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Aequationes Mathematicae

On monohedral tilings of a regular polygon

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Abstract. A tiling of a topological disc by topological discs is called monohedral if all tiles are congruent. Maltby (J Comb Theory Ser A 66:40–52, 1994) characterized the monohedral tilings of a square by three topological discs. Kurusa et al. (Mediterr J Math 17:156, 2020) characterized the monohedral tilings of a circular disc by three topological discs. The aim of this note is to connect these two results by characterizing the monohedral tilings of any regular n-gon with at most three tiles for any $n \geq 5$.

Mathematics Subject Classification. 52C20, 52B45, 52A10.

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1. Introduction

Subsets of the Euclidean plane $\Re^2$ homeomorphic to the Euclidean closed circular unit disc $\mathbf{B}^2$ centered at the origin o are usually called *topological discs* or *Jordan regions*.[1] A family of topological discs $\{D_1, D_2, \dots, D_k\}$ whose union is a topological disc D and whose elements are mutually nonoverlapping (i.e. their interiors are mutually disjoint), is called a *tiling*, *decomposition*, or *dissection* of D, and the elements of the family are called *tiles*. A tiling is called *monohedral*, if all tiles are congruent to a given topological disc, which is often called *prototile* [3].

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[1]The set $\mathbf{B}^2$ is the set of points in the plane whose Euclidean distance from o is at most one. To distinguish them from topological dics, we call the sets similar to $\mathbf{B}^2$ circular discs, or Euclidean circular discs.

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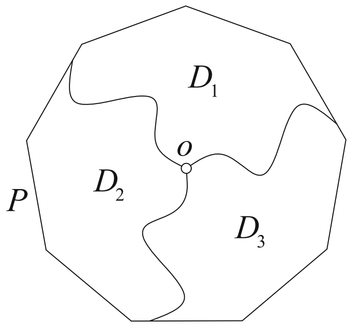

Figure 1. A rotationally generated tiling of a regular 9-gon P with three tiles

The history of the investigation of tilings goes back to ancient times and well beyond the boundary of mathematics (see e.g. [6,19]). The aim of this paper is to examine one such problem. A result of Maltby [14] in 1994 states that a square cannot be dissected into three non-rectangular congruent topological discs. Along the same line, Yuan et al. [20] proved, answering a question of Danzer, that in any monohedral tiling of a square by five *convex* tiles, the prototile is a rectangle, and conjectured that the same holds if the number of tiles is an odd prime. This question has been recently answered in [17] in the special case that the prototile is a q-gon with $q \geq 6$ or it is a right-angled trapezoid, and a computer-assisted proof has been given in [15] for seven or nine tiles.

We intend to investigate a similar question, also based on the result of Maltby in [14]. To state our main result, we call a monohedral tiling of a regular n-gon P, centered at the origin o, by tiles $D_1, D_2, \ldots, D_k$ *rotationally generated* if the rotation around o and with angle $\frac{2\pi}{k}$ leaves P invariant, and permutes the tiles (cf. Fig. 1).

Theorem 1. *Let P be a regular n-gon with $n \geq 5$, and let $\mathcal{F}$ be a monohedral tiling of P by k topological discs, where $2 \leq k \leq 3$. Then either $k = 2$, n is odd and $\mathcal{F}$ contains the two halves of P dissected by a line of symmetry of P, or n is divisible by k and $\mathcal{F}$ is rotationally generated.*

We note that the same theorem with the Euclidean circular disc in place of P was proved in [13], and monohedral tilings, with at most 3 tiles, of a convex disc with strictly convex and smooth boundary were partially characterized in [16]. Theorem 1 can be regarded as a result connecting the one in [14] for squares and the one in [13] for circular discs. The proof of Theorem 1 is based on (geometric, combinatorial and topological) tools from both [13] and [14], and also on some new ideas.

