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Aequationes Mathematicae

Functional equations stemming from 'scientific laws'

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Abstract. Functional equations involving certain generalized forms of the so-called 'laws of sciences' are considered. The resolution of these equations is linked to the concept of comparison meaningfulness that appears in measurement theory and dimension theory. The results obtained are stated without assuming any topological requirement.

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1. Introduction

In a foundational paper by Luce on psychophysical laws (see [14]), a functional equation related to the general form of a scientific law was introduced. As argued in [14], the form of a scientific law is greatly restricted by the knowledge of 'admissible transformations' of the dependent and independent variables. Admissible transformations refer to the types of scales used to measure the variables, being independent *interval scales* in the context of Luce's functional equation. In a subsequent article (see [15]), Luce provided a solution of the mentioned equation under certain regularity conditions. Later, in a seminal paper, Aczél, Roberts and Rosenbaum (see [3]) studied twelve functional equations related to 'laws of sciences'. In particular, they reconsidered and resolved Luce's functional equation without assuming any regularity condition. By the way, the solutions turn out to be affine functions of a single real variable.

In the current paper we deal with relevant and meaningful functions used when the (input and output) variables are defined with respect to specific ordinal scales. The topic considered here has been thoroughly investigated since the sixties. As a novelty, we present certain generalizations of Luce's functional equation by considering *ordinal scales*[1] instead of interval scales. The equations

[1]A scale type is defined by means of a class of admissible transformations. In the case of ordinal scales, these transformations are precisely the strictly increasing functions. An interval scale is an affine ordinal scale.

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studied are gradually introduced going from the simple one-dimensional case to the more general one involving aggregation operators. To be more precise, let us denote by Δ the set of all strictly increasing real-valued functions of a single real variable. Then, we begin by investigating the solutions of the following functional equation $(*)$ given by $f(\phi(x)) = T(\phi)(f(x))$, where the independent variables are $x \in \mathbb{R}$ and $\phi \in \Delta$. Here, the unknowns are $f : \mathbb{R} \to \mathbb{R}$ and the operator $T : \Delta \to \Delta$. Equation $(*)$ is a generalization of the so-called *ordinal invariance functional equation*, which is recovered whenever $T(\phi) = \phi$, for all $\phi \in \Delta$, and will be studied in distinct and more general scenarios. One of the main consequences, derived from the resolution of equation $(*)$, concerns the endomorphism classification, with respect to function composition, of the set of increasing affine functions of a single variable.

As will be seen later, the article links the solutions of these functional equations with the concept of *comparison meaningfulness* that arises in *measurement theory* and *dimension theory*.[2] In addition, it should be remarked that our results are stated without assuming any topological condition. Functional equations as the ones considered in this paper have interesting applications in decison sciences; in particular, in social choice theory. In this setting, a core problem consists in aggregating individual utility functions into a social utility, the aggregator often called a *social evaluation functional*, under certain rational requirements. Here, the concept of meaningfulness has great significance because its interpretation in terms of utility measurability and inter/intra-comparability of well-being among individuals (for a thorough discussion of these items see [5–7]).

Comparison meaningful aggregation functions mapping ordinal scales into an ordinal scale have been deeply studied in the literature (e.g., see [10,16–20]). They include important classes of functions such as lattice polynomial functions and, as will be shown below, appear as solutions of certain generalizations of equation $(*)$. As far as we can report, both the general formulation of equation $(*)$ and its resolution in distinct scenarios are novel. Thus, the new findings do not seem to be derived from the papers mentioned above. However, certain results stated in the multi-dimensional context strongly depend upon the results shown in [18] and the references therein. This point will be clarified in the corresponding sections.

2. Basic definitions

Let $n \in \mathbb{N}$ and $N := \{1, \ldots, n\}$. As usual, $\mathbb{R}^n$ will denote the n-dimensional Euclidean space, i.e., $\mathbb{R}^n = \{x = (x_j) : x_j \in \mathbb{R}, \text{for any } j \in N\}$. The set of

[2] See [13] for a thorough treatment of these theories. A detailed and interesting discussion of the meaningfulness concept, along with a proposal for its formal definition, can be seen in [8,9].

permutations on N will be denoted by $S(N)$, and σ will stand for a typical permutation.

Given $x = (x_j)$, $y = (y_j) \in \mathbb{R}^n$, we will write $x \leq y$ whenever $x_j \leq y_j$ for all $j \in N$, and $x \ll y$ whenever $x_j < y_j$ for all $j \in N$.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be:

(i) *increasing* if $x \leq y$ entails $f(x) \leq f(y)$, for any $x, y \in \mathbb{R}^n$,
(ii) *decreasing* if $-f$ is increasing,
(iii) *monotone* whenever it is increasing, or decreasing,
(iv) *strictly increasing* if it is increasing and, in addition, $x \ll y$ entails $f(x) < f(y)$, for any $x, y \in \mathbb{R}^n$,
(v) *strictly decreasing* if $-f$ is strictly increasing,
(vi) *strictly monotone* whenever it is strictly increasing, or strictly decreasing.

From now on $\Omega := \mathbb{R}^{\mathbb{R}}$ will be used to denote the set of all real-valued functions of a single variable. Let there be given $\psi \in \Omega$ and $x = (x_j) \in \mathbb{R}^n$. Then $\psi(x)$ stands for the following vector of $\mathbb{R}^n$, $\psi(x) := (\psi(x_j))$, $j \in N$. In a similar way, if $\Psi = (\psi_j) \in \Omega^n$ is an n-tuple of real-valued functions defined on $\mathbb{R}$, then $\Psi(x) := (\psi_j(x_j)) \in \mathbb{R}^n$.

The sub-domain of Ω which consists of all strictly increasing (respectively, decreasing) functions will be denoted by Δ (respectively, Γ). An important sub-domain of Δ is $\Delta_{ia} := \{\phi \in \Delta : \phi(t) = at + b, a > 0, b \in \mathbb{R}\}$ which includes all increasing affine real-valued functions. Thus, $\Delta_{ia} \subset \Delta \subset \Omega$. The term *operator* (respectively, *aggregation operator*) will be used when referring to a map from Δ (respectively, Δ^n), or a subset of Δ (respectively, Δ^n), into Δ. The notations id and Id will refer to the identity function in Ω, and the identity operator from Δ into Δ, respectively. That is, $\mathrm{id}(x) = x$, for all $x \in \mathbb{R}$, and $\mathrm{Id}(\phi) = \phi$, for all $\phi \in \Delta$. If X is a nonempty set and $w \in \mathbb{R}$, then 1_w will denote the constant function given by $1_w(x) = w$, for all $x \in X$. As usual the symbol "$\circ$"stands for function composition.[3]

Definition 2.2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be:

(i) *order invariant* whenever it satisfies the following functional equation[4] $f(\phi(x)) = \phi(f(x))$, for any $x \in \mathbb{R}^n$, $\phi \in \Delta$,
(ii) *idempotent* provided that $f(x, \ldots, x) = x$, for any $x \in \mathbb{R}$,
(iii) *symmetric* whenever $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$, for any $x = (x_j) \in \mathbb{R}^n$, for any permutation $\sigma \in S(N)$,
(iv) a *projection* if there is $k \in N$ such that $f(x_1, \ldots, x_n) = x_k$, for any $x = (x_j) \in \mathbb{R}^n$.

[3]The notation used in these two paragraphs is not standard. However, we have decided to keep it because it is often employed in social sciences (see, e.g., [5–7]).
[4]We will refer to this equation as *the ordinal invariance functional equation.*

3. The one-dimensional case

In this section, as already said in the Introduction, we will pay attention to the following functional equation:

$$f(\phi(x)) = T(\phi)(f(x)) \qquad (*)$$

where the independent variables are $x \in \mathbb{R}$ and $\phi \in \Delta$, and the unknown function and the unknown operator are $f : \mathbb{R} \to \mathbb{R}$ and $T : \Delta \to \Delta$, respectively.

It should be noted that every function $g \in \Delta$ induces two operators from Δ into Δ; namely, the left operator and the right operator. These operators will be denoted by L_g and R_g, respectively, and are defined in the following way: $L_g(\phi) = g \circ \phi$, and $R_g(\phi) = \phi \circ g$, for any $\phi \in \Delta$. Note that both $(L_g)^{-1}$ and $(R_g)^{-1}$ are well-defined and, obviously, $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. For simplicity, we will write $L_g^{-1} := (L_g)^{-1}$ and $R_g^{-1} := (R_g)^{-1}$.

Theorem 3.1. *The solutions of the functional equation* $(*)$ *are:*

(i) $f = 1_w$, *for some* $w \in \mathbb{R}$, *and then* T *is any operator such that* $T(\phi)(w) = w$, *for any* $\phi \in \Delta$,

(ii) $f \in \Delta$, *and then there are half-closed half-open, or half-open half closed, real intervals* I_i *such that* $T(\phi)(z) = (L_f \circ R_{f^{-1}})(\phi)(z)$, *for any* $z \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} I_i$, *for any* $\phi \in \Delta$,

(iii) $f \in \Gamma$, *and then there are half-closed half-open, or half-open half closed, real intervals* J_i *such that* $T(\phi)(z) = (L_f \circ R_{f^{-1}})(\phi)(z)$, *for any* $z \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} J_i$, *for any* $\phi \in \Delta$.

