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Aequationes Mathematicae

Remarks on a linearization of Koopmans recursion

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Dedicated to Professors Maciej Sablik and László Székelyhidi on their 70^{th} *birthday.*

Abstract. Let X be a metric space and $U : X^{\infty} \to \mathbb{R}$ be a continuous function satisfying the Koopmans recursion $U(x_0, x_1, x_2, \ldots) = \varphi(x_0, U(x_1, x_2, \ldots))$, where $\varphi : X \times I \to I$ is a continuous function and I is an interval. Denote by $\succeq$ a preference relation defined on the product X^{∞} represented by a function $U : X^{\infty} \to \mathbb{R}$, called a utility function, that means $(x_0, x_1, \ldots) \succeq (y_0, y_1, \ldots) \Leftrightarrow U(x_0, x_1, \ldots) \geq U(y_0, y_1, \ldots)$. We consider a problem when the preference relation $\succeq$ can be represented by another utility function V satisfying the affine recursion $V(x_0, x_1, x_2, \ldots) = \alpha(x_0)V(x_1, x_2, \ldots) + \beta(x_0)$. Under suitable assumptions on relation $\succeq$ we determine the form of the functions φ defining the utility functions possessing the above property. The problem is reduced to solving a system of simultaneous functional equations. The subject is strictly connected to a problem of preference in economics. In this note we extend the results obtained in Zdun (Aequ Math 94, 2020).

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1. Introduction

This note is closely related to the study of a problem of preference in economics. We present a method of solving some economic problems applying functional equations.

Let X be a metric space. Let $U : X^{\infty} \to \mathbb{R}$ be a continuous, non constant function such that the range I is a non-trivial interval. Define on X^{∞} the following relation:

$$(x_0, x_1, \ldots) \succeq (y_0, y_1, \ldots) \Leftrightarrow U(x_0, x_1, \ldots) \geq U(y_0, y_1, \ldots).$$

The mapping U is said to be a **utility function** and "$\succeq$" a **preference relation** represented by U. This relation is transitive, reflexive and connected. In preference theory the space X is treated as a set of consumption products. The sequence $(x_0, x_1, x_2, \ldots) \in X^\infty$ describes a consumption program over time. The preference relation "$\succeq$" describes how an individual consumer would rank all consumption programs.

In the considered subject economists assume that the preference relation is **stationary** (see e.g. [4]), that is

$$\forall x \in X \ \forall a, b \in X^\infty \quad a \succeq b \Longleftrightarrow (x, a) \succeq (x, b).$$

If the relation $\succeq$ represented by the utility function U is stationary, then there exists a unique function $\varphi : X \times I \to I$ weakly increasing with respect to the second variable such that

$$U(x_0, x_1, x_2, \ldots) = \varphi(x_0, U(x_1, x_2, \ldots)), \tag{1}$$

where $I = U[X^\infty]$.

This statement has a simple explanation. Let $\succeq$ be stationary. Note that if $U(a) = U(b)$ for some $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$, then $U(x, a) = U(x, b)$ for every $x \in X$. This is obvious since $U(a) \geq U(b)$ and $U(b) \geq U(a)$ imply that $U(x, a) \geq U(x, b)$ and $U(x, b) \geq U(x, a)$ thus $U(x, a) = U(x, b)$.

Let $x \in X$ and $t \in I$. By the surjectivity of U there exists an $a \in X^\infty$ such that $t = U(a)$. Define $\varphi(x, t) := U(x, a)$. This definition is correct since the value $\varphi(x, t)$ does not depend on the choice of an element a. Directly by this definition we get that $\varphi(x, U(a)) = U(x, a)$ for all $a \in X^\infty$ and $x \in X$.

Let $s < t$. There exist $a, b \in X^\infty$ such that $s = U(a)$ and $t = U(b)$. We have $b \succeq a$ and $(x, b) \succeq (x, a)$ i.e. $U(x, a) \leq U(x, b)$. Since $U(x, a) = \varphi(x, U(a)) = \varphi(x, s)$ and $U(x, b) = \varphi(x, U(b)) = \varphi(x, t)$ we have $\varphi(x, s) \leq \varphi(x, t)$.

To prove the uniqueness assume that (1) holds also with a function $\psi : X \times I \to I$. Let $x \in X$ and $t \in I$. By the surjectivity of U, there exists $(x_1, x_2, \ldots) \in X^\infty$ such that $t = U(x_1, x_2, \ldots)$. Then we have $\psi(x, t) = \psi(x, U(x_1, x_2, \ldots)) = U(x, x_1, x_2, \ldots) = \varphi(x, U(x_1, x_2, \ldots)) = \varphi(x, t)$. Thus $\psi = \varphi$.

We also have the reverse statement. If φ is strictly increasing with respect to its second variable then relation $\succeq$ is stationary. In fact, if $U(a) \leq U(b)$ then $U(x, a) = \varphi(x, U(a)) \leq \varphi(x, U(b)) = U(x, b)$. Conversely, if $U(x, a) \leq U(x, b)$ then $\varphi(x, U(a)) \leq \varphi(x, U(b))$, so $U(a) \leq U(b)$.

The recursion (1) was introduced in paper [5] by Koopmans T.C., Diamond P.A. and Willson R.E.. They gave there a system of axioms on the preference relation which is equivalent to the fact that $\varphi : X \times \mathrm{I} \to \mathrm{I}$ is a continuous function strictly increasing in its second variable.

Definition 1. The recursion (1) is called **Koopmans recursion**. Moreover, the function φ in (1) is said to be an **aggregator** of U.

Further we will consider the aggregator φ in the form of one parameter family of continuous strictly increasing functions $\{f_x : I \to I, x \in X\}$, where

$$f_x(t) := \varphi(x,t).$$

Moreover, the recursion (1) will be written in a shorten form

$$U(x,a) = f_x(U(a)),\ x \in X,\ a \in X^\infty. \tag{2}$$

Example 1. Let $L \in \mathbb{N}$, $L \geq 2$ and $X := \{0,..,L-1\}$. Define the function

$$U(x_0,x_1,x_2,...) := \sum_{i=0}^{\infty} \frac{x_i}{L^{i+1}},\quad (x_0,x_1,\dots) \in X^\infty.$$

The expansion of every $t \in [0,1]$ on base L has the form $t = \sum_{i=0}^{\infty} \frac{x_i}{L^{i+1}}$ for some $x_i \in X$, $i \in \mathbb{N}$. Hence U is a utility function with range $[0,1]$.