Finally, we remark that in the past few years a 'dual version' of this problem, namely the investigation of dissecting the Euclidean plane into mutually *incongruent* tiles with equal area under various constraints has also gained significant interest. For related results the interested reader is referred to [2–5,10,11]. The number of dissections of a square into equal area rectangles was estimated in [1,7]. For the investigation of monohedral dissections of geometric figures using a different notion of dissection, see e.g. [8,9].

The structure of our paper is as follows. In Sect. 2 we introduce the necessary notation and tools to prove our main result. In Sect. 3 we present the proof of Theorem 1. Finally, in Sect. 4 we collect some additional remarks.

2. Preliminaries

In the paper, for any set $X \subset \Re^2$, we denote the interior, the boundary, the closure and the convex hull of X by $\mathrm{int}(X), \partial X$, $\mathrm{cl}(X)$ and $\mathrm{conv}(X)$, respectively. Furthermore, if X is bounded and nonempty, then $\mathrm{diam}(X)$ denotes its diameter. For any $x, y \in \Re^2$, by $[x, y]$ we denote the closed segment with endpoints x, y. By a simple curve we mean a continuous curve which does not cross itself, and a simple, closed curve is a simple curve whose two endpoints coincide. With a little abuse of notation, we call the points of a simple, not closed curve, different from its endpoints, interior points of the curve. Finally, for brevity, we call a topological disc simply a *disc*.

In the proof, $n \geq 5$ and P denotes a regular n-gon with unit side-length centered at o, and vertices $p_1, p_2, \ldots, p_n$ in counterclockwise order. We set $\mathcal{F} = \{D_1, D_2, \ldots, D_k\}$ with $k \geq 2$, where all the D_i are congruent to a disc D, and for $i = 1, 2, \ldots, k$ let $S_i = D_i \cap \partial P$. For any value of $i \neq 1$, we choose an isometry $g_{1i} : \Re^2 \to \Re^2$ satisfying $g_{1i}(D_1) = D_i$, and set $g_{i1} = g_{1i}^{-1}$, and define g_{ij} by $g_{ij}(\cdot) = g_{1j}(g_{1i}^{-1}(\cdot))$ for all i, j.

We remark that every disc is compact, and thus, it is Lebesgue measurable. On the other hand, the boundary of a disc is not necessarily rectifiable; as an example we may choose e.g. the Koch snowflake (for more 'esoteric' examples, see [18]). In the proof, for any disc D we use the notation $\mathrm{area}(D)$ and $\mathrm{perim}(D)$ for the area and the perimeter of D, respectively, and we use the latter one only if ∂D is clearly rectifiable. If Γ is a rectifiable curve, then by $l(\Gamma)$ we mean the length of Γ. In particular, this yields that if ∂D is rectifiable for some disc D, then we have $l(\partial D) = \mathrm{perim}(D)$.

We start with some preliminary lemmas and remarks.

Remark 1. Since any D_i is a disc, any two points of D_i can be connected by a continuous curve which contains only interior points of D_i, apart from its endpoints. In the paper, we call such a curve an *in-curve* of D_i.

Remark 2. We note that if some isometry g_{ij} is a reflection about a line L, then L separates D_i and D_j. Indeed, suppose for contradiction that there are points $x, y \in D_i$ in different open half planes bounded by L, and let Γ be an in-curve of D_i connecting x and y. Then there is a point z of Γ on L. Thus, $g_{ij}(z) = z$, implying that $\mathrm{int}(D_i) \cap \mathrm{int}(D_j) \neq \emptyset$; a contradiction.