Proof. If f is a constant function, say $f = 1_w$, for some $w \in \mathbb{R}$, then it is immediate to check that T satisfies the condition given in case (i). Thus, suppose that f is not constant. To prove the other two cases we will use the following property:

(&) Suppose f and T satisfy equation $(*)$ and let $a, b, c \in \mathbb{R}$ such that $a < b < c$. Then:

(1) $f(a) < f(b) \Longrightarrow f(b) < f(c)$,

(2) $f(a) = f(b) \Longrightarrow f(b) = f(c)$,

(3) $f(b) < f(a) \Longrightarrow f(c) < f(b)$, and

(4) $f(b) = f(c) \Longrightarrow f(a) = f(b)$.

In order to show claim (&), assume $f(a) < f(b)$, the remaining cases being entirely analogous. Consider a function, say $\phi \in \Delta$, such that $\phi(a) = b$ and $\phi(b) = c$. Then $f(\phi(a)) = f(b) = T(\phi)(f(a))$, and $f(\phi(b)) = f(c) = T(\phi)(f(b))$. So, because $f(a) < f(b)$ and $T(\phi) \in \Delta$, it follows that $f(b) < f(c)$. Note that, as

a consequence of (&), if $a, b \in \mathbb{R}$ are such that $a < b$, and $f(a) = f(b) = r$, for some $r \in \mathbb{R}$, then $f = 1_r$ (i.e., f is a constant function).
To see now that $f \in \Delta$ or Γ, we argue in the following way. Let $x, y \in \mathbb{R}$, $x < y$, such that $f(x) \neq f(y)$. Assume $f(x) < f(y)$. Then, we show that $f \in \Delta$. To that end, let $u, v \in \mathbb{R}$, $u < v$, be arbitrary real numbers. By analyzing all possibilities of $\{u, v\}$ in relation with $\{x, y\}$, together with claim (&), the conclusion follows. For example, if $u < x < v < y$, then $f(u) < f(x)$ (otherwise, by claim (&), it would follow that $f(y) < f(x)$, leading to contradiction). But then, by claim (&) again, it follows that $f(u) < f(v)$, as desired. The case $f(y) < f(x)$ is similar leading to $f \in \Gamma$.

To finish the proof, it remains to prove the expression of operator T in cases (ii) and (iii). We only consider case (ii), because (iii) is completely similar to (ii). Since $f \in \Delta$, it is well-known that the possible discontinuities of f can only occur on a contable number of points and, in addition, they are jump discontinuities. Thus, the image of f can be written as follows: $f(\mathbb{R}) = \mathbb{R} \setminus \bigcup_{i=1}^{\infty} I_i$, where I_i is a half-closed half-open, or a half-open half closed, real interval. Let now $z \in f(\mathbb{R})$ be an arbitrary point. Then there is a single point $x \in \mathbb{R}$ such that $f(x) = z$. Because f fulfills functional equation $(*)$, it holds that $f(\phi(x)) = T(\phi)(f(x))$, for any $\phi \in \Delta$. Therefore, $T(\phi)(z) = f(\phi(f^{-1}(z))) = (f \circ \phi \circ f^{-1})(z) = (L_f \circ R_{f^{-1}})(\phi)(z)$, for all $\phi \in \Delta$, which completes the proof. $\square$

Remark 3.2. (i) Note that, in the three situations above, operator T depends upon f. The reason for this to happen is that T is an unknown operator in functional equation $(*)$ and the resolution of this equation involves both f and T. In contrast, particular specifications of equation $(*)$ emerge whenever operator T is known. For example, if $T = \mathrm{Id}$, then equation $(*)$ turns out to be the one corresponding to order invariant real-valued functions defined on $\mathbb{R}$ (i.e., $f(\phi(x)) = \phi(f(x)), x \in \mathbb{R},\ \phi \in \Delta$). Looking at the statement of Theorem 3.1 it is clear that the only case in which T is the identity operator corresponds to the situation $f(x) = \mathrm{id}$. In other words, it has been argued that the only solution of the functional equation $f(\phi(x)) = \phi(f(x))$, $(x \in \mathbb{R},\ \phi \in \Delta)$, is $f(x) = \mathrm{id}$.[5]

(ii) Operator T arising in case (i) is somehow undefined. The following two examples allow us to illustrate this situation. First, consider the constant operator defined as $T(\phi) = \mathrm{id}$, for any $\phi \in \Delta$. Clearly, in this case, it holds that $T(\phi)(w) = w$, for all $w \in \mathbb{R}$, for all $\phi \in \Delta$. As a second example, let $w \in \mathbb{R}$ and consider the operator defined by $T(\phi)(x) = \phi(x) + w - \phi(w)$,

[5] An alternative way to prove this observation is as follows. Suppose, by way of contradiction, that there is $a \in \mathbb{R}$ such that $f(a) \neq a$. Assume, without loss of generality, that $a < f(a)$ and consider $\phi \in \Delta$ such that $\phi(a) = a$, and $\phi(f(a)) = f(a) + 1$. Then $f(\phi(a)) = f(a) < f(a)+1 = \phi(f(a))$, which contradicts the fact that f satisfies the ordinal invariance functional equation.

($x \in \mathbb{R}$, $\phi \in \Delta$). Note that $T(\phi)(w) = w$, for any $\phi \in \Delta$. Observe that, in both examples, the function $f = 1_w$ is the only solution of the corresponding functional equation $(*)$.

(iii) Operator T has an interesting property; to wit, it is a *partial endomorphism.* Indeed, in case (i) it holds that $T(\phi_1 \circ \phi_2)(w) = T(\phi_1) \circ T(\phi_2)(w)$, for any $\phi_1, \phi_2 \in \Delta$. In case (ii) (respectively, (iii)), it holds that $T(\phi_1 \circ \phi_2)(z) = T(\phi_1) \circ T(\phi_2)(z)$, for any $z \in \mathbb{R}\backslash \cup_{i=1}^{\infty} I_i$, for any $\phi_1, \phi_2 \in \Delta$ (respectively, $T(\phi_1 \circ \phi_2)(z) = T(\phi_1) \circ T(\phi_2)(z)$, for any $z \in \mathbb{R}\backslash \cup_{i=1}^{\infty} J_i$, for all $\phi_1, \phi_2 \in \Delta$).

Theorem 3.1, together with Remark 3.2(iii), allow for providing a method to generate endomorphisms, with respect to function composition "$\circ$", of Δ. This result is presented in the next corollary whose proof follows directly from Theorem 3.1.

Corollary 3.3. *Let $T : \Delta \to \Delta$ be an endomorphism for which there is a function $f \in \Omega$ such that $f(\phi(x)) = T(\phi)(f(x))$, for any $x \in \mathbb{R}$, $\phi \in \Delta$. Then:*

(i) There is $w \in \mathbb{R}$ such that $T(\phi)(w) = w$, for all $\phi \in \Delta$, and, a fortiori, $f = 1_w$, or

(ii) There is a collection $(I_i)_{i=1}^{\infty}$ of half-closed half-open, or half-open half closed, real intervals such that $T(\phi)(z) = L_f \circ R_{f^{-1}}(\phi)(z)$, for any $z \in \mathbb{R}\backslash \bigcup_{i=1}^{\infty} I_i$, for any $\phi \in \Delta$, and, a fortiori, either $f \in \Delta$ or $f \in \Gamma$.

Remark 3.4. (i) In view of the satement of Corollary 3.3 the most interesting examples of endomorphisms of Δ arise in case (ii) with the entire domain being $\mathbb{R}$ (i.e., the intervals I_i are empty). To illustrate case (ii), let $\alpha \neq 0$ and consider T defined as follows: $T(\phi)(x) = \alpha\phi(x/\alpha)$, $x \in \mathbb{R}$. Then, $T(\phi_1 \circ \phi_2)(x) = \alpha(\phi_1 \circ \phi_2)(x/\alpha) = \alpha\phi_1(\phi_2(x/\alpha)) = \alpha\phi_1(\alpha\phi_2(x/\alpha)/\alpha) = T(\phi_1)(\alpha\phi_2(x/\alpha)) = T(\phi_1)(T(\phi_2)(x)) = T(\phi_1) \circ T(\phi_2)(x)$. Thus, $T(\phi_1 \circ \phi_2) = T(\phi_1) \circ T(\phi_2)$. Note that this endomorphism corresponds to the situation $f(x) = \alpha x$, $x \in \mathbb{R}$.