Put $f_i(t) = \frac{t}{L} + \frac{i}{L}$, where $t \in [0,1]$ and $i \in \{0,\dots,L-1\}$. It is easy to see that

$$f_i(U(x_1,x_2,...)) = U(i,x_1,x_2,...),\ i \in X,\ (x_1,x_2,\dots) \in X^\infty,$$

which means that U satisfies the Koopmans recursion (1) with aggregator $\varphi(t,x) := \frac{t}{L} + \frac{x}{L}$, $t \in [0,1]$, $x \in X$.

A basic role in the study of problems of preference is played by the property of "impatience" introduced by Koopmans in [4].

Definition 2. We will say that the preference relation "$\succeq$" satisfies **impatience** if for all $n \geq 1$, $\hat{a},\hat{b} \in X^n$ and all $x \in X^\infty$

$$(\hat{a},\hat{a},\hat{a},\dots) \succeq (\hat{b},\hat{b},\hat{b},\dots) \Leftrightarrow (\hat{a},\hat{b},x) \succeq (\hat{b},\hat{a},x).$$

In simple terms this means that, if the repeated consumption $\hat{a} \in X^n$ is preferred over the repeated consumption $\hat{b} \in X^n$, so that $\hat{a}$ is "better" than $\hat{b}$, then the individual would sooner consume $\hat{a}$ than $\hat{b}$.

Koopmans set the problem how the relation of preference should be represented to satisfy impatience. This property has relations represented by the utility functions having the affine aggegator $f_x(t) = \alpha_x t + \beta_x$. We consider the problem when the preference relation satisfying impatience can be represented by a utility function with affine aggregator. For this purpose we will focus on studying the properties of continuous solutions of systems of simultaneous linear functional equations. A partial answer to the above mentioned problem is given in [11]. In this note we extend previous results.

2. Preliminaries

Define on the infinite Cartesian product X^∞ the following metric $\varrho(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_n,y_n)}{1+d(x_n,y_n)}$, where d is the metric on the space X, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Note that the metric space (X^∞, ϱ) has the following property.

Remark 1. The convergence of the sequences is equivalent to the convergence with respect to coordinates, that is $\lim_{k\to\infty}(x_{0,k}, x_{1,k}, \dots) = (x_0, x_1, \dots)$, if and only if, $\lim_{k\to\infty} x_{n,k} = x_n$ for every $n \in \mathbb{N}$.

Let φ satisfy (1) with a utility function U. Introduce the notation

$$f_a := \varphi(a, \cdot), \quad a \in X.$$

From now on we assume that, for any $a \in X$ the functions $f_a : I \to I$ are strictly increasing and continuous. We extend this notation

$$f_{(x_0,x_1,\dots,x_k)} := f_{x_0} \circ f_{x_1} \circ \dots f_{x_k}, \quad x_i \in X,\ k \in \mathbb{N}.$$

Note that, by (2),

$$U(a_1, a_2, \dots, a_k, x) = f_{a_1} \circ f_{a_2} \circ, \dots f_{a_k}(U(x)),\ x \in X^\infty,\ a_1, \dots, a_k \in X.$$

Thus (1) is equivalent to

$$U(\hat{a}, x) = f_{\hat{a}}(U(x)), \quad \hat{a} \in X^k,\ x \in X^\infty, \quad k \in \mathbb{N}. \tag{3}$$

We have the following generalization of Remark 1 from [11].

Remark 2. Every function $f_{\hat{a}}$ for $\hat{a} \in \bigcup_{k\geq 1} X^k$ has a unique fixed point. This fixed point is attractive.

Proof. Note that

$$p_{\hat{a}} := U(\hat{a}, \hat{a} \dots)$$

is a fixed point of $f_{\hat{a}}$. In fact, by (1), we have

$$p_{\hat{a}} := U(\hat{a}, \hat{a} \dots) = \varphi(\hat{a}, U(\hat{a}, \hat{a} \dots)) = f_{\hat{a}}(p_{\hat{a}}).$$

Let $p \in I$. By the surjectivity of U there exists a sequence $(c_1, c_2, \dots) \in X^\infty$ such that $U(c_1, c_2, \dots) = p$. It follows, by (3), that

$$f_{\hat{a}}(p) = f_{\hat{a}}(U(c_1, c_2, \dots)) = U(\hat{a}, c_1, c_2, \dots).$$

Hence $f_{\hat{a}}^2(U(c_1, c_2, \dots)) = f_{\hat{a}}(U(\hat{a}, c_1, c_2, \dots)) = U(\hat{a}, \hat{a}, c_1, c_2, \dots)$. Further, by induction, we get

$$f_{\hat{a}}^n(p) = U(\underbrace{\hat{a}, \dots, \hat{a}}_{n}, c_1, c_2, \dots).$$

By Remark 1

$$(\hat{a}, \dots, \hat{a}, c_1, c_2, \dots) \to (\hat{a}, \hat{a}, \hat{a}, \dots),$$

the continuity of U implies that

$$f_{\hat{a}}^n(p) = U(\hat{a},\ldots,\hat{a},c_1,c_2,\ldots) \to U(\hat{a},\hat{a},\hat{a},\ldots) = p_{\hat{a}}.$$

□

Especially if p is a fixed point, then $f_{\hat{a}}^n(p) = p$, so $p = p_{\hat{a}}$. Moreover, $p_{\hat{a}}$ is an attractive fixed point.

Since $p = p_{\hat{a}}$ is a unique attractive fixed point we get

Remark 3. For every $\hat{a} \in X^k$ and $k \in \mathbb{N}$

(H) $f_{\hat{a}}(t) < t$ for $t > p_{\hat{a}}$ and $f_{\hat{a}}(t) > t$ for $t < p_{\hat{a}}$.

Remark 4. The family $G := \{f_{\hat{a}} : \hat{a} \in \bigcup_{k\geq 1} X^k\}$ is a semigroup of strictly increasing continuous functions possessing property (H).

