In the remaining classes (b), (c) and (d) the right hand side of the Eq. (1.3) is (in (b) we take $\mu := \chi$):

$$\begin{aligned} f(x)h(y) + g(x)f(y) &= f(x)[-cf + \chi](y) + [cf + \mu](x)f(y) \\ &= f(x)\chi(y) + \mu(x)f(y), \end{aligned}$$

so (1.3) boils down to the Eq. (1.5), i.e., to

$$f(xy) = f(x)\chi(y) + \mu(x)f(y). \tag{7.6}$$

Class (b) We find that (recall that $\mu = \chi$ in (7.6) here in (b))

$$\begin{aligned} f(x)\chi(y) + \chi(x)f(y) &= A(x)\chi(x)\chi(y) + \chi(x)A(y)\chi(y) \\ &= (A(x) + A(y))\chi(x)\chi(y) = A(xy)\chi(xy) = f(xy), \end{aligned}$$

which is (7.6). Since $A \neq 0$ it follows that $f = A\chi \neq 0$. Furthermore $f(e) = A(e)\chi(e) = 0 \cdot 1 = 0$, so Proposition 6.4 gives the non-degeneracy.

Class (c) It is easy to compute that $f = c_1(\chi - \mu)$ satisfies (7.6). Now $f \neq 0$ because $c_1 \in \mathbb{C}^*$ by assumption. Proposition 6.4 gives the non-degeneracy of (f, g, h), since $f(e) = 0$.

Class (d) $\mathcal{A}$ satisfies (7.6) = (1.5) by Proposition 5.3(a). As is easy to check so does $c_0(\chi - \mu)$, so by linearity $f = \mathcal{A} + c_0(\chi - \mu)$ satisfies (7.6).

We show by contradiction that $f \neq 0$. If $f = 0$, then $\mathcal{A} = -c_0(\chi - \mu)$, so $\mathcal{A}$ is central. From Proposition 5.3(b) we see that $\mathcal{A} = 0$, contradicting that $\mathcal{A} \neq 0$ in (d). Furthermore $f(e) = 0$, so Proposition 6.4 gives the non-degeneracy of (f, g, h).

This finishes the proof of part II.

Part III takes care of the proof of the statements after (a)-(d).

About (a). The functions f, g, h and k are linearly independent as a consequence of the irreducibility of ρ (the space of matrix coefficients of ρ is 4-dimensional). We show that the function f in (a) is not central by contradiction. First we note that $h \neq g$, because $\{g, h\}$ is linearly independent. If f were central we would find from (1.3) that $f(x)h(y) + g(x)f(y) = f(y)h(x) + g(y)f(x)$, or equivalently that $f(x)[h(y) - g(y)] = f(y)[h(x) - g(x)]$, so that $f = \alpha(h - g)$ for some $\alpha \in \mathbb{C}$. But then the set $\{f, g, h\}$ is linearly dependent, which is not the case in class (a).

About (b) and (c). The formulas in (b) and (c) show that f, g and h are abelian.

About (d). We show that the function f is not central by contradiction: If it is central, then $\mathcal{A} = f - c_0(\chi - \mu)$ is also central, and so $\mathcal{A}$ vanishes by Proposition 5.3(b), contradicting that $\mathcal{A} \neq 0$ in (d). By Proposition 5.3(c) $\mathcal{A}$, χ and μ are linearly independent. Noting that

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} 1 & c_0 & -c_0 \\ c & c_0 c & 1 - c_0 c \\ -c & 1 - c_0 c & c_0 c \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \chi \\ \mu \end{pmatrix} \tag{7.7}$$

and that the matrix has determinant $-1 \neq 0$ we infer that also f, g and h are linearly independent.

Part IV proves the four classes (a)-(d) are mutually disjoint.

(a) and (b) have no solution (f, g, h) in common, because f is central in (b), but not in (a). Similarly for (a) and (c).

(a) and (d) Let (f, g, h) be a solution which is both in (a) and (d). An assertion in class (a) is that f, g, h and k are linearly independent. We shall arrive at a contradiction by showing that $k \in \operatorname{span}\{f, g, h\}$.

To make the computations more transparent we calculate in $C(G)$ modulo the subspace $\operatorname{span}\{f, g, h\}$. For $F_1, F_2 \in C(G)$ we write $F_1 \equiv F_2$ if and only if $F_1 - F_2 \in \operatorname{span}\{f, g, h\}$. It is clear from (d) that $\chi, \mu \in \operatorname{span}\{f, g, h\}$, so $\chi \equiv 0$ and $\mu \equiv 0$. We determine the function $k \in C(G)$ in (a) by the help of the formulas of (d) and (6.2), in which we have chosen $x_0 \in G$ such that $f(x_0) \neq 0$. We let $y \in G$ be a dummy variable in G.