(ii) We now include an example showing that, in general, the intervals I_i are nonempty (in other words, T is only a partial endomorphism). Let $f \in \Delta$ such that $f(\mathbb{R}) = (0, 1)$. Define the operator T as follows:

$$T(\phi)(z) = \begin{cases} L_f \circ R_{f^{-1}}(\phi)(z) & \text{if } z \in (0,1) \\ z & \text{if } z \in (-\infty, 0] \cup [1, \infty) \end{cases}.$$

It is straightforward to see that T is a well-defined endomorphism of Δ. Note that, in this example, there are two intervals I_i'; namely, the half-open half closed real interval $(-\infty, 0]$ and the half-closed half-open real interval $[1, \infty)$.

To the best of our knowledge, a characterization of the endomorphisms of Δ remains an open problem. Nevertheless, in a significant particular case, as will be seen below, a full characterization can be provided. Indeed, we conclude this section by stating a characterization of the endomorphisms, with respect to function composition, of Δ_{ia}. This result will be obtained by using a functional equation approach in combination with Theorem 3.1.

Corollary 3.5. *Let $T : \Delta_{ia} \to \Delta_{ia}$ be an endomorphism. Then:*

(i) Either there are $w \in \mathbb{R}$, and a multiplicative function $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $T(ax + b) = g(a)x + w(1 - g(a))$, $g \neq id$, for all $a > 0$, $b \in \mathbb{R}$, or

(ii) There are $\alpha, \beta \in \mathbb{R}$ such that $T(ax + b) = ax + \alpha b + \beta(1 - a)$, $x \in \mathbb{R}$, for all $a > 0$, $b \in \mathbb{R}$.

Proof. Let $T : \Delta_{ia} \to \Delta_{ia}$ be an endomorphism, and consider the functional equation $(*)$ with the following specification: $f(\phi(x)) = T(\phi)(f(x))$, $\phi \in \Delta_{ia}$.

Note that, in this setting, functional equation $(*)$ re-writes in the following manner: $f(ax+b) = r(a, b)f(x)+s(a, b)$, where the independent real variables are x, b, and $a > 0$, and the unknown functions are $f : \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{++}$, and $s : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$. The solutions of this equation were obtained by Aczél, Roberts and Rosenbaum in [3] (see also [14]). They are of the form $f(x) = \alpha x + \beta$, $x \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$.

It should be observed that Theorem 3.1 also applies to this functional equation since the arguments used in its proof remain true under the weaker condition of considering only strictly increasing affine functions. Thus, operator T must meet the following three mutually exclusive conditions derived from the statement of Theorem 3.1:

(1) There is $w \in \mathbb{R}$ such that $T(\phi)(w) = w$, for all $\phi \in \Delta_{ia}$, or
(2) There are both a function $f \in \Delta$ and a collection $(I_i)_{i=1}^{\infty}$ of half-closed half-open, or half-open half closed, real intervals such that $T(\phi)(z) = L_f \circ R_{f^{-1}}(\phi)(z)$, for any $z \in \mathbb{R}\setminus \cup_{i=1}^{\infty} I_i$, for any $\phi \in \Delta_{ia}$, or
(3) There are both a function $f \in \Gamma$ and a collection $(J_i)_{i=1}^{\infty}$ of half-closed half-open, or half-open half closed, real intervals such that $T(\phi)(z) = L_f \circ R_{f^{-1}}(\phi)(z)$, for any $z \in \mathbb{R}\setminus \cup_{i=1}^{\infty} J_i$, for any $\phi \in \Delta_{ia}$.

In addition, note that operator T can be written as follows: $T(ax + b) = \varphi(a, b)x + \psi(a, b)$, for any $x, a, b \in \mathbb{R}$, $a > 0$, where $\varphi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{++}$ and $\psi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$. Moreover, because T is an endomorphism with respect to $\circ$, by denoting $\phi_1 = ax+b$, $\phi_2 = cx+d \in \Delta_{ia}$, and imposing that $T(\phi_1 \circ \phi_2) = T(\phi_1) \circ T(\phi_2)$, the following equalities are found: $T(\phi_1 \circ \phi_2) = T(a(cx + d) + b) = T(acx + ad + b) = \varphi(ac, ad + b)x + \psi(ac, ad + b) = T(\phi_1) \circ T(\phi_2) = \varphi(a, b)(\varphi(c, d)x + \psi(c, d)) + \psi(a, b) = \varphi(a, b)\varphi(c, d)x + \varphi(a, b)\psi(c, d) + \psi(a, b)$, for any $x, a, b, c, d \in \mathbb{R}$, $a, c > 0$. Hence, the two conditions $\varphi(ac, ad + b) = \varphi(a, b)\varphi(c, d)$, and $\psi(ac, ad + b) = \varphi(a, b)\psi(c, d) + \psi(a, b)$, hold true for any $x, a, b, c, d \in \mathbb{R}$, $a, c > 0$.

Suppose that case (1) above holds. Note that this situation appears for a constant solution of the functional equation $f(ax + b) = r(a, b)f(x) + s(a, b)$

(i.e., with the notation above, it would be $\alpha = 0$, hence $f = \beta = w$). Now, because, for each $a, b \in \mathbb{R}$, $a > 0$, it holds that $T(ax + b)(w) = w$, it follows that $\psi(a, b) = (1 - \varphi(a, b))w$. Thus, $T(ax + b) = \varphi(a, b)x + (1 - \varphi(a, b))w$, for some $w \in \mathbb{R}$, for any $x, a, b \in \mathbb{R}$, $a > 0$, and note that the second condition above; namely, $\psi(ac, ad + b) = \varphi(a, b)\psi(c, d) + \psi(a, b)$, is a consequence of the first one. Indeed, $\psi(ac, ad+b) = (1-\varphi(ac, ad+b))w = (1-\varphi(a, b)\varphi(c, d))w = (\varphi(a, b)(1 - \varphi(c, d)) + 1 - \varphi(a, b))w = \varphi(a, b)(1 - \varphi(c, d))w + (1 - \varphi(a, b))w = \varphi(a, b)\psi(c, d)+\psi(a, b)$, holds true for any $x, a, b, c, d \in \mathbb{R}$, $a, c > 0$. Therefore, in case (1) the following functional equation arises: $\varphi(ac, ad+b) = \varphi(a, b)\varphi(c, d)$, for any $x, a, b, c, d \in \mathbb{R}$, $a, c > 0$. The solutions of this equation, the proof of which is given in Lemma 6.1 of the "Appendix", are of the form $\varphi(a, b) = g(a)$, $a > 0$, (i.e., they are independent of b), where $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a multiplicative function. So, $T(ax+b) = g(a)x+w(1-g(a))$, for any $x, a, b \in \mathbb{R}$, $a > 0$, which proves (i) of the statement of Corollary 3.5 provided that $g \neq \mathrm{id}$[6].

Now, suppose that case (2) above holds. In this case, by the previous argument, $f(x) = \alpha x + \beta$, $x \in \mathbb{R}$, where $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$. Note that, a fortiori, the intervals I_i are empty since $f(\mathbb{R}) = \mathbb{R}$. Moreover, since $f^{-1}(t) = \dfrac{1}{\alpha}t - \dfrac{\beta}{\alpha}$, with $t = f(x)$, an easy calculation shows that $T(ax+b)(z) = L_f \circ R_{f^{-1}}(ax+b)(z) = az + \alpha b + \beta(1 - a)$, for all $z \in \mathbb{R}$, $a, b \in \mathbb{R}$, $a > 0$. Therefore, $T(ax + b) = ax + \alpha b + \beta(1 - a)$, $x, a, b \in \mathbb{R}$, $a > 0$, where $\alpha > 0$ and $\beta \in \mathbb{R}$. Case (3) leads to a similar conclusion, but now with $\alpha < 0$. Thus, statement (ii) of Corollary 3.5 is reached which ends the proof. $\square$

Remark 3.6. (i) Note that, by Corollary 3.5(i), if $g(a) = 1$, for all $a > 0$, then $T(ax + b) = x$, for all $a > 0$, $b \in \mathbb{R}$. So, $T(\phi) = \mathrm{Id}$, for any $\phi \in \Delta_{ia}$.

(ii) In the proof of Corollary 3.5 the following system of functional equations appears:

$$\begin{cases} \varphi(ac, ad + b) = \varphi(a, b)\varphi(c, d) \\ \psi(ac, ad + b) = \varphi(a, b)\psi(c, d) + \psi(a, b) \end{cases}$$

where $\varphi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{++}$, $\psi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$, $a, c > 0$. Then, as shown above, the solutions of this system are given by:

(1) Either $\varphi(a, b) = g(a)$ and $\psi(a, b) = w(1 - g(a))$, for some $w \in \mathbb{R}$, where $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is multiplicative, $g \neq \mathrm{id}$, or

(2) $\varphi(a, b) = a$ and $\psi(a, b) = \alpha b + \beta(1 - a)$, for some $\alpha, \beta \in \mathbb{R}$.