Lemma 1. *If* $\mathrm{diam}(D) = \mathrm{diam}(P)$*, then* $k = 2$*. Furthermore, either* n *is odd and* $\mathcal{F}$ *contains the two halves of* P *dissected by a line of symmetry of* P*, or* n *is divisible by* 2 *and* $\mathcal{F}$ *is rotationally generated.*

Proof. Under our conditions, each D_i contains a diametrically opposite pair of points of P, or in other words, the two endpoints of a longest diagonal of P. First, observe that if $p_{i_1}, p_{i_2} \in D_i$ and $p_{j_1}, p_{j_2} \in D_j$ are mutually distinct diametrically opposite points of P where $p_{i_1}, p_{j_1}, p_{i_2}, p_{j_2}$ are in this cyclic order in ∂P, then any in-curve of D_i connecting p_{i_1} and p_{i_2} would cross any in-curve of D_j connecting p_{j_1} and p_{j_2}, leading to a contradiction. Thus, there is a vertex of P contained in any diameter of any D_i. Without loss of generality, let us assume that p_1 is such a vertex.

First, consider the case that $n = 2m$ for some integer $m \geq 3$. Since $[p_1, p_{m+1}]$ is the unique diameter of P containing p_1, it follows that $p_{m+1} \in D_i$ for all values of i. Furthermore, any isometry mapping a longest diagonal of P into a longest diagonal of P is a symmetry of P, implying that g_{ij} is a symmetry of P for all i, j. Thus, g_{ij} is the reflection about the line L through $[p_1, p_{m+1}]$, or about the bisector L' of $[p_1, p_{m+1}]$, or about o. On the other hand, if g_{ij} is the reflection about L', the fact that p_1 and p_{m+1} are in different open half planes bounded by L' contradicts Remark 2. Thus, g_{ij} is the reflection about L or about o for any $i \neq j$. Since $g_{il}(\cdot) = g_{jl}(g_{ij}(\cdot))$ for all i, j, l by definition, the fact that there are only two possible isometries as g_{ij} implies that $k \leq 2$. If g_{12} is the reflection about o, then we are done. If g_{12} is the reflection about L, then from Remark 2 it follows that L separates D_1 and D_2, and the tiling is rotationally generated.

Finally, consider the case that $n = 2m + 1$ for some integer $m \geq 2$, and let L denote the line through $[o, p_1]$. By our conditions, any tile D_i contains p_{m+1} or p_{m+2}. Suppose for contradiction that a tile contains both p_{m+1} and p_{m+2}. Then any tile contains either p_1, p_{m+1} and p_{m+2}, or p_1, p_2 and p_{m+2}, or p_1, p_n and p_{m+1}. However, this would give points $p_{i_1}, p_{j_1}, p_{i_2}, p_{j_2}$ as in the first paragraph of the proof, which was shown to be impossible. Thus, any tile contains either p_{m+1} or p_{m+2}.

Assume that there are at least two tiles containing one of them, say $p_{m+1} \in D_1, D_2$. Then g_{12} is the reflection about either the line L' through $[p_1, p_{m+1}]$, or the bisector of $[p_1, p_{m+1}]$, or the midpoint of $[p_1, p_{m+1}]$. Here the second case contradicts Remark 2. In the first and the third cases we have that $D_1, D_2 \subset P \cap P'$, where $P' = g_{12}(P)$ (cf. Fig. 2). Thus, $P' \cap P \subsetneq P$ yields that $k \geq 3$. If $k = 3$, then $P \backslash P' \subseteq D_3$, and the compactness of D_3 implies that

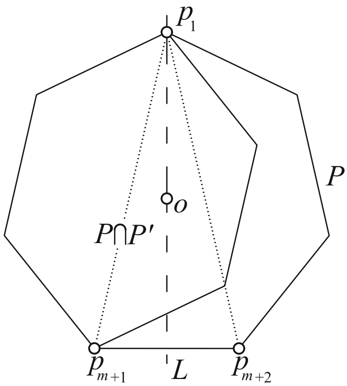

FIGURE 2. An illustration for the proof of Lemma 1

$p_{m+1}, p_{m+2} \in D_3$; a contradiction, as in the previous paragraph. If $k > 3$, then there are at least two tiles containing p_{m+2}, which, by the previous argument, are contained in $P \cap P''$, where P'' is the reflected copy of P about the line through $[p_1, p_{m+2}]$. Thus, in this case the midpoint of $[p_{m+1}, p_{m+2}]$ does not belong to any tile; a contradiction.