The mappings $f_a : I \to I$ need not be surjections (see Example 1).

Put $I_a := f_a[I]$. We have

$$\bigcup_{a\in X} I_a = I.$$

In fact, $f_a[I] = f_a[U(X^\infty)] = U(a,X^\infty)$ and $\bigcup_{a\in X} U(a,X^\infty) = U(X^\infty) = I$.

Theorem 1. *If a utility function U satisfies (2) then for every sequence $(x_1, x_2,\ldots)$ and every $t \in I$ there exists the limit $\lim_{n\to\infty} f_{x_1} \circ f_{x_2} \circ,\ldots, f_{x_n}(t)$. This limit does not depend on t and*

$$U(x_1,x_2,\ldots) = \lim_{n\to\infty} f_{x_1} \circ f_{x_2} \circ,\ldots, f_{x_n}(t).$$

Proof. Let $t \in I$ and $(x_1,\ldots) \in X^\infty$. Then there exists $a \in X^\infty$ such that $t = U(a)$. In view of Remark 1 there exists the limit $\lim_{n\to\infty}(x_1,x_2,\ldots,x_n,a) = (x_1,x_2,\ldots)$. The continuity of U implies that

$$\lim_{n\to\infty} U(x_1,x_2,\ldots,x_n,a) = U(x_1,x_2,\ldots).$$

On the other hand, by (3),

$$f_{x_1} \circ f_{x_2} \circ,\ldots, f_{x_n}(t) = f_{(x_1,x_2,\ldots,x_n)}(U(a)) = U(x_1,x_2,\ldots,x_n,a).$$

Hence we get our assertion. □

Corollary 1. *For a given family of continuous injections $f_x, x \in X$ there exists at most one utility function U fulfilling (2). If it exists then every f_x satisfies (H).*

Corollary 2. *If the functions $f_x; I \to I$, $x \in X$ are the agreggator of a utility function, then for every $t \in I$ the set $\{f_{x_1} \circ f_{x_2} \circ,\ldots, f_{x_n}(t), x_i \in X, n \in \mathbb{N}\}$ is dense in I.*

In fact, let $s \in I$ and U be a utility function satisfying (2). By the surjectivity of U there exists $x = (x_1,x_2,\ldots)$ such that $s = U(x)$. In view of Theorem 1, for every $t \in I$ the sequence $f_{x_1} \circ f_{x_2} \circ,\ldots, f_{x_n}(t)$ converges to s.

3. Conjugacy

Definition 3. We say that two utility functions U, $V : X^\infty \to \mathbb{R}$ are **equivalent** if they represent the same preference relation.

By the definition of a utility function the sets $I := U[X^\infty]$ and $J := V[X^\infty]$ are intervals.

Lemma 1. (See [11].) *The utility functions U and V are equivalent if and only if there exists an increasing homeomorphism $\Phi : J \to I$ such that $U = \Phi \circ V$.*

Let a utility function V satisfy the recursion

$$V(x, a) = g_x(V(a)), \quad x \in X, \; a \in X^\infty. \tag{4}$$

Theorem 1 and Corollary 2 let us to generalize Th.3 from [11]. We have the following statement

Remark 5. The utility functions U and V satisfying, respectively, recursions (2) and (4) are equivalent if and only if there exists an increasing homeomorphism $\Phi : I \to J$ such that

$$\Phi \circ f_x = g_x \circ \Phi, \quad x \in X. \tag{5}$$

Proof. The necessity of (5) was proved in [11, Theorem 3]. Furthermore, in the same theorem the sufficiency of (5) was shown under the assumption of uniqueness of the continuous solution of (4). Now, in view of Corollary 1, we see that this assumption is satisfied. □

Remark 6. The continuous solution of equation (5) is unique.

In fact, let $a = (x_1, x_2, \dots)$ and $\hat{x}_n := (x_1, x_2, \dots, x_n)$. By (5) we get inductively

$$\Phi(f_{\hat{x}_n}(t)) = g_{\hat{x}_n}(\Phi(t)), \; n \in \mathbb{N}, \; t \in I.$$

Letting $n \to \infty$ in both sides of the equality we get, by Theorem 1, that $\Phi(U(a)) = V(a)$. If a continuous function Ψ satisfies (5) then similarly $\Psi(U(a)) = V(a)$, so $\Psi(U(a)) = \Phi(U(a))$ for every $a \in X^\infty$. Since $U[X^\infty] = I$ we have $\Phi = \Psi$.

Let us consider a particular case, where $g_x(t) = \alpha_x t + \beta_x$, $\alpha_x \in (0, \infty)$ and $\beta_x \in \mathbb{R}$ for $x \in X$. Then (4) has the following form

$$V(x_0, x_1, x_2, \dots) = \alpha_{x_0} V(x_1, x_2, \dots) + \beta_{x_0}, \quad x_0, x_1, \dots \in X. \tag{6}$$

Remark 7. If V is a utility function, then $0 < \alpha_x < 1$ for $x \in X$.

In fact, Remark 3 implies that all functions g_x satisfy (H), so $0 < \alpha_x < 1$.

Definition 4. The function V satisfying recurrence (6) is called **affine utility function**.

Definition 5. The preference relation defined by a utility function equivalent to an affine function is said to be an **affine relation**.

In economic literature such a relation is also called Uzawa-Epstein preference relation (see [1]).

As a consequence of Remarks 5, 6 and 7 we get the new, more general version of Th.4 in [11].

Theorem 2. *Let U satisfy (2) and V satisfy (6) with coefficients α_x and β_x. If U and V are equivalent then the system*

$$\Phi(f_x(t)) = \alpha_x \Phi(t) + \beta_x, \ t \in I, \ x \in X \tag{7}$$

has a unique strictly increasing and continuous solution and $0 < \alpha_x < 1$. Conversely, if (7) has a strictly increasing continuous solution then U and V are equivalent.