(iii) Corollary 3.5 can be given the following algebraic interpretation. Let $X := \mathbb{R}_{++} \times \mathbb{R}$ and define the binary operation $*$ on X as follows: $(x, y) * (z, t) = (xz, xt + y)$, where $(x, y), (z, t) \in X$. Then, it is easy to see that $(X, *)$ is a noncommutative group with identity $(1, 0)$. Note that, for any

[6]The case $g = \mathrm{id}$ leads to an endomorphism of the kind $T(ax + b) = ax + w(1 - a)$, for some $w \in \mathbb{R}$, for any $x, a, b \in \mathbb{R}$, $a > 0$. Note that such an endomorphism appears in case (ii) of the statement of Corollary 3.5 provided that $\alpha = 0$ and $\beta = w$.

$(x, y) \in X$, the inverse element is given by $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$. Thus, Corollary 3.5 states that an endomorphism T of $(X, *)$ is of one of the following two kinds:

(1) Either there are $w \in \mathbb{R}$, and a multiplicative function $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $g \neq \mathrm{id}$ such that $T(a, b) = (g(a), w(1 - g(a)))$, for any $(a, b) \in X$, or

(2) There are $\alpha, \beta \in \mathbb{R}$ such that $T(a, b) = (a, \alpha b + \beta(1 - a))$, for all $(a, b) \in X$.

4. The n-dimensional case

In this section we will consider the functional equation $(*)$ in several variables; i.e.,

$$f(\phi(x)) = T(\phi)(f(x)) \qquad (**)$$

where the independent variables are $x \in \mathbb{R}^n$ and $\phi \in \Delta$, and the unknown function and the unknown operator are $f : \mathbb{R}^n \to \mathbb{R}$ and $T : \Delta \to \Delta$, respectively, and $1 \leq n \in \mathbb{N}$.

The approach followed to deal with equation $(**)$ in this more general context is based upon a key property of *measurement theory*; to wit, comparison meaningfulness with respect to ordinal scales. We now define this important concept.

Definition 4.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *comparison meaningful with respect to a single ordinal scale* whenever $f(x) \leq f(y) \Rightarrow f(\phi(x)) \leq f(\phi(y))$, for any $x, y \in \mathbb{R}^n$, $\phi \in \Delta$.

A thorough study on comparison meaningful functions from $\mathbb{R}^n$ into $\mathbb{R}$, and even in a more general domains, was done by Marichal, Mesiar and Rückschlossová in [19], and a state-of-the-art survey on such functions and related ones were given by Marichal and Mesiar in [18] (see also [10,16,19]). The next result, which is a rephrasing of Proposition 5.4 in [18], will be very useful in what follows.

Proposition 4.2. *Any increasing and comparison meaningful with respect to a single ordinal scale function $f : \mathbb{R}^n \to \mathbb{R}$ is of the form $f = g \circ p$, where $g : \mathbb{R} \to \mathbb{R}$ is a constant or a strictly increasing real-valued function, and $p : \mathbb{R}^n \to \mathbb{R}$ is a lattice polynomial function.*

Remark 4.3. Lattice polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ were first studied by Birkhoff in [4]. They are defined inductively in the following manner (see [11]):

(i) For every $k \in N$, the projection on the k-th coordinate is a lattice polynomial function.

(ii) If p and q are lattice polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$, then $p \wedge q$ and $p \vee q$ are also lattice polynomial functions, where $\wedge$ and $\vee$ denote the minimum and the maximum lattice operations on $\mathbb{R}$, respectively.

(iii) Every lattice polynomial function from $\mathbb{R}^n$ to $\mathbb{R}$ is constructed by finitely many applications of the rules (i) and (ii).

As shown in [4], lattice polynomial functions can be written in disjuntive and conjuntive forms as Boolean max-min functions (for details, see [4,16,18]). The notation $\mathcal{L}_n$ will stand for the set of lattice polynomial functions from $\mathbb{R}^n$ into $\mathbb{R}$. It is clear that if $p \in \mathcal{L}_n$, then p is order invariant (see Proposition 4.5 in [18]).

We now establish the increasing solutions of functional equation $(**)$.

Theorem 4.4. *The increasing solutions of functional equation* $(**)$ *are:*

(i) $f = 1_w$, *for some* $w \in \mathbb{R}$, *and then* T *is any operator such that* $T(\phi)(w) = w$, *for all* $\phi \in \Delta$,

(ii) $f = g \circ p$, *where* $g \in \Delta$, *and* $p \in \mathcal{L}_n$, *and then there are half-closed half-open, or half-open half closed, real intervals* I_i *such that* $T(\phi)(z) = (L_g \circ R_{g^{-1}})(\phi)(z)$, *for any* $z \in \mathbb{R} \setminus \cup_{i=1}^{\infty} I_i$, *for any* $\phi \in \Delta$.

Proof. The proof follows from a combination of Proposition 4.2 and the simple fact that if a function f is a solution of equation $(**)$, then it is comparison meaningful with respect to a single ordinal scale. Indeed, let $x, y \in \mathbb{R}^n$, and $\phi \in \Delta$. Assume $f(x) \le f(y)$. Then, since f satisfies equation $(**)$ it follows that $f(\phi(x)) = T(\phi)(f(x)) \le T(\phi)(f(y)) = f(\phi(y))$, the latter inequality being true because $T(\phi) \in \Delta$. Thus, f is comparison meaningful with respect to a single ordinal scale. Now, by Proposition 4.2, either f is constant, or there are a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, and a lattice polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ such that $f = g \circ p$. If f is constant, then case (i) of the statement of Theorem 4.4 is met. Otherwise, let $z \in g \circ p(\mathbb{R}^n)$ be an arbitrary point. Then, there is $x \in \mathbb{R}^n$ such that $p(x) = g^{-1}(z)$. Hence $T(\phi)(z) = g \circ p(\phi(x)) = g \circ \phi(p(x)) = g \circ \phi \circ g^{-1}(z) = (L_g \circ R_{g^{-1}})(\phi)(z)$, for any $\phi \in \Delta$, where the second equality is true since p is order invariant. Finally, by noting that $p(\mathbb{R}^n) = \mathbb{R}$, that g is strictly increasing, and arguing as in the last part of the proof of Theorem 3.1, it follows that $z \in \mathbb{R} \setminus \bigcup_{i=1}^{\infty} I_i$, where I_i are half-closed half-open, or half-open half closed, real intervals. $\square$

Remark 4.5. (i) A similar characterization for the decreasing solutions of $(**)$ can be given simply replacing condition (ii) of the statement of Theorem 4.4 by the following one: $f = g \circ p$, where $g \in \Gamma$, and $p \in \mathcal{L}_n$, and then there are half-closed half-open, or half-open half closed, real intervals J_i such that $T(\phi)(z) = (L_g \circ R_{g^{-1}})(\phi)(z)$, for any $z \in \mathbb{R} \setminus \cup_{i=1}^{\infty} J_i$, for any $\phi \in \Delta$.

(ii) The monotonicity conditions that appear in the statement of Theorem 4.4, and Remark 4.5(i) above, cannot be dropped. Indeed, consider the classical mode function, mode : $\mathbb{R}^n \to \mathbb{R}$, defined by $\text{mode}(x) = \text{argmax}_{r\in\mathbb{R}} \sum_{i=1}^{n} \chi_{\{0\}}(x_i - r)$, where $x = (x_i)$, and χ_E denotes the characteristic function of $E \subseteq \mathbb{R}$ (in case of multiple values for argmax, take the smallest one). Note that the mode function is a solution of equation $(**)$ since it is order invariant. However, it is not of the form as stated in Theorem 4.4, or Remark 4.5(i).

(iii) Theorem 3.1 is a particular case of Theorem 4.4 (together with Remark 4.5(i)) since it can be proved that the only comparison meaningful with respect to a single ordinal scale functions from $\mathbb{R}$ into $\mathbb{R}$ are constant, or strictly increasing, or strictly decreasing.

(iv) Taking advantage of certain results shown in [18], important classes of functions appear as solutions of equation $(**)$ as long as certain qualifications are imposed on them. For example, the increasing, idempotent and symmetric solutions of equation $(**)$ are *the order statistic functions*[7], and then $T = \text{Id}$. So, in this case, equation $(**)$ reduces to the ordinal invariance functional equation.

In the context of several variables it makes sense to consider the following version of functional equation $(**)$:

$$f(\Phi(x)) = T(\Phi)(f(x)) \qquad (***)$$

where the independent variables are $x \in \mathbb{R}^n$ and $\Phi \in \Delta^n$, and the unknowns are $f : \mathbb{R}^n \to \mathbb{R}$ and $T : \Delta^n \to \Delta$, where $1 \leq n \in \mathbb{N}$.

The solutions of equation $(***)$ involve a class of functions tighter than the one which consists only of comparison meaningful with respect to a single ordinal scale functions. So, the following definition, which mimics Definition 4.1 in this new setup, is introduced.