By (6.2) and $g = cf + \mu$ we see that

$$\begin{aligned} f(x_0)k(y) &= g(x_0y) - g(x_0)g(y) \equiv g(x_0y) = cf(x_0y) + \mu(x_0y) \\ &= cf(x_0y) + \mu(x_0)\mu(y) \equiv cf(x_0y) + 0 = cf(x_0y), \end{aligned}$$

so $k \equiv 0$ if $f(x_0y) \equiv 0$. We see that $f(x_0y) \equiv 0$ by the help of the following computation in which we use Proposition 5.3(a).

$$\begin{aligned} f(x_0y) &= \mathcal{A}(x_0y) + c_0(\chi - \mu)(x_0y) \\ &= \mathcal{A}(x_0)\chi(y) + \mu(x_0)\mathcal{A}(y) + c_0\chi(x_0)\chi(y) - c_0\mu(x_0)\mu(y) \\ &\equiv 0 + \mu(x_0)\mathcal{A}(y) + 0 - 0 = \mu(x_0)\mathcal{A}(y) \\ &= \mu(x_0)[f(y) - c_0(\chi - \mu)(y)] \equiv 0. \end{aligned}$$

(b) and (c). Here $f = A\chi_1 = c_0(\chi - \mu)$ for some characters $\chi_1, \chi, \mu \in C(G)$ and an additive function $A \in C(G) \setminus \{0\}$. It is well known and elementary to verify that $A\chi_1 = 0$ (for details see Ajebbar and Elqorachi [2, Lemma 4.4 on pp. 1122–1123]). But then $A = 0$, contradicting that $A \neq 0$.

(b) and (d). We get a contradiction, because f is central in (b), but not in (d). Similarly for (c) and (d). $\square$

Corollary 7.3. *If G is a connected, solvable, topological group, then the continuous, non-degenerate solutions (f, g, h) of (1.3) are the following, where $c, c_0 \in \mathbb{C}$ and $c_1 \in \mathbb{C}^*$ are constants, and $\chi, \mu \in C(G)$ denote characters of G. The classes (1), (2) and (3) are mutually disjoint.*

(1) $f = A\chi$, $g = cf + \chi$, $h = -cf + \chi$, where $A \in C(G)$ is a non-zero, additive function on G.
(2) $f = c_1(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$, where $\chi \neq \mu$.
(3) $f = \mathcal{A} + c_0(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$.
Here $\chi \neq \mu$ and $\mathcal{A} := A([y_0, \cdot])\chi$, where $A \in C([G, G])\setminus\{0\}$ is an additive function satisfying the transformation law (5.1), and $y_0 \in G$ is chosen such that $\chi(y_0) \neq \mu(y_0)$.

In (1) and (2) the functions f, g and h are abelian, while the f in (3) is not central (and so in particular is not abelian).

In (3) the functions $\mathcal{A}$, χ and μ are linearly independent, as are f, g and h.

Proof. By Lie's theorem (a) of Theorem 7.2 is void. □

Corollary 7.4. *If G is a connected, nilpotent, topological group, then the continuous, non-degenerate solutions (f, g, h) of (1.3) are the following. The classes (1) and (2) are disjoint.*

(1) $f = A\chi$, $g = cf + \chi$, $h = -cf + \chi$, *where $A \in C(G)$ is a non-zero additive function on G, $\chi \in C(G)$ is a character of G, and $c \in \mathbb{C}$ is a constant.*

(2) $f = c_1(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$, *where $\chi, \mu \in C(G)$ are distinct characters of G, and $c_1 \in \mathbb{C}^*$ and $c \in \mathbb{C}$ are constants.*

f, g and h are abelian.

Proof. Apply Proposition 5.2 to Corollary 7.3. □

Corollary 7.5. *If G is an abelian topological group, then the continuous, non-degenerate solutions (f, g, h) of (1.3) are the following. The classes (1) and (2) are disjoint.*

(1) $f = A\chi$, $g = cf + \chi$, $h = -cf + \chi$, *where $A \in C(G)$ is a non-zero additive function on G, $\chi \in C(G)$ is a character of G, and $c \in \mathbb{C}$ is a constant.*

(2) $f = c_1(\chi - \mu)$, $g = cf + \mu$, $h = -cf + \chi$, *where $\chi, \mu \in C(G)$ are distinct characters of G, and $c_1 \in \mathbb{C}^*$ and $c \in \mathbb{C}$ are constants.*

Proof. Apply Proposition 4.3 to Theorem 7.2. □

Example 7.6. Consider Theorem 7.2 for the matrix group $G := SL(2, \mathbb{R})$. It is known that $[G, G] = G$, so that 0 is the only additive function on G, and 1 its only character. It follows that the classes (b)–(d) are void for $G = SL(2, \mathbb{R})$, so only (a) remains. It is also known, although not quite as well, that $\pi(x) := x$, $x \in G$, up to equivalence is the only continuous, irreducible representations of G on $\mathbb{C}^2$. In other words the continuous, irreducible representations of G on $\mathbb{C}^2$ are the representations $\pi_A(x) := AxA^{-1}, x \in G$, parametrised by $A \in GL(2, \mathbb{C})$. This gives (a). We skip the details.