Hence we get

Corollary 3. *A preference relation is affine if and only if for every $x \in X$ there exist $\alpha_x \in (0, 1)$ and $\beta_x \in \mathbb{R}$ such that system (7) has a continuous and strictly increasing solution.*

Note that in general a continuous solution $\Phi : I \to \mathbb{R}$ need not be surjective. Usually system (7) has a one parameter family of continuous strictly increasing solutions (see e.g. [8]). This is a sum of a particular solution of (7) and functions $c\gamma$ for $c > 0$, where γ is a continuous increasing solution of $\gamma \circ f_x = \alpha_x \gamma$. In our case the assumption that f_x satisfies (2) with a utility function U implies that $\gamma = 0$.

Remark 8. If (7) holds then $\frac{\beta_x}{1-\alpha_x} \in J := \Phi[I]$ for $x \in X$.

In fact, put $g_x(t) := \alpha_x t + \beta_x$. It follows, by (7), that $g_x[J] \subset J$. This inclusion is equivalent to the fact that the fixed point of the mapping g_x belongs to J. Note that $\frac{\beta_x}{1-\alpha_x}$ is a fixed point of g_x.

Since the composition of affine functions is affine, it is easy to see, that for the functions $f_x, x \in X$ satisfying system (7), for every $\hat{x} = (x_1, \ldots, x_{n-1})$ we have

$$\Phi(f_{\hat{x}}(t)) = \alpha_{\hat{x}} \Phi(t) + \beta_{\hat{x}},$$

where $\alpha_{\hat{x}} = \alpha_{x_1} \alpha_{x_2} \ldots \alpha_{x_n}$ and a $\beta_{\hat{x}} \in \mathbb{R}$.

Knowing the solution of system (7) allows us to determine the utility function with given aggregator functions. This shows the following.

Theorem 3. *If a utility function U satisfies recursion (2) and system (7) with coefficients $0 < \alpha_x < 1$ has a not constant continuous solution Φ, then* $\lim_{n\to\infty} a_{x_0}\cdots a_{x_n} = 0$ *and the series*

$$\beta_{x_0} + \sum_{k=0}^{\infty}\prod_{i=0}^{k}\alpha_{x_i}\beta_{x_{k+1}} =: S(x_0, x_1, \dots)$$

is convergent. Moreover,

$$\Phi(U(x)) = S(x), \quad x \in X^\infty$$

and $S(x_0, x_1, \dots) = \alpha_{x_0} S(x_1, x_2, \dots) + \beta_{x_0}$ *for* $(x_0, x_1, \dots) \in X^\infty$.

The function S is known in economic literature as Uzawa-Epstein function.

Proof. Let Φ satisfy system (7), then it is easy to verify that

$$\Phi(f_{x_0}\circ f_{x_1}\circ \dots f_{x_n}(t)) = \alpha_{x_0}\dots\alpha_{x_n}\Phi(t) + \beta_{x_0} + \sum_{k=0}^{n-1}\prod_{i=0}^{k}\alpha_{x_i}\beta_{x_{k+1}}.$$

Let $x = (x_0, x_1, \dots)$. It follows, by Theorem 1, that for every $t \in I$ there exists $\lim_{n\to\infty} f_{x_0}\circ f_{x_1}\circ \dots f_{x_n}(t)$. Since $0 < \alpha_x < 1$ there exists the limit $\lim_{n\to\infty} a_{x_0}\cdots a_{x_n} =: A(x)$. Letting $n \to \infty$ in the last equality we get

$$\Phi(U(x)) = A(x)\Phi(t) + S(x) \text{ for } t \in I.$$

Thus, the series $S(x)$ is convergent. If $A(\overline{x}) \neq 0$ for an $\overline{x} \in X^\infty$, then $\Phi(t) = \frac{\Phi(U(\overline{x}))-S(\overline{x})}{A(\overline{x})}$, $t \in I$. This is a contradiction, since Φ is an injective function. Thus $A(x) = 0$ for $x \in X^\infty$.

It is easy to see that $S(x_0, a) = \alpha_{x_0} S(a) + \beta_{x_0}$ for $a = (x_1, x_2, \dots) \in X^\infty$ and $x_0 \in X$. On the other hand $S[X^\infty] = \Phi[U[X^\infty]] = \Phi[I]$. Thus the range of S is an interval, since Φ is continuous and not constant. Thus S is an affine utility function. □

We have the following property which is inverse to that given in Theorem 3

Remark 9. If a utility function U satisfies (2) and there exists a function φ such that $\varphi \circ U = S$, then

$$\varphi(f_x(t)) = \alpha_x\varphi(t) + \beta_x, \quad \text{for} \quad x \in X.$$

Indeed, let $\varphi \circ U = S$ and $t \in I$. By the surjectivity of U there exists $a \in X^\infty$ such that $t = U(a)$. Since S is an affine utility function, we have

$$\begin{aligned}\varphi(f_x(t)) &= \varphi(f_x(U(a))) = \varphi(U(x,a)) = S(x,a) = \alpha_x S(a) + \beta_x \\ &= \alpha_x\varphi(U(a)) + \beta_x = \alpha_x\varphi(t) + \beta_x.\end{aligned}$$

An affine preference relation can be determined by different affine aggregators with different coefficients α_x and β_x. The relationship between these coefficients gives the following.

Proposition 1. *Let f_x for $x \in X$ be the aggregator functions of a utility function U. If Φ and Ψ are continuous and strictly increasing solutions of (7) and the system of equations*

$$\Psi(f_x(t)) = \overline{\alpha}_x \Psi(t) + \overline{\beta}_x, \tag{8}$$

respectively, then

$$\overline{\alpha}_x = \alpha_x \text{ and } \overline{\beta}_x = a\beta_x + b(1-\alpha_x)$$

for some $a > 0$ and $b \in \mathbb{R}$. Moreover, $\Psi = a\Phi + b$.