Definition 4.6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *comparison meaningful with respect to independent ordinal scales*[8] provided that $f(x) \leq f(y) \Rightarrow f(\Phi(x)) \leq f(\Phi(y))$, for every $x, y \in \mathbb{R}^n$, $\Phi \in \Delta^n$.

The next result, which generalizes Theorem 1 in [12], can be found in [6] (see also [10, 17, 18]).

[7]For any $k \in N$, the order statistic function $OS_k : \mathbb{R}^n \to \mathbb{R}$ associated with the k-th argument is given by $OS_k(x) = x_{\sigma(k)}$, $x \in \mathbb{R}^n$, where σ, which depends upon $x = (x_j)$, denotes a permutation such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$.

[8]If in Definition 4.6 $\Phi \in \Delta^n_{ia}$, then f is said to be *comparison meaningful with respect to independent interval scales*.

Proposition 4.7. *A comparison meaningful with respect to independent ordinal scales function $f : \mathbb{R}^n \to \mathbb{R}$ is of the form $f = g \circ p$, where $g : \mathbb{R} \to \mathbb{R}$ is a constant or a strictly monotone real-valued function, and $p : \mathbb{R}^n \to \mathbb{R}$ is a projection.*

By using Proposition 4.7 functional equation $(***)$ can be resolved.

Theorem 4.8. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a solution of functional equation $(***)$ then:*

(i) Either $f = 1_w$, for some $w \in \mathbb{R}$, and then T is any aggregation operator such that $T(\Phi)(w) = w$, for all $\Phi \in \Delta^n$, or

(ii) There are $j \in N$ and a strictly monotone function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g(x_j)$, for any $x = (x_i) \in \mathbb{R}^n$. In addition, there are half-closed half-open, or half-open half closed, real intervals I_i in such a way that $T(\Phi)(z) = (L_g \circ R_{g^{-1}})(\phi_j)(z)$, for any $z \in \mathbb{R} \backslash \cup_{i=1}^{\infty} I_i$, for any $\Phi = (\phi_i) \in \Delta^n$.

Proof. The proof follows from Proposition 4.7, from the fact that any solution of equation $(***)$ turns out to be comparison meaningful with respect to independent ordinal scales, and from the arguments used in the proof of Theorem 4.4. $\square$

Remark 4.9. It is an obvious consequence of Theorem 4.8 the fact that the idempotent solutions of equation $(***)$ are the projections.

A further generalization of equation $(***)$ can be studied provided that only partial independence of the components of $\Phi \in \Delta^n$ is demanded.[9] To begin with this case, some notations are still needed. Let $n = n_1 + \cdots + n_l$, where all numbers involved are in $\mathbb{N}$. Then, $N = \{1, \ldots, n\}$ is meant to be partitioned into l subgroups which will be denoted by N_k, each with cardinality n_k, $1 \leq k \leq l$. For each k, let φ_k denote a bijection from $\{1, \ldots, n_k\}$ onto N_k. In this manner, each vector $x \in \mathbb{R}^n$ can be expressed by $x = (x_{N_k})_{k=1}^{l}$, where $x_{N_k} \in \mathbb{R}^{n_k}$, and $x_{N_k i} = x_{\varphi_k(i)}$. This decomposition will be also used below for other n-tuples in an n-fold Cartesian product of a certain set.

Let then be given the following functional equation:

$$f(\Phi(x)) = T(\Phi)(f(x)) \qquad (****)$$

where the independent variables are $x \in \mathbb{R}^n$ and $\Phi \in \mathcal{S}$, and the unknown function and the unknown aggregation operator are $f : \mathbb{R}^n \to \mathbb{R}$ and $R : \mathcal{S} \to \Delta$, respectively. Here, $\mathcal{S} := \{\Phi = (\Phi_{N_k}) \in \Delta^n : \Phi_{N_k} = (\phi_{j_k}) \in \Delta^{n_k}$ and $\phi_{i_k} = \phi_{j_k}$, for all $i_k, j_k \in N_k, 1 \leq k \leq l\}$. Thus, for each $\Phi = (\Phi_{N_k}) \in \mathcal{S}$ all the components of the corresponding Φ_{N_k} are equal though they are allowed to vary from one k to another. Note that, in this way, each $\Phi = (\Phi_{N_k}) \in \mathcal{S}$ can be identified with an l-tuple $(\phi_k) \in \Delta^l$, where $\phi_k = \phi_{i_k}$, for all $i_k \in N_k$. This identification will be recurrently used.

[9]Functional equations involving partial invariance of multi-attributes have been recently studied in [1,2].

The corresponding definition, that mimics those of Definition 4.1 and Definition 4.6 in this context, is now in order.

Definition 4.10. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *comparison meaningful with respect to partially independent ordinal scales* provided that $f(x) \leq f(y) \Rightarrow f(\Phi(x)) \leq f(\Phi(y))$, for any $x, y \in \mathbb{R}^n$, $\Phi \in \mathcal{S}$.

While the resolution of equation $(* * * *)$ remains an open problem in general, we consider an interesting case where it is possible to offer such a solution. It is based on the following result which, at the same time, gives an answer to a question posed by Kim in [12].

Proposition 4.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that is increasing, idempotent and comparison meaningful with respect to partially independent ordinal scales. Then, there are $\bar{k} \in \{1, \ldots, l\}$ and $g \in \mathcal{L}_{n_{\bar{k}}}$ such that $f(x) = g(x_{N_{\bar{k}}})$, for every $x = (x_{N_k}) \in \mathbb{R}^n$. That is, f is a lattice polynomial function which depends only upon a subgroup of variables indexed in $N_{\bar{k}}$, for some $1 \leq \bar{k} \leq l$.*

Proof. By Proposition 4.2, it is sufficient to show that f depends only upon a subgroup of variables indexed in some N_k. In order to do that, consider the following subset of $\mathbb{R}^n$; $A := \{x = (x_{N_k}) \in \mathbb{R}^n : x_{i_k} = x_{j_k},$ for all $i_k, j_k \in N_k, 1 \leq k \leq l\}$. Note that each vector $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$ can be expanded to a vector, denoted by $\hat{x} \in A$, in the following way: $\hat{x} = (\hat{x}_{N_k})$, where $\hat{x}_{i_k} = \hat{x}_{j_k} = x_k$, for all $i_k, j_k \in N_k, 1 \leq k \leq l$. Define then $g : \mathbb{R}^l \to \mathbb{R}$ as $g(x_1, \ldots, x_l) = f(\hat{x})$, for every $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$. It should be observed that the function g is increasing, idempotent and comparison meaningful with respect to independent ordinal scales. Thus, by Proposition 4.7 (see also Remark 4.9), g turns out to be a projection. Therefore, there is $\bar{k}$, $1 \leq \bar{k} \leq l$, such that $g(x_1, \ldots, x_l) = x_{\bar{k}}$, for every $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$. In other words, f, when restricted to A, depends only on a subgroup of variables indexed in some $N_{\bar{k}}$, $1 \leq \bar{k} \leq l$. It remains to show that f depends only on this subgroup of variables on the whole $\mathbb{R}^n$. To that end, let $\bar{x} \in \mathbb{R}^{n_{\bar{k}}}$ be fixed, and consider the following subset of $\mathbb{R}^n$; $B_{\bar{x}} = \{x = (x_{N_k}) \in \mathbb{R}^n : x_{N_{\bar{k}}} = \bar{x},$ and $x_{N_k} = 1_a,$ for all $k \neq \bar{k},$ for some $a \in \mathbb{R}, 1 \leq k \leq l\}$. We now prove that, when restricted to $B_{\bar{x}}$, f is a constant function. Let $x, y \in B_{\bar{x}}$. Then, $x_{N_{\bar{k}}} = y_{N_{\bar{k}}} = \bar{x}$, $x_{N_k} = 1_a$ and $y_{N_k} = 1_b$, for any $k \neq \bar{k}$, for some $a, b \in \mathbb{R}$. Let $r := f(x)$, and denote by $\tilde{r} \in \mathbb{R}^n$ the following vector: $\tilde{r} = (\tilde{r}_{N_k})$, where $\tilde{r}_{N_k} = 1_r$, for all $1 \leq k \leq l$. That is, $\tilde{r} = (r, \ldots, r)$. Then, by idempotency, it holds that $f(x) = f(\tilde{r})$. Consider the following k-tuple of strictly increasing functions $\tilde{\Phi} = (\tilde{\phi}_k) \in \mathcal{S}$, where $\tilde{\phi}_{\bar{k}} = \mathrm{id}$ and $\tilde{\phi}_k(c) = c + (b - a)$, $c \in \mathbb{R}$, for all $k \neq \bar{k}$. Then, since f is comparison meaningful with respect to partially independent ordinal scales, it follows that $f(\tilde{\Phi}(x)) = f(\tilde{\Phi}(\tilde{r}))$. Note that, $\tilde{\Phi}(x) = (\tilde{\phi}_k(x_{N_k})) = y$, because $\tilde{\phi}_{\bar{k}}(x_{N_{\bar{k}}}) = x_{N_{\bar{k}}}$ and $\tilde{\phi}_k(x_{N_k}) = 1_b$, for all $k \neq \bar{k}$. Similarly, $\tilde{\Phi}(\tilde{r}) = (\tilde{\phi}_k(\tilde{r}_{N_k})) \in A$, because $\tilde{\phi}_{\bar{k}}(\tilde{r}_{N_{\bar{k}}}) = \tilde{r}_{N_{\bar{k}}}$ and

$\tilde{\phi}_k(\tilde{r}_{N_k}) = 1_{r+b-a}$, for all $k \neq \bar{k}$. Therefore, $f(y) = f(\tilde{\phi}_k(\tilde{r}_{N_k})) = g(\alpha)$, where $\alpha = (\alpha_k) \in \mathbb{R}^l$ is such that $\alpha_{\bar{k}} = r$ and $\alpha_k = r + b - a$, for all $k \neq \bar{k}$. Now, by the argument above, g is the $\bar{k}$-projection function. Hence, $g(\alpha) = r$, and so $f(x) = r = f(y)$.