We have shown that $k = 2$, and there is a unique tile containing p_{m+1} and a unique tile containing p_{m+2}. Let these tiles be D_1 and D_2, respectively. Let q be the midpoint of $[p_{m+1}, p_{m+2}]$ and assume, without loss of generality, that $q \in D_1$. Then the only congruent copy of P containing p_1, p_{m+1}, q is P, implying that $g_{12}(P) = P$. Since we also have $g_{12}([p_1, p_{m+1}]) = [p_1, p_{m+2}]$, this yields that g_{12} is the reflection about the line L through $[o, p_1]$, from which the assertion readily follows. □

Next, we recall Lemma 2.3 from [13].

Lemma 2. *Let $\{\bar{D}_1, \bar{D}_2, \bar{D}_3\}$ be a tiling of the disc $\bar{D}$ where, for $i = 1, 2, 3$, $\bar{D}_i$ is a disc such that $\bar{S}_i = \bar{D}_i \cap \partial \bar{D}$ is a nondegenerate simple continuous curve. Then $\bar{D}_1 \cap \bar{D}_2 \cap \bar{D}_3$ is a singleton $\{q\}$, and for any $i \neq j, \bar{D}_i \cap \bar{D}_j$ is a simple continuous curve connecting q and a point in $\partial \bar{D}$.*

Our next lemma is a generalization of Lemma 2.

Lemma 3. *Let the disc $\bar{D}$ be decomposed into three discs $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$. For $i = 1, 2, 3$, set $\bar{S}_i = \bar{D}_i \cap \partial \bar{D}$. Then, with a suitable choice of indices, exactly one of the following holds (cf. Fig. 3).*

(1) $\bar{S}_3$ contains at most two points, and $\bar{S}_1$ and $\bar{S}_2$ are connected arcs whose union covers $\partial \bar{D}$.

(2) $\bar{S}_1$ is the union of two disjoint, connected, nondegenerate arcs, the sets $\bar{S}_2, \bar{S}_3, \bar{D}_1 \cap \bar{D}_2$, $\bar{D}_1 \cap \bar{D}_3$ are connected arcs, and $\bar{D}_2 \cap \bar{D}_3 = \emptyset$.

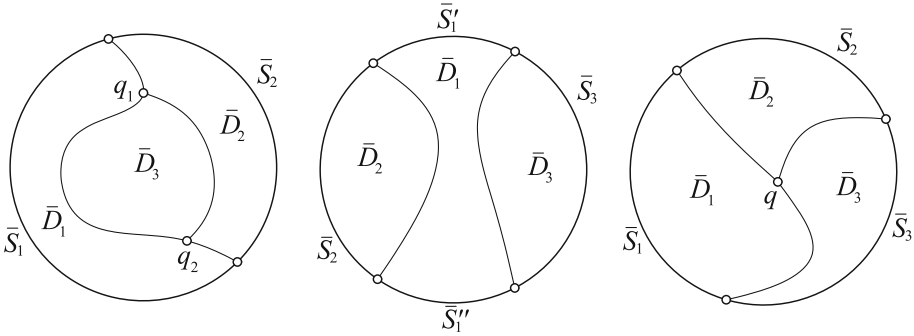

Figure 3. The topological types described in Lemma 3 with a Euclidean disc as $\bar{D}$. We note that the points q_1 and q_2 in the left panel, and q in the right panel may lie on $\partial \bar{D}$

(3) $\bar{S}_2$, $\bar{S}_3$, $\bar{D}_1 \cap \bar{D}_2$, $\bar{D}_1 \cap \bar{D}_3$, $\bar{D}_2 \cap \bar{D}_3$ *are connected arcs,* $\bar{D}_1 \cap \bar{D}_2 \cap \bar{D}_3$ *is a singleton* $\{q\}$*, and* $\bar{S}_1$ *is either a connected arc, or the union of a connected arc and* $\{q\}$*.*