8. A sine subtraction law

In this section we find the continuous solutions of the sine subtraction law (8.1) on topological groups by the help of our main result (Theorem 7.2).

The sine subtraction law which is named after the formula $\sin(x - y) = \sin x \cos y - \cos x \sin y$ where $x, y \in \mathbb{R}$, from elementary trigonometry, has a long history, but new facets of it are still being discovered. Thus Aserrar and Elqorachi [3,4] and Ebanks [8,9,11] have recently studied various extensions of it from $\mathbb{R}$ to semigroups. For the classic case of an abelian group see Aczél and Dhombres [1, pp. 216–217].

We shall for a given continuous involution $x \mapsto x^*$ (see Definition 3.4) of a topological group G solve the sine subtraction law

$$f(xy^*) = f(x)g(y) - g(x)f(y), \; x, y \in G, \tag{8.1}$$

where the pair of functions $f, g \in C(G)$ is the unknown.

Ebanks [11, Theorem 3.2] solved (8.1) on any semigroup, but under the restriction that f was central. When $f \neq 0$ this excludes the non-abelian solutions. We get rid of the restriction. Theorem 8.3 below presents all solutions $f, g \in C(G)$ of (8.1) such that $f \neq 0$. The price we pay is that our underlying space must be a group, not a general semigroup.

Lemma 8.1. *Let G be a group endowed with an involution $x \mapsto x^*$. If the pair $f, g \in \mathcal{F}(G, \mathbb{C})$ satisfies the sine subtraction law (8.1), then f is odd with respect to the involution $x \mapsto x^*$.*

Proof. Putting $y = e$ in (8.1) gives us $f = g(e)f - f(e)g$. Taking $x = e$ in (8.1) we get $f^* = f(e)g - g(e)f = -(g(e)f - f(e)g) = -f$. □

Remark 8.2 is the key to the proof of the principal result of this section (Theorem 8.3).

Remark 8.2. Let $f, g \in C(G)$ with $f \neq 0$. Then (f, g) is a solution of the sine subtraction law (8.1) $\iff$ (f, g, g^*) is a solution of (1.3) and f is odd.

In Theorem 8.3(a) we encounter the adjugate map adj : $M(2, \mathbb{C}) \to M(2, \mathbb{C})$. We recall that it is given by

$$\operatorname{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{C}).$$

Theorem 8.3. *On a topological group G endowed with a continuous involution $x \mapsto x^*$ the solutions $f, g \in C(G)$ of the sine subtraction law (8.1) such that $f \neq 0$ are the following. The classes (a)–(d) are mutually disjoint.*

(a) There exists a function $k \in C(G)$ such that the matrix valued function $\rho := \begin{pmatrix} g & f \\ k & g^* \end{pmatrix}$ is a continuous, irreducible representation of G on $\mathbb{C}^2$ and such that $\rho(x^*) = adj(\rho(x))$ for all $x \in G$.

(b) $f = A\chi$ and $g = cf + \chi$, where $A \in C(G)$ is a non-zero additive function on G such that $A^* = -A$, $\chi \in C(G)$ is a character of G such that $\chi = \chi^*$, and $c \in \mathbb{C}$. (c)thm:9.3b $f = c_0(\chi - \chi^*)$, $g = cf + \chi^*$, where $c_0 \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $\chi \in C(G)$ is a character of G such that $\chi \neq \chi^*$. (d)thm:9.3c $f = \mathcal{A} + c_0(\chi - \chi^*)$, $g = cf + \chi^*$, where

(i) $c_0, c \in \mathbb{C}$ are constants.

(ii) $\chi \in C(G)$ is a character of G such that $\chi \neq \chi^*$, and $y_0 \in G$ is chosen such that $\chi(y_0) \neq \chi^*(y_0)$.