To finish the proof, assume, by way of contradiction, that f does not depend on the subgroup of variables indexed in $N_{\bar{k}}$. This means that there are vectors $x = (x_{N_k}), y = (y_{N_k}) \in \mathbb{R}^n$ such that $x_{N_{\bar{k}}} = y_{N_{\bar{k}}}$, and $f(x) < f(y)$. Denote by a, b the following real numbers; $a := \min_{i \neq N_{\bar{k}}} x_i$, $b := \max_{i \neq N_{\bar{k}}} y_i$. Consider the vectors $\dot{x} = (\dot{x}_{N_k})$ and $\dot{y} = (\dot{y}_{N_k})$ defined as follows; $\dot{x}_{N_{\bar{k}}} = \dot{y}_{N_{\bar{k}}} = x_{N_{\bar{k}}} = y_{N_{\bar{k}}}$, and $\dot{x}_{N_k} = 1_a$, $\dot{y}_{N_k} = 1_b$, for all $k \neq \bar{k}$. Note that $\dot{x} \leq x$ and $y \leq \dot{y}$. Further, $\dot{x}, \dot{y} \in B_{x_{N_{\bar{k}}}} = B_{y_{N_{\bar{k}}}}$, and so, by the previous argument, it follows that $f(\dot{x}) = f(\dot{y})$. Then, since f is increasing, it holds that $f(\dot{x}) \leq f(x) < f(y) \leq f(\dot{y})$, which gives the desired contradiction and the proof is complete. □

With the help of Proposition 4.11 the following theorem, stated with no proof, is reached.

Theorem 4.12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an increasing and idempotent solution of functional equation $(* * * *)$. Then, there is $\bar{k} \in \{1, \ldots, l\}$ such that f is a lattice polynomial function which depends only upon the subgroup of variables indexed in $N_{\bar{k}}$. Moreover, $T(\Phi)(z) = \phi_j(z)$, for any $\Phi = (\phi_k) \in \mathcal{S}$, for any $z \in \mathbb{R}$, where $j \in N_{\bar{k}}$.*

Remark 4.13. Note that, for the particular case of increasing and idempotent functions, Theorem 4.12 is a rephrasing of Theorem 4.4 and Theorem 4.8 provided that $l = 1$ and $l = n$, respectively.

5. The general case

The results presented till now can be extended to the aggregation operator setting. Since the most general statement shown in Sect. 4 corresponds to Theorem 4.12, we will provide a version of this result in this new scenario. To that end, we first introduce some notations and definitions that mimic those presented above.

Let X be a nonempty finite set. A typical function, from X to $\mathbb{R}$, will be denoted by v. The set of all real-valued functions from X to $\mathbb{R}$, usually denoted by $\mathbb{R}^X$, will be here denoted by $\mathcal{V}$.

Let $n \in \mathbb{N}$. An *n-tuple* (also called a *profile*) of real-valued functions will be denoted by $V = (v_i)_{i \in N}$ (or simply, by $V = (v_i)$), and $\mathcal{V}^n$ will stand for the set of all possible n-tuples. A profile $V = (v_i) \in \mathcal{V}^n$ can also be viewed as a real-valued map defined on $X \times N$ in the following manner: $(x, i) \in X \times N \longrightarrow V(x, i) = v_i(x) \in \mathbb{R}$. In order to present some basic definitions the following notation will be useful. For a given $x \in X$, and $V = (v_i) \in \mathcal{V}^n$, $V(x)$ will denote the following vector in $\mathbb{R}^n$, $V(x) := (v_i(x))$. If $\Psi = (\psi_i) \in \Omega^n$ and $V = (v_i) \in \mathcal{V}^n$, then $\Psi(V) := (\psi_i \circ v_i) \in \mathcal{V}^n$. Note that, with the notation introduced, it holds that $\Psi(V)(x) = (\psi_i(v_i(x)))$, for every $x \in X$. For a given $v \in \mathcal{V}$, $V_v := (v, \dots, v) \in \mathcal{V}^n$. Similarly, for a given $\psi \in \Omega$, $\Psi_\psi := (\psi, \dots, \psi) \in \Omega^n$.

Let there be given $\Theta \subseteq \Omega$, and $\mathcal{U} \subseteq \mathcal{V}$. Then $\mathcal{U}$ is said to be Θ-*stable* provided that $\theta \circ u \in \mathcal{U}$, for every $u \in \mathcal{U}$, $\theta \in \Theta$. In this section, two basic requirements for $\mathcal{U}$ are demanded. On the one hand, $\mathcal{U}$ must include all constant functions defined on X. On the other hand, $\mathcal{U}$ has to be a Δ-stable set.

Let $\mathcal{U} \subseteq \mathcal{V}$. An *n-dimensional aggregation operator* or, simply, an aggregation operator (or, also, an aggregator) is a map $F : \mathcal{U}^n \to \mathcal{V}$.

Let there be given an aggregator F and $x \in X$. Then the pair(F, x) induces a real-valued function defined on $\mathcal{U}^n$, denoted by F^x, in the following way: $U \in \mathcal{U}^n \to F^x(U) = F(U)(x) \in \mathbb{R}$. Let $m \in \mathbb{N}$ denote the cardinality of X. Then, any profile $U = (u_j) \in \mathcal{U}^n$ can be expressed in a matrix notation where the entry (x, j) corresponds to the value $U(x, j) = u_j(x)$. In addition, for each $x \in X$, the function F^x can be viewed as a function from $(\mathbb{R}^m)^n = \mathbb{R}^{m\times n}$ into $\mathbb{R}$, where, for every $j \in N$, the j-th group of m variables of F^x corresponds to the variables of U indexed in $\{(y, j) : y \in X\}$. In other words, the order of the variables in F^x is determined following the columns of U. Alternatively, the function F^x can be viewed as a function from $(\mathbb{R}^n)^m = \mathbb{R}^{m\times n}$ into $\mathbb{R}$, where, for every $y \in X$, the y-th group of n variables of F^x corresponds to the variables of U indexed in $\{(y, j) : j \in N\}$. Thus, the order of the variables in F^x is now determined following the rows of U. It should be noted that, unless otherwise stated, we will use the first representation.

We still carry on with the notations $n = n_1 + \cdots + n_l \in \mathbb{N}$, $N = (N_k)_{k=1}^l$, for a partition of n into l subgroups (each with cardinality n_k), and $x = (x_{N_k})$, for a vector $x \in \mathbb{R}^n$. Similarly, we will use $U = (U_{N_k})$, for $U \in \mathcal{U}^n$, and $\Phi = (\phi_k)$, for $\Phi \in \mathcal{S}$. In this way, for given $U = (U_{N_k}) \in \mathcal{U}^n$ and $\Phi = (\phi_k) \in \mathcal{S}$, $\Phi(U) = (\phi_k(U_{N_k}))$, where $\phi_k(U_{N_k}) = (\phi_k \circ u_{j_k})_{j_k \in N_k} \in \mathcal{U}^{n_k}$.

The concepts of monotonicity, idempotency and comparison meaningfulness, that will be used in the context of aggregators, are collected in the next definition.

Definition 5.1. An aggregation operator $F : \mathcal{U}^n \to \mathcal{V}$ is said to be *increasing*, or *idempotent*, or *comparison meaningful with respect to partially independent ordinal scales* whenever so is F^x, for any $x \in X$.

We now consider the following generalization of functional equation $(***\,*)$ in the context of aggregators:

$$F(\Phi(U)) = T(\Phi)(F(U)) \qquad (\$)$$

where the independent variables are $U \in \mathcal{U}^n$ and $\Phi \in \mathcal{S}$, and the unknown operators are $F : \mathcal{U}^n \to \mathcal{V}$ and $T : \mathcal{S} \to \Delta$, respectively.

The next result, which is stated with no proof, is a generalization of Proposition 4.11 above.