Proof. Assume that one of the $\bar{S}_i$, say $\bar{S}_1$, has more than one component, and let $q_1, q_2, \ldots, q_m$ be points of $\bar{S}_1$, in this cyclic order, contained in different components of $\bar{S}_1$. Let $r_1, r_2, \ldots, r_m$ be points in $\partial \bar{D} \setminus \bar{S}_1$ such that $q_1, r_1, q_2, \ldots, q_m, r_m$ are in this cyclic order in $\partial \bar{D}$. Note that every r_j belongs to $\bar{S}_2$ or $\bar{S}_3$, and no two of them belongs to the same set. Indeed, if, say, r_{j_1} and r_{j_2} belong to $\bar{S}_2$, where $j_1 \neq j_2$, then any in-curve in $\bar{D}_2$ connecting them, and any in-curve in $\bar{S}_1$ connecting q_{j_1} and q_{j_2} would cross, which is a contradiction. Thus, we have $m = 2$, which also yields, by the same argument, that $\bar{S}_2$ and $\bar{S}_3$ are connected. If neither component of $\bar{S}_1$ is a singleton, then the closure of $(\partial \bar{D}_1) \setminus \bar{S}_1$ contains two disjoint, simple curves which, apart from their endpoints, are contained in $\mathrm{int}(\bar{D})$. Since in this case $\bar{D}_2$ and $\bar{D}_3$ can be separated by an in-curve of $\bar{D}_1$ disjoint from $\bar{D}_2 \cup \bar{D}_3$, it follows by compactness that the two components of $\mathrm{cl}((\partial \bar{D}_1) \setminus \bar{S}_1)$ coincide with $\bar{D}_1 \cap \bar{D}_2$ and $\bar{D}_1 \cap \bar{D}_3$, implying (2). If exactly one component of $\bar{S}_1$ is a singleton, a similar argument can be applied, implying (3). Finally, if both components of $\bar{S}_1$ are singletons, the conditions in (1) are satisfied with a suitable relabeling of the tiles.

Assume that all the $\bar{S}_i$ are connected. Since $\bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3 = \partial \bar{D}$ and every $\bar{S}_i$ is a simple connected arc properly contained in $\partial \bar{D}_i$, we have that at least two of the $\bar{S}_i$ contain more than one point. If one of them, say $\bar{S}_3$, contains at most one point, then (1) follows. If $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$ are nondegenerate, simple arcs, then the conditions of Lemma 2 are satisfied, implying (3). □

Definition 1. Let $\{\bar{D}_1, \bar{D}_2, \bar{D}_3\}$ be a tiling of the disc $\bar{D}$. If the discs $\bar{D}_1, \bar{D}_2, \bar{D}_3$ satisfy the conditions in (k) of Lemma 3 with $k = 1$, $k = 2$ or $k = 3$, we say that the tiling is a *Type k decomposition* of $\bar{D}$.

Remark 3. Assume that $\bar{D}$ is decomposed into two discs $\bar{D}_1, \bar{D}_2$, and for $i = 1, 2$, set $\bar{S}_i = \bar{D}_i \cap \partial\bar{D}$. Note that since the number of tiles is more than one, we have that no $\bar{S}_i$ coincides with $\partial\bar{D}$. Furthermore, by the argument in the proof of Lemma 3 we also have that $\bar{S}_1$ and $\bar{S}_2$ are connected. Motivated by this property, we call any tiling of $\bar{D}$ with two discs a Type 1 decomposition of $\bar{D}$.

Lemmas 4–6 and Definitions 2–3 are from [13].