Proof. Assume that V and W are affine utility functions such that V satisfies (6) and W satisfies

$$W(x_0, x_1, x_2, \dots) = \overline{\alpha}_{x_0} W(x_1, x_2, \dots) + \overline{\beta}_{x_0}. \tag{9}$$

By Theorem 2 the utility functions U and W are equivalent, as well as U and V. Thus V and W are equivalent. By Theorem 3 V and W are Uzawa-Epstein functions. They have the following forms.

$$V(x_0, x_1, x_2, \dots) := \beta_{x_0} + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \alpha_{x_i} \beta_{x_{k+1}},$$
$$W(x_0, x_1, x_2, \dots) := \overline{\beta}_{x_0} + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \overline{\alpha}_{x_i} \overline{\beta}_{x_{k+1}}.$$

Bommier et al., in [1], proved that $W = aV + b$ for some $a > 0$ and $b \in \mathbb{R}$ (see also [2]). The recursions (6) and (9) can be written in the shorter form

$$V(x, y) = \alpha_x V(y) + \beta_x, \quad x \in X,\ y \in X^\infty,$$
$$W(x, y) = \overline{\alpha}_x W(y) + \overline{\beta}_x, \quad x \in X,\ y \in X^\infty.$$

Since $W = aV + b$ we have

$$W(x, y) = aV(x, y) + b = a(\alpha_x V(y) + \beta_x) + b = a\alpha_x V(y) + a\beta_x + b,$$
$$W(x, y) = \overline{\alpha}_x W(y) + \overline{\beta}_x = \overline{\alpha}_x (aV(y) + b) + \overline{\beta}_x = a\overline{\alpha}_x V(y) + \overline{\alpha}_x b + \overline{\beta}_x.$$

Thus

$$a\alpha_x V(y) + a\beta_x + b = a\overline{\alpha}_x V(y) + \overline{\alpha}_x b + \overline{\beta}_x \text{ for } y \in X^\infty.$$

Since V is a surjection of X^∞ onto I, we have $\overline{\alpha}_x = \alpha_x$ and $a\beta_x + b = \overline{\alpha}_x b + \overline{\beta}_x$, so $\overline{\alpha}_x = \alpha_x$ and $\overline{\beta}_x = a\beta_x + b(1-\alpha_x)$.

It is easy to verify that the function $a\Phi + b$ satisfies (8) with $\overline{\alpha}_x = \alpha_x$ and $\overline{\beta}_x = a\beta_x + b(1-\alpha_x)$. By Theorem 2, system (8) has a unique continuous strictly increasing solution, so $\Psi = a\phi + b$. □

As a corollary of Proposition 1 we conclude that the coefficients α_x are determined uniquely, whereas the coefficients β_x depend on two parameters.

Remark 10. If every f_x is differentiable at its fixed point p_x and solution Φ of (7) is differentiable and $\Phi' > 0$, then

$$\alpha_x = f_x'(p_x) \quad \text{and} \quad \beta_x = (1 - f_x'(p_x))\Phi(p_x).$$

In fact, putting in (7) $x = p_x$ we get $\Phi(p_x) = \frac{\beta_x}{1-\alpha_x}$. Next, differentiating both sides of (7) at p_x, we get $f_x'(p_x) = \alpha_x$.

Lemma 2. *If system (7) has an injective solution Φ and $\frac{\beta_x}{1-\alpha_x} \in \Phi[I]$ for $x \in Y \subset X$, then the following conditions are equivalent:*
(i) $f_x, x \in Y$ have a common fixed point,
(ii) $f_x, x \in Y$ pairwise commute,
(iii) $g_x(t) := \alpha_x t + \beta_x, x \in Y$ have a common fixed point.

Proof. If $f_x(p) = p$, then $\Phi(p) = g_x(\Phi(p)) = \frac{\beta_x}{1-\alpha_x}$. Hence g_x commute as affine functions having a common fixed point. Since $\Phi \circ f_x \circ f_y = g_x \circ g_y \circ \Phi = g_y \circ g_x \circ \Phi = \Phi \circ f_y \circ f_x$ we have $f_x \circ f_y = f_y \circ f_x$. By the same equality the commutativity of f_x, $x \in Y$ implies the commutativity of g_x, $x \in Y$. However, commuting affine functions have a common fixed point. If $g_x(q) = q$, then $q = \frac{\beta_x}{1-\alpha_x} \in \Phi[I]$, so $q = \Phi(p)$ for a $p \in I$. We have $\Phi(f_x(p)) = g_x(\Phi(p)) = g_x(q) = q = \Phi(p)$. So the functions f_x have a common fixed point. □

Theorem 4. *Let $f_x, x \in X$ be the aggregator functions of a utility function U and satisfy system (7) with $\alpha_x \in (0, 1), \beta_x \in \mathbb{R}$ and a continuous, strictly increasing function Φ. If f_x for $x \in Y \subset X$ have a common fixed point, then U restricted to Y^∞ is constant.*

Proof. Let $x = (x_0, x_1, \ldots) \in Y^\infty$. By Lemma 3 the functions f_{x_i} pairwise commute, as well as g_{x_i}, where $g_{x_i}(t) = \alpha_{x_i} t + \beta_{x_i}$ for $t \in I$, $i \in \mathbb{N}$. By (7)

$$\Phi(f_{x_0} \circ f_{x_1} \circ \ldots f_{x_n}(t)) = w_n(x_0, x_1, \ldots x_n)(\Phi(t)),$$

where $w_n(x_0, x_1, \ldots x_n)(t) = \alpha_{x_0}\alpha_{x_1} \ldots \alpha_{x_n} t + S_n(x_0, x_1, \ldots x_n)$ and

$$S_n(x_0, x_1, \ldots x_n) = \beta_{x_0} + \sum_{k=0}^{n-1} \prod_{i=0}^{k} \alpha_{x_i}\beta_{x_{k+1}}.$$

By Lemma 2, the functions w_n commute and have a common fixed point. Denote it by q. Thus we have

$$q(1 - \alpha_{x_0}\alpha_{x_1} \ldots \alpha_{x_n}) = S_n(x_0, x_1, \ldots x_n).$$

By Theorem 3, $\lim_{n\to\infty} a_{x_0} \cdots a_{x_n} = 0$. Letting $n \to \infty$ in the last formula we get that $S_n(x_0, x_1, \ldots x_n)$ converges to q. On the other hand, by Theorem 3, $\lim_{n\to\infty} S_n(x_0, x_1, \ldots x_n) = S(x)$, where S is the Uzawa-Epstein function. Also Theorem 3 implies that $\Phi(U(x)) = S(x) = q$ for $x \in Y^\infty$. Thus U is constant on Y^∞. □

Putting $Y = X$ we get the following.

Corollary 4. *If $f_x, x \in X$ pairwise commute and system (7) has a continuous and strictly increasing solution, then there is no utility function satisfying (2).*

4. Impatience

The following characterization of impatience is given in [11].