Proposition 5.2. *For an aggregation operator $F : \mathcal{U}^n \to \mathcal{V}$ the following statements are equivalent:*

(i) F is comparison meaningful with respect to partially independent ordinal scales, increasing and idempotent,

(ii) There are functions $\psi : X \to \{1, \dots, l\}$ and $(g_x)_{x \in X}$ such that:
(a) $g_x \in \mathcal{L}_{m \times n_{\psi(x)}}$, for every $x \in X$,
(b) $F(U)(x) = g_x(U_{N_{\psi(x)}})$, for every $x \in X$, $U \in \mathcal{U}^n$.

The corresponding generalization of Theorem 4.12 is now in order.

Theorem 5.3. *Let $F : \mathcal{U}^n \to \mathcal{V}$ be an increasing and idempotent solution of functional equation (\$). Then, there are $\bar{k} \in \{1, \dots, l\}$ and lattice polynomial functions $g_x \in \mathcal{L}_{m \times n_{\bar{k}}}$ such that $F(U)(x) = g_x(U_{N_{\bar{k}}})$, for any $x \in X$, $U \in \mathcal{U}^n$. Moreover, $T(\Phi)(z) = \phi_j(z)$, for any $\Phi = (\phi_k) \in \mathcal{S}$, for any $z \in \mathbb{R}$, where $j \in N_{\bar{k}}$.*

Proof. We first show that F is comparison meaningful with respect to partially independent ordinal scales. To that end, let there be given $x \in X$, $U, V \in \mathcal{U}^n$ such that $F(U)(x) \leq F(V)(x)$, and $\Phi \in \mathcal{S}$. We have to prove that $F(\Phi(U))(x) \leq F(\Phi(V))(x)$. Now, since F satisfies (\$) and $T(\Phi) \in \Delta$, it holds that $F(\Phi(U))(x) = T(\Phi)(F(U))(x) \leq T(\Phi)(F(V))(x) = F(\Phi(V))(x)$. Thus F is comparison meaningful with respect to partially independent ordinal scales. Therefore, in view of Proposition 5.2, there are functions $\psi : X \to \{1, \dots, l\}$ and $(g_x)_{x \in X}$ such that:
(a) $g_x \in \mathcal{L}_{m \times n_{\psi(x)}}$, for every $x \in X$,
(b) $F(U)(x) = g_x(U_{N_{\psi(x)}})$, for every $x \in X$, $U \in \mathcal{U}^n$.

We now prove that ψ is a constant function. To see this, note that, from the fact $F(\Phi(U)) = T(\Phi)(F(U))$, it follows that $g_x(\phi_{\psi(x)}(U_{N_{\psi(x)}})) = T(\Phi)$ $(g_x(U_{N_{\psi(x)}}))$, for any $x \in X$, $U \in \mathcal{U}^n$ and $\Phi \in \mathcal{S}$. Now, because g_x is order invariant, it follows that $g_x(\phi_{\psi(x)}(U_{N_{\psi(x)}})) = \phi_{\psi(x)}(g_x(U_{N_{\psi(x)}}))$. Thus, $\phi_{\psi(x)}(g_x(U_{N_{\psi(x)}})) = T(\Phi)(g_x(U_{N_{\psi(x)}}))$, for any $x \in X$, $U \in \mathcal{U}^n$ and $\Phi \in \mathcal{S}$. Let now there be given $\hat{x} \in X$ and $\hat{\Phi} = (\hat{\phi_k}) \in \mathcal{S}$. Then, because, obviously, $\{g_{\hat{x}}(U_{N_{\psi(\hat{x})}}) : U \in \mathcal{U}^n\} = \mathbb{R}$, it follows that $T(\hat{\Phi}) = \hat{\phi}_{\psi(\hat{x})}$. So, $T(\Phi) = \phi_{\psi(x)}$, for any $x \in X$, for any $\Phi \in \mathcal{S}$. In addition, note that, by definition, $T(\Phi)$ does not depend on x. This clearly implies that $\psi(x) = \psi(y)$, for any $x, y \in X$. So, ψ is a constant function. Let $\bar{k} = \psi(x)$. Then, it holds that $T(\Phi) = \phi_j$, for any $\Phi = (\phi_k) \in \mathcal{S}$, where $j \in N_{\bar{k}}$. Finally, (b) above says that $F(U)(x) = g_x(U_{N_{\bar{k}}})$, for every $x \in X$, $U \in \mathcal{U}^n$, which completes the proof. □

Proposition 5.2 can be refined provided that the following strengthening of idempotency is required. This assumption will provide a significant reduction in the number of variables which the functions g_x depend upon.

Definition 5.4. An aggregation operator $F : \mathcal{U}^n \to \mathcal{V}$ is said to be *strong idempotent* whenever $F(U_u) = u$, for all $u \in \mathcal{U}$.

Proposition 5.5. *For an aggregation operator $F : \mathcal{U}^n \to \mathcal{V}$ the following statements are equivalent:*

(i) F is comparison meaningful with respect to partially independent ordinal scales, increasing and strong idempotent,
(ii) There are functions $\psi : X \to \{1, \dots, l\}$ and $(h_x)_{x \in X}$ such that:
(a) $h_x \in \mathcal{L}_{n_{\psi(x)}}$, for every $x \in X$,
(b) $F(U)(x) = h_x(U(x)_{N_{\psi(x)}})$, for every $x \in X$, $U \in \mathcal{U}^n$, where the notation $U(x)_{N_{\psi(x)}}$ means $U(x)_{N_{\psi(x)}} := (U(x,j))_{j \in N_{\psi(x)}}) \in \mathbb{R}^{n_{\psi(x)}}$.

Proof. We only prove (i) implies (ii) because the other implication is routine. In order to make things as easy as possible, we introduce the subscript notation for X. That is, $X = \{x_1, \dots, x_m\}$. By Proposition 5.2, there are functions $\psi : X \to \{1, \dots, l\}$ and $(g_{x_i})_{x_i \in X}$ such that:
(a) $g_{x_i} \in \mathcal{L}_{m \times n_{\psi(x_i)}}$, for every $x_i \in X$,
(b) $F(U)(x_i) = g_{x_i}(U_{N_{\psi(x_i)}})$, for every $x_i \in X$, $U \in \mathcal{U}^n$.

Let $x_p \in X$ be fixed. By using the alternative row-expression of g_{x_p}, it is possible to write $g_{x_p}(U_{N_{\psi(x_p)}}) = h_{x_p}(U(x_i)_{N_{\psi(x_p)}})$, for every $x_i \in X$, $U \in \mathcal{U}^n$. Take now $u \in \mathcal{U}$, and consider the following profile $V = U_u \in \mathcal{U}^n$. Because F is strong idempotent, it follows that $F(V) = u$. In particular, $F(V)(x_p) = u(x_p)$. Now, $F(V)(x_p) = h_{x_p}(\tilde{u}(x_1), \dots, \tilde{u}(x_i), \dots, \tilde{u}(x_m))$, where $\tilde{u}(x_i) = (u(x_i), \dots, u(x_i)) \in \mathbb{R}^{n_{\psi(x_p)}}$. Note that the latter equality holds true for all $u \in \mathcal{U}$. Moreover, since $h_{x_p}(\tilde{u}(x_1), \dots, \tilde{u}(x_i), \dots, \tilde{u}(x_m)) \in u(x_r)$, for some $r \in \{1, \dots, m\}$, it holds that h_{x_p} only depends upon the p-th subgroup of variables[10]. Thus, $h_{x_p}(U(x_i)_{N_{\psi(x_p)}}) = h_{x_p}(U(x_p)_{N_{\psi(x_p)}})$ holds true for any $x_p \in X$, $U \in \mathcal{U}^n$. Therefore, $h_{x_p} \in \mathcal{L}_{n_{\psi(x_p)}}$, and the proof is complete. □

If the condition $l = n$ is added to the statement of Proposition 5.5, then the conclusion becomes even sharper.

Corollary 5.6. *Assume $l = n$. For an aggregation operator $F : \mathcal{U}^n \to \mathcal{V}$ the following statements are equivalent:*

(i) F is comparison meaningful with respect to partially independent ordinal scales, increasing and strong idempotent,
(ii) There is a function $\psi : X \to \{1, \dots, n\}$ such that $F(U)(x) = U(x, \psi(x))$, for every $x \in X$, $U \in \mathcal{U}^n$.