(iii) $\mathcal{A} := A([y_0,\cdot])\chi$*, where* $A \in C([G,G])$ *is a non-zero additive function on* $[G,G]$ *satisfying the transformation law* (5.1) *with* $\mu = \chi^*$ *and that*

$$A([y_0,x^*]) = A([y_0,x^{-1}]) \text{ for all } x \in G. \tag{8.2}$$

Further information:

f *is odd with respect to the involution* $x \mapsto x^*$*.*

In (b) and (c) the functions f *and* g *are abelian, while the* f*'s in (a) and (d) are not central (and so in particular not abelian).*

f *and* g *are linearly independent.*

In (a) the functions f*,* g*,* g^* *and* k *are linearly independent.*

In (d) both sets $\{\mathcal{A},\ \chi,\chi^*\}$ *and* $\{f,g,g^*\}$ *are linearly independent.*

Proof. According to Remark 8.2 we obtain the solutions $f,g \in C(G)$, $f \neq 0$, of (8.1), when we take the solutions of Theorem 7.2 of the form (f,g,g^*), $f \neq 0$, and incorporate that f is odd. Copying the formulas in Theorem 7.2(a)–(d) with h replaced by g^* it remains to incorporate that f is odd.

(a) We choose $x_0 \in G$ such that $f(x_0) \neq 0$. Supposing f odd we get for $y \in G$ from (6.3) and then (6.2) that

$$\begin{aligned} k^*(y) = k(y^*) &= \frac{1}{f(x_0)}[h(y^*x_0) - h(x_0)h(y^*)] \\ &= \frac{1}{f(x_0)}[g(x_0^*y) - g(x_0^*)g(y)] \\ &= \frac{1}{f(x_0)}f(x_0^*)k(y) = \frac{1}{f(x_0)}(-f(x_0))k(y) = -k(y), \end{aligned}$$

which implies that $\rho(x^*) = W(\rho(x))$ for all $x \in G$. Conversely, $\rho(x^*) = W(\rho(x))$ for all $x \in G$ implies that f is odd.

(b) Supposing f odd we get $cf + \chi = g = (g^*)^* = (-cf+\chi)^* = cf + \chi^*$, so that $\chi = \chi^*$. Since $f = A\chi$ we read that A is odd. The converse is also true: If A is odd and $\chi = \chi^*$ then f is odd.

(c) Supposing f odd we get $cf + \mu = g = (g^*)^* = (-cf+\chi)^* = cf + \chi^*$, so that $\mu = \chi^*$. Conversely, if $\mu = \chi^*$ then f is odd.

(d) Supposing f odd we get just like in the previous point that $\mu = \chi^*$. From the formula $f = \mathcal{A} + c_0(\chi - \mu)$ we get that $\mathcal{A}^* = -\mathcal{A}$, which implies the formula (8.2) due to Proposition 5.3(e). The converse is true as well.

The remaining statements come from Theorem 7.2 and Lemma 8.1, except for the linear independence of $\{f,g\}$. But if $\{f,g\}$ is linearly dependent, then the right hand side of (8.1) vanishes, giving the contradiction $f = 0$. $\square$

Theorem 8.3 simplifies a little bit, when the involution is the group inversion $x^* = x^{-1}$: In (a) the condition $\rho(x^*) = \operatorname{adj}(\rho(x))$ for all $x \in G$ can be replaced by $\det(\rho(x)) = 1$ for all $x \in G$ (use the formula $A \operatorname{adj}(A) = (\det A)I$ for

$A \in M(2,\mathbb{C})$), while in (b) and (d) the conditions $A = -A^*$ and (8.2) are automatically satisfied.

The main news about (8.1) compared with the literature is the emergence of the solutions in points (a) and (d) in Theorem 8.3.

Ebanks [11, Example 3.3] presents on the $(ax+b)$-group for $x^* = x^{-1}$ a solution of (8.1) such that f is not central. Thus his assumption about f being central is a real restriction.

Corollary 8.4. *The continuous solutions (f,g) with $f \neq 0$ of the sine subtraction law (8.1) on a connected, nilpotent, topological group G endowed with a continuous involution $x \mapsto x^*$ are the ones described in (b) and (c) of Theorem 8.3. In particular f and g are abelian functions.*

Proof. Combine Theorem 8.3 with Lie's theorem (= Theorem 4.1) and Proposition 5.2. □

Example 8.5. The Heisenberg group H_3 (defined in [14, Example A.17(a)]) is a connected, nilpotent group. Let us as involution take the group inversion, so $x^* := x^{-1}$ for $x \in H_3$.

From Corollary 8.4 we read that the solutions $f, g \in C(H_3)$ with $f \neq 0$ of the sine subtraction law (8.1) on H_3 are

(i) $f = A$, $g = cA + 1$, where $c \in \mathbb{C}$ is a constant and where $A \in C(H_3)$ is an additive function on H_3 such that $A \neq 0$. Such functions A are described in [14, Example 2.11].

(ii) $f = c_0(\chi - \check{\chi})$, $g = cf + \check{\chi}$, where $c_0 \in \mathbb{C}^*$ and $c \in \mathbb{C}$ are constants and where $\chi \in C(H_3)$ is a character of H_3 such that $\chi \neq 1$. Such functions χ are described in [14, Example 3.14].

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