The preference relation $\succeq$ defined by a utility function with aggregator functions $\{f_x, x \in X\}$ satisfies impatience if and only if

$$\text{(P)} \qquad \forall_{k \geq 1} \forall_{a,b \in X^k} \;\; p_a \geq p_b \Leftrightarrow f_a \circ f_b \geq f_b \circ f_a,$$

where p_a is the only one fixed point of f_a.

Hence, if $\succeq$ satisfies impatience, then $f_a \circ f_b = f_b \circ f_a$ if and only if f_a and f_b have a joint fixed point.

In [11] it is proved that every affine relation satisfies impatience. The proof is to check the condition (P) for the functions $f_{\hat{x}}(t) = \alpha_{\hat{x}} t + \beta_{\hat{x}}$.

As a simple consequence of the above properties and Theorem 2 we get the following statement.

If $f_x, x \in X$ are the aggregator functions of U and system (7) has a strictly increasing continuous solution, for some $\alpha_x \in (0,1)$, $\beta_x \in R$, then the preference relation defined by U satisfies impatience (see also Th.5 in [11]).

We consider the inverse problem: When is a preference relation satisfying impatience affine? To answer this problem we determine the form of all aggregators satisfying condition (P) such that the system of simultaneous equations (7) has a continuous strictly increasing solution.

Some necessary and sufficient conditions for the surjective aggregator functions f_x to satisfy condition (P) are given in [11]. They ensure the affinity of the preference relation. In this section we do not assume the surjectivity of f_x. We complete and extend the results presented in [11].

If the relation $\succeq$ is affine then, by Corollary 3, system (7) has an injective solution. Then, similarly as in [11] one can prove the following property: If $\succeq$ is affine and $f_{\hat{a}} \neq f_{\hat{b}}$, then their graphs are either disjoint or intersect in one point.

We have $f_a^n = f_{\underbrace{(a,a,\ldots,a)}_{n}}$, so if $f_a^n \neq f_b^m$ then $f_a^n(t) \neq f_b^m(t)$ for all $t \in I$, except for one point at most.

Recall that the functions f_a and f_b are said to be iteratively incommensurable if $f_a^n(t) \neq f_b^m(t)$ for all $t \in I$ and all $n, m \in \mathbb{N}$ (see [3,9]).

Hence we get

Corollary 5. *If the relation $\succeq$ is affine then for every $a, b \in X$ the functions $f_a \neq f_b$, are iteratively incommensurable except for one point at most.*

A direct checking of iterative incommensurability is a difficult task. In the considered problem the following property of the relation $\succeq$ is very useful.

Define on X^∞ the relation $x \sim y \Leftrightarrow x \succeq y \wedge y \succeq x$. This means that $x \sim y$ if and only if $U(x) = U(y)$.

Consider the following axiom concerning the properties of relation $\sim$.

(A) If for $a, b \in X$ there exist $\overline{x}, \underline{x} \in X^\infty$ such that $\overline{x} \nsim \underline{x}$, $(a, \overline{x}) \sim (b, \overline{x})$, $(a, \underline{x}) \sim (b, \underline{x})$, then for every $x \in X^\infty$, $(a, x) \sim (b, x)$.

Notice that (A) is equivalent to the following property of the utility function: If for $a, b \in X$ there exist $\overline{x}, \underline{x} \in X^\infty$ such that $U(a, \overline{x}) = U(b, \overline{x})$ and $U(a, \underline{x}) = U(b, \underline{x})$ and $U(\underline{x}) \neq U(\overline{x})$, then for every $x \in X^\infty$, $U(a, x) = U(b, x)$.

Lemma 3. *Let $\succeq$ be the relation generated by U with the aggregators $f_x, x \in X$. The graphs of f_x are either disjoint or intersect in one point if and only if the relation $\succeq$ satisfies* (A).

Proof. Let (A) hold and $f_a(t_1) = f_b(t_1)$ and $f_a(t_2) = f_b(t_2)$ for some $t_1 \neq t_2$. Then there exist $\overline{x}, \underline{x} \in X^\infty$ such that $t_1 = U(\overline{x})$ and $t_2 = U(\underline{x})$. Hence, by (2), $U(a, \overline{x}) = U(b, \overline{x})$ and $U(a, \underline{x}) = U(b, \underline{x})$ so, by (A), $U(a, x) = U(b, x)$ for all $x \in X^\infty$. Further, by (2), $f_a = f_b$.

Conversely, let $U(a, \overline{x}) = U(b, \overline{x})$, $U(a, \underline{x}) = U(b, \underline{x})$ and $U(\underline{x}) \neq U(\underline{x})$ for some $\underline{x}, \overline{x} \in X^\infty$. By (2) we have $f_a(U(\underline{x})) = f_b(U(\underline{x}))$ and $f_a(U(\overline{x})) = f_b(U(\overline{x}))$, so $f_a = f_b$. Thus $f_a(U(x)) = f_b(U(x))$ for $x \in X^\infty$ and, by (2), $U(a, x) = U(b, x)$, that is $(a, x) \sim (b, x)$. □

Note that axiom (A) implies that any functions $f_a \neq f_b$ are iteratively incommensurable.

Remark 11. If the preference relation $\succeq$ is affine then it satisfies (A).

Proof. Let an affine relation $\succeq$ be represented by U and $f_x, x \in X$ be its aggregator. In view of Theorem 2, system (7) has an injective solution Φ. Suppose that there exist $t_1 \neq t_2$ such that $f_x(t_1) = f_y(t_1)$ and $f_x(t_2) = f_y(t_2)$. It follows, by (7), that

$$(\alpha_x - \alpha_y)\Phi(t_1) = (\beta_y - \beta_x) \text{ and } (\alpha_x - \alpha_y)\Phi(t_2) = (\beta_y - \beta_x).$$

The injectivity of Φ implies that $\alpha_x = \alpha_y$ and $\beta_y = \beta_x$, so $f_x = f_y$. Thus the graphs of $f_x, x \in X$ are either disjoint or intersect in one point so, by Lemma 3, condition (A) holds. □

Assume that the preference relation satisfies impatience and consider the following two cases:

(I) There exist $a, b \in X$, $a \neq b$ such that f_a and f_b have a common fixed point.