Proof. Indeed, if $l = n$, then $n_1 = \cdots = n_l = 1$. Therefore, $n_{\psi(x)} = 1$, and so $N_{\psi(x)}$ is a singleton, for any $x \in X$, where ψ is the function provided in the statement of Proposition 5.5. Moreover, by Proposition 5.5 again, $h_x \in \mathcal{L}_1$

[10]It is a well-known fact that if $h \in \mathcal{L}_n$, then $h(a) \in \{a_1, \dots, a_n\}$, for all $a = (a_j) \in \mathbb{R}^n$ (see, e.g., [16]).

and $F(U)(x) = h_x(U(x)_{N_{\psi(x)}})$, for every $x \in X$, $U \in \mathcal{U}^n$. Note that the only lattice polynomial function of a single variable is the identity. So, $h_x = \mathrm{id}$, and therefore $F(U)(x) = U(x)_{N_{\psi(x)}} = U(x, \psi(x))$, for every $x \in X$, $U \in \mathcal{U}^n$. □

Remark 5.7. Strong idempotency is an essential assumption for Corollary 5.6 to be true. Indeed, assume $X = \{x, y\}$, $N = \{1, 2\}$, $F : \mathcal{V}^2 \to \mathcal{V}$, and $V = (v_1, v_2) \in \mathcal{V}^2$. Thus, here, $\mathcal{U} = \mathcal{V}$. Consider the aggregator F defined as follows: $F(V)(x) = v_1(y)$ and $F(V)(y) = v_2(x)$. Clearly, F, so-defined, is increasing, idempotent and comparison meaningful with respect to partially independent ordinal scales. However, it is not of the form as stated in Corollary 5.6. Note that it fails to be strong idempotent.

As a direct consequence of Theorem 5.3, Proposition 5.5 and Corollary 5.6, the following result is reached.

Theorem 5.8. *Assume $l = n$. Then, the following statements are equivalent:*

(i) F is an increasing and strong idempotent solution of functional equation (\$),

(ii) There is $j \in N$ such that $F(U)(x) = U(x, j)$, for every $x \in X$, $U \in \mathcal{U}^n$. Moreover, $T(\Phi)(z) = \phi_j(z)$, for any $\Phi = (\phi_i) \in \Delta^n$, for any $z \in \mathbb{R}$.

We now include some examples showing that the monotonicity property and the strong idempotency assumption for the aggregation operator F cannot be ruled out from the statement of Theorem 5.8.

Examples 5.9. Consider the same environment as in Remark 5.7. That is, $X = \{x, y\}$, $N = \{1, 2\}$, $F : \mathcal{V}^2 \to \mathcal{V}$ and $V = (v_1, v_2) \in \mathcal{V}^2$.
(1) Define $F(V)(x) = \max\{v_1(x), v_1(y)\}$, and $F(V)(y) = \min\{v_1(x), v_1(y)\}$. Note that F is increasing and satisfies (\$) with $T(\Phi) = \phi_1$, for any $\Phi = (\phi_1, \phi_2) \in \Delta^2$. However, it is not a projection. Obviously, F fails to be strong idempotent.
(2) Define $F(V) = v_1$, provided that ($v_1(x) > v_1(y)$ and $v_2(x) > v_2(y)$) or ($v_1(x) < v_1(y)$ and $v_2(x) < v_2(y)$), and $F(V)(x) = v_1(y)$, $F(V)(y) = v_1(x)$, otherwise. Obviously, F, so-defined, is not a projection. It is easy to see that F is strong idempotent and also that it fulfils equation (\$) (with $T(\Phi) = \phi_1$, for any $\Phi = (\phi_1, \phi_2) \in \Delta^2$). However F fails to be increasing. Indeed, let $V = (v_1, v_2), U = (u_1, u_2) \in \mathcal{V}^2$ be two profiles defined as follows: $v_1(x) = 1$, $v_1(y) = 0$, $v_2(x) = 3$, $v_2(y) = 2$, $u_1(x) = 4$, $u_1(y) = 0$, $u_2(x) = 4$, $u_2(y) = 6$. Clearly, $v_j(z) \le u_j(z)$, for every $j \in N$, $z \in X$. However, $F(V)(x) = 1 > F(U)(x) = 0$.

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6. Appendix

Lemma 6.1. *Let* φ *be a solution of the following functional equation:*

$$f(ac, ad+b) = f(a,b)f(c,d)$$

where $f : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{++},\ a, b, c, d \in \mathbb{R}, a, c > 0.$
Then, $\varphi(a,b) = g(a)$*, where* $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *is multiplicative. In particular,* φ *is independent of* b*.*

Proof. By taking $a = 1$ and $d = 0$, the equation writes as $\varphi(c,b) = \varphi(1,b)\varphi(c,0)$, for any $b, c \in \mathbb{R}$, $c > 0$. By denoting $p(c) = \varphi(c,0)$ and $h(b) = \varphi(1,b)$, it follows that $\varphi(c,b) = p(c)h(b)$, for any $b, c \in \mathbb{R}$, $c > 0$, hence φ is multiplicatively separable (i.e., it is the product of two functions of a single variable). Note that, since φ never vanishes, nor do p and h. In addition, the signs of p and h agree because φ is strictly positive.

Consider now the following replacements in the functional equation. Firstly, let $a = 1$ and $b = 0$. Then, $\varphi(c,d) = \varphi(1,0)\varphi(c,d)$, for any $c, d \in \mathbb{R}$, $c > 0$. Since φ does not vanish, it holds that $\varphi(1,0) = p(1)h(0) = 1$.

Secondly, let $b = d = 0$. Then, $p(ac)h(0) = p(a)h(0)p(c)h(0)$, for any $a, c > 0$. Thus, since h does not vanish, it holds that $p(ac) = h(0)p(a)p(c)$, for any $a, c > 0$.

Thirdly, let $a = 1$ and replace y by $d + b$. Then, it holds that $\varphi(c, y) = \varphi(1, b)\varphi(c, y - b)$, for any $c, b, y \in \mathbb{R}$, $c > 0$. Thus, $p(c)h(y) = \varphi(c, y) = \varphi(1, b)\varphi(c, y - b) = p(1)h(b)p(c)h(y - b)$, for any $c, b, y \in \mathbb{R}$, $c > 0$. Therefore, since p does not vanish, it follows that $h(y) = p(1)h(b)h(y - b)$ and so $h(y - b) = \dfrac{1}{p(1)}\dfrac{h(y)}{h(b)} = h(0)\dfrac{h(y)}{h(b)}$, for any $b, y \in \mathbb{R}$, since h does not vanish either.

Finally, consider the functional equation written in the following form $p(ac)h(ad + b) = p(a)h(b)p(c)h(d)$, for all $a, b, c, d \in \mathbb{R}$, $a, c > 0$. By taking into account the identities obtained above, namely, $p(ac) = h(0)p(a)p(c)$, $h(ad + b) = h(ad - (-b)) = h(0)\dfrac{h(ad)}{h(-b)}$ and $p(1)h(0) = 1$, it follows that $h(ad) = \dfrac{1}{h(0)^2}h(-b)h(b)h(d)$, for all $a, b, d \in \mathbb{R}$, $a > 0$. Hence, by making $b = 0$, it holds that $h(ad) = h(d)$, for any $a, d \in \mathbb{R}$, $a > 0$. Now, by letting $d = 1$ in the latter equality it follows that $h(a) = h(1)$, for all $a > 0$. Similarly, for $d = -1$, it follows that $h(-a) = h(-1)$, for all $a > 0$. Now, $h(-2b) = h(-b - b) = h(0)\dfrac{h(-b)}{h(b)}$. But, since $h(b) = h(1)$, if $b > 0$, and $h(b) = h(-1)$, if $b < 0$, it turns out that $h(-2b) = h(-b)$, holds true for any $b \in \mathbb{R}$. Thus, $h(-2b) = h(0)\dfrac{h(-b)}{h(b)} = h(-b)$, for all $b \in \mathbb{R}$, hence $h(b) = h(0)$, for all $b \in \mathbb{R}$ and, therefore, h is a constant function.

To finish the proof, note that $\varphi(a, b) = p(a)h(b) = p(a)h(0)$, for any $a, b \in \mathbb{R}$, $a > 0$, where $p : \mathbb{R}_{++} \to \mathbb{R}\backslash\{0\}$ is a real-valued function satisfying $p(ac) = h(0)p(a)p(c)$, for any $a, c > 0$. Let $w = h(0)$ and $g = wp$. In addition, observe that g is strictly positive because the signs of p and h agree. Moreover, it holds that $g(ac) = (wp)(ac) = wp(ac) = wwp(a)p(c) = wp(a)wp(c) = (wp)(a)(wp)(c) = g(a)g(c)$, for all $a, c > 0$. Thus, g is multiplicative and, therefore, $\varphi(a, b) = wp(a) = g(a)$, for any $a, b \in \mathbb{R}$, $a > 0$, which completes the proof. $\square$

7. Conclusions and related literature

In the current article certain generalized versions of a classical functional equation that appears in the context of 'scientific laws' is presented. The functional equation we refer to was first formulated by Luce, in [14], in the setup of psychophysical laws, and then studied in detail by Aczél, Roberts and Rosenbaum in [3]. The main significance of the present approach is twofold. On the one hand, the resolution of the functional equations considered here is linked to the fulfilment of an important property in measurement theory and dimension theory; to wit, comparison meaningfulness. On the other hand, and as a by-product, some results concerning endomorphisms, with respect to function composition, of certain subsets of strictly increasing functions of a single variable are established. It should be noted that no topological requirements (such as continuity) are used to establish our results.

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