(II) For any $a, b \in X$, $a \neq b$ f_a and f_b have not common fixed point.

Case (I)

Let $f_a(p) = f_b(p) = p$. Then f_a and f_b commute. Assume that $\succeq$ satisfies (A). Then f_a, f_b are iteratively incommensurable except for the point p or

$f_a^n = f_b^m$ for some $n, m \in \mathbb{N}$. This second case is trivial and has been considered in [11]. Further we assume that $f_a^n \neq f_b^m$ for $n, m \in \mathbb{N}$.

Consider the following system of simultaneous equations

$$\begin{cases} \Psi \circ f_a = \alpha_a \Psi + \beta_a, \\ \Psi \circ f_b = \alpha_b \Psi + \beta_b. \end{cases} \tag{10}$$

A necessary and sufficient condition for the existence of homeomorphic solutions of the systems (10), is given in Th.7 in [11]. There it is assumed that f_a and f_b are iteratively incommensurable. The non-surjective case is considered in the comment at the end of this paper.

Moreover, if f_a and f_b belong to the same continuous iteration semigroup, then system (10) has a continuous and strictly increasing solution. This case occurs if f_a and f_b are sufficiently regular. For example if they are of class C^2 and $\log f_a'(p)/\log f_b'(p) \notin \mathbb{Q}$, then a solution of (10) is also of class C^2 (see [7] Th.10.2 and Th.6.1).

Note that, if system (10) has a solution then, obviously, $\frac{\beta_a}{1-\alpha_a} = \frac{\beta_b}{1-\alpha_b}$. Putting $G := \Psi - \frac{\beta_a}{1-\alpha_a}$ we get the equivalent system of simultaneous Schröder equations

$$\begin{cases} G \circ f_a = \alpha_a G, \\ G \circ f_b = \alpha_b G. \end{cases} \tag{11}$$

Introduce the notation $I^- := I \cap (-\infty, p]$ and $I^+ := [p.\infty)$. By Remark 3 $f_x[I^-] \subset I^-$ and $f_x[I^+] \subset I^+$ for $x \in X$. In each of the intervals I^- and I^+ the continuous solution of (11) is uniquely determined up to a multiplicative constant (see [9,11]). Hence a two parameter family of functions

$$G(t) = \begin{cases} \eta_1 G_-(t), \, t \in I_- \\ \eta_2 G_+(t), \, t \in I_+ \end{cases}$$

where, G_-, G_+ are the particular solutions of (11), respectively on I_- and I_+, gives the general form of continuous solutions of (11) on I.

The injective, continuous solutions of (10) allow us to determine the aggregator functions of a given utility function. We have

Proposition 2. *Let a utility function U satisfying (2) be equivalent to an affine utility function. If Ψ is an injective continuous solution of (10), then, there exist $\mu_1 > 0$ and $\mu_2 > 0$ such that the formula*

$$f_x(t) = \begin{cases} \Psi^{-1}(\alpha_x \Psi(t) + \mu_1 \beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu_1)), \, x \in X, \; t \in I_- \\ \Psi^{-1}(\alpha_x \Psi(t) + \mu_2 \beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu_2)), \, x \in X, \; t \in I_+ \end{cases} \tag{12}$$

expresses the aggregator functions f_x of U for which $\frac{\mu\beta_x}{1-\alpha_x} + \frac{\beta_a}{1-\alpha_a}(1-\mu_i) \in \Psi[I], \; i = 1, 2$.

Proof. Let Ψ be a continuous solution of system (10). Then $G := \Psi - \frac{\beta_a}{1-\alpha_a}$ is a particular continuous solution of (11). The constant function $\frac{\beta_a}{1-\alpha_a}$ is a particular solution of (11), thus two parameter family of functions

$$F_{\eta_1,\eta_2}(t) = \begin{cases} \eta_2 G_-(t) + \frac{\beta_a}{1-\alpha_a}, \, t \in I_- \\ \eta_2 G_+(t) + \frac{\beta_a}{1-\alpha_a}, \, t \in I_+ \end{cases}$$

gives the general continuous solution of (10). By Theorem 2 system (7) has a continuous solution. Let Φ be a continuous solution of (7). Note that Φ satisfies also system (10), so $\Phi = F_{\eta_1,\eta_2}$ for some $\eta_1 > 0$ and $\eta_2 > 0$. Thus for $t \in I_-$ we have $\eta_1 G_-(f_x(t)) + \frac{\beta_a}{1-\alpha_a} = \Phi(f_x(t)) = \alpha_x \Phi(t) + \beta_x = \eta_1 \alpha_x G_-(t) + \frac{\alpha_x \beta_a}{1-\alpha_a} + \beta_x$. Putting $\mu_1 = 1/\eta_1$ we get

$$\frac{1}{\mu_1}(G_-(f_x(t)) - \alpha_x G_-(t)) = \frac{\beta_a(\alpha_x - 1)}{1-\alpha_a} + \beta_x, \; t \in I_-.$$

Thus

$$G_-(f_x(t)) - \alpha_x G_-(t) = \mu_1(\beta_x - \frac{\beta_a(1-\alpha_x)}{1-\alpha_a}), \quad t \in I_-.$$

Since $\Psi = G + \frac{\beta_a}{1-\alpha_a}$, we have $\Psi(f_x(t)) = G_-(f_x(t)) + \frac{\beta_a}{1-\alpha_a} = \alpha_x G_-(t) + \mu_1(\beta_x - \frac{\beta_a(1-\alpha_x)}{1-\alpha_a}) + \frac{\beta_a}{1-\alpha_a} = \alpha_x G_-(t) + \mu_1 \beta_x + \frac{\beta_a}{1-\alpha_a}(1 - \mu_1(1-\alpha_x)) = \alpha_x(\Psi(t) - \frac{\beta_a}{1-\alpha_a}) + \mu_1 \beta_x + \frac{\beta_a}{1-\alpha_a}(1 - \mu(1-\alpha_x)) = \alpha_x \Psi(t) + \mu \beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu_1)$, so

$$\Psi(f_x(t)) = \alpha_x \Psi(t) + \mu_1 \beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu_1), \; x \in X, \; t \in I_-.$$

Similarly we get

$$\Psi(f_x(t)) = \alpha_x \Psi(t) + \mu_2 \beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu_2), \; x \in X, \; t \in I_+.$$

Hence we get (12). The condition limiting the coefficients β_x and α_x is a consequence of Remark 8. □

From the above facts we get the following final result.

Theorem 5. *Let a relation $\succeq$ defined by a utility function U fulfilling* (2) *satisfy impatience. Suppose that Ψ is a strictly increasing continuous solution of* (10)*. Then the relation $\succeq$ is affine if and only if f_x for $x \in X$ are given by formula* (12)*.*

Proof. Let an affine relation $\succeq$ be defined by a function U satisfying (2). By Theorem 2 system (7) has a continuous and strictly increasing solution. Then, by Proposition 1, the functions f_x are given by (12).

Inversely, if f_x are given by (12), then Ψ satisfies system (7) with coefficients α_x and $\mu\beta_x + \frac{\beta_a}{1-\alpha_a}(1-\alpha_x)(1-\mu)$. Hence, in view of Theorem 2, the relation $\succeq$ is affine. □

Case II

The case where the mappings $f_x : I \to I$ are homeomorphisms was considered in [11]. Now we study the case of non-surjective mappings. Let $a = \inf I$ and $b = \sup I$. We allow that $a = -\infty$ and $b = \infty$. If I is not closed then we may extend f_x continuously on $cl\, I$.

In this section we consider the special case where $X = \{0, 1, \dots, n-1\}$, $Int f_p[I] \cap\ Int f_q[I] = \emptyset$ for $p, q \in X$, $p \neq q$ and $\bigcup_{k\in X} f_k[I] = I$.

Assume the hypothesis

(i) $f_0, \dots, f_{n-1} : [a, b] \to [a, b]$ are continuous strictly increasing mappings and $f_0(a) = a,\ f_{n-1}(b) = b,\ f_{k+1}(a) = f_k(b),\ k = 0, \dots, n-2.$

Let us start from the following

Remark 12. Let the aggregators $f_0, \dots f_{n-1}$ of a utility function U satisfy (i). Then the preference relation defined by U satisfies impatience.

Proof. To show this we verify that the functions $f_0, \dots, f_{n-1}$ satisfy condition (P). Let $\hat{a} = (a_1, a_2 \dots)$ and $\hat{b} = (b_1, b_2, \dots)$, where $a_k, b_k \in \{0, 1, \dots, n-1\}$, $k \in \mathbb{N}$. We have that $f_{\hat{a}}$ maps I onto I_{a_1} and $f_{\hat{b}}$ maps I onto I_{b_1}.

Let $f_{\hat{a}}(p_{\hat{a}}) = p_{\hat{a}}$ and $f_{\hat{b}}(p_{\hat{b}}) = p_{\hat{b}}$. Since $p_{\hat{a}} \in I_{a_1}$ and $p_{\hat{b}} \in I_{b_1}$ the inequality $p_{\hat{a}} < p_{\hat{b}}$ occurs if and only if $\sup I_{a_1} \leq \inf I_{b_1}$. On the other hand, since $f_{\hat{a}} \circ f_{\hat{b}}(t) \in I_{a_1}$ and $f_{\hat{b}} \circ f_{\hat{a}}(t) \in I_{b_1}$ for $t \in I$ we have $f_{\hat{a}} \circ f_{\hat{b}} \leq \sup I_{a_1}$ and $\inf I_{b_1} \leq f_{\hat{b}} \circ f_{\hat{a}}$. Hence $f_{\hat{a}} \circ f_{\hat{b}} \leq f_{\hat{b}} \circ f_{\hat{a}}$ if and only if $\sup I_{a_1} \leq \inf I_{b_1}$. Thus condition (P) holds. □

Let $J := [c, d]$. Define on J the functions

$$H_k(t) = \alpha_k t + \beta_k$$

for $k = 0, 1, \dots, n-1$. Assume that $H_0, \dots, H_{n-1}$ satisfy the following hypothesis.

(ii) $H_0(c) = c = \mu_0,\ H_{n-1}(d) = d =: \mu_{n-1},\quad H_{k-1}(d) = H_k(c) =: \mu_k,$ $k = 1, \dots, n-1.$

The functions $H_k : J \to [\mu_k, \mu_{k+1}]$ are determined uniquely by the parameters μ_k. The coefficients α_k and β_k are the solutions of the systems of equations

$$\alpha_k c + \beta_k = \mu_k,\quad \alpha_k d + \beta_k = \mu_{k+1},\quad k = 0, 1, \dots, n-1. \tag{13}$$

Applying the results from [10] we get the following criterion for affinity of the preference relation.

Theorem 6. *Let $f_0, \dots, f_{n-1}$ be aggregator functions of a utility function U and satisfy (i). If*

$$|f_k(s) - f_k(t)| < |s - t|,\quad s \neq t,\ k = 0, \dots, n-1, \tag{14}$$

then the preference relation defined by the function U is affine.

Proof. It was proved in [10] (see Th.4 and Th.5) that if $f_0, \dots, f_{n-1}$ satisfy (i) and $H_0, \dots H_{n-1}$ satisfy (ii), then the system

$$\varphi(f_k(t)) = H_k(\varphi(t)), \quad k = 0, \dots, n-1, \ t \in I \tag{15}$$

has a unique bounded solution φ. This solution is monotonic, $\varphi(a) = c$ and $\varphi(b) = d$. If, in addition, condition (14) holds, then the solution φ is continuous and strictly increasing. In our case system (15) has the form

$$\varphi(f_k(t)) = \alpha_k \varphi(t) + \beta_k, \ k = 0, \dots, n-1.$$

Thus, by Theorem 2, U is affine. □

Corollary 6. *If the assumptions of Theorem 6 are satisfied then for every $n-2$ parameters $c < \mu_1 <, \cdots < \mu_{n-2} < d$ there exists a unique homeomorphic solution Φ of system (7) such that $\Phi(f_i(a)) = \mu_i$ for $i = 1, \dots n-2$. The coefficients α_k and β_k in (7) are uniquely determined by the system of linear equations (13).*

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