Lemma 4. *Let G and C be simple curves. Then G contains at most finitely many congruent copies of C which are mutually disjoint, apart from possibly their endpoints.*

Definition 2. A *multicurve* (see also [12]) is a finite family of simple curves, called the *members of the multicurve*, which are parameterized on nondegenerate closed finite intervals, and any point of the plane belongs to at most one member, or it is the endpoint of exactly two members. If $\mathcal{F}$ and $\mathcal{G}$ are multicurves, $\bigcup \mathcal{F} = \bigcup \mathcal{G}$, and every member of $\mathcal{F}$ is the union of some members of $\mathcal{G}$, we say that $\mathcal{G}$ is a *partition* of $\mathcal{F}$.

Definition 3. Let $\mathcal{F}$ and $\mathcal{G}$ be multicurves. If there are partitions $\mathcal{F}'$ and $\mathcal{G}'$ of $\mathcal{F}$ and $\mathcal{G}$, respectively, and a bijection $f \colon \mathcal{F}' \to \mathcal{G}'$ such that $f(C) \in \mathcal{G}'$ is congruent to C for all $C \in \mathcal{F}'$, we say that $\mathcal{F}$ and $\mathcal{G}$ are *equidecomposable.*

Lemma 5. *If $\mathcal{F}$ and $\mathcal{G}$ are multicurves with $\bigcup \mathcal{F} = \bigcup \mathcal{G}$, then $\mathcal{F}$ and $\mathcal{G}$ are equidecomposable.*

Lemma 6. *If $\mathcal{F}$ and $\mathcal{G}$ are equidecomposable, and their subfamilies $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{G}' \subseteq \mathcal{G}$ are equidecomposable, then $\mathcal{F} \backslash \mathcal{F}'$ and $\mathcal{G} \backslash \mathcal{G}'$ are equidecomposable.*

We finish with a remark and a definition.

Remark 4. Let $\{D_1, D_2, D_3\}$ be a monohedral tiling of the regular n-gon P with unit side-length and $n \geq 5$. For $i = 1, 2, 3$, set $S_i = D_i \cap \partial P$. Note that for any $i \neq j$, $g_{ij}(S_i) \subset (\partial \operatorname{conv}(D_j)) \cap (\partial D_j)$. Furthermore, we have the following:

(1) If $S_i^* \subseteq S_i$ and $S_j^* \subseteq S_j$ are maximal nondegenerate segments in S_i and S_j, respectively, such that the interiors of S_i^* and $g_{ji}(S_j^*)$ intersect, then $S_i^* = g_{ji}(S_j^*)$.
(2) If some vertex p_t of P lies in the interiors of both S_i and $g_{ji}(S_j)$, then $S_i = g_{ji}(S_j)$, and $P = g_{ji}(P)$.
(3) If $S_i \cap g_{ji}(S_j)$ contains a segment of unit length, then $S_i = g_{ji}(S_j)$, and $P = g_{ji}(P)$.

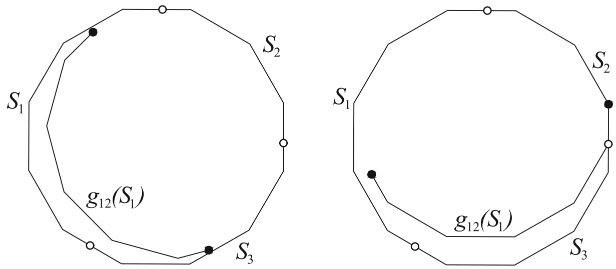

Figure 4. An illustration for Remark 4. In the left panel S_2 and $g_{12}(S_1)$ do not overlap; in the right panel S_2 and $g_{12}(S_1)$ slightly overlap. The endpoints of the arcs S_i are denoted by empty circles, and the endpoints of $g_{12}(S_1)$ are denoted by full circles

If the interiors of S_i and $g_{ji}(S_j)$ are disjoint, we say that S_i and $g_{ji}(S_j)$ are *nonoverlapping*. Based on our above observations and Lemma 3, if S_i and $g_{ji}(S_j)$ overlap but they do not coincide, then their intersection contains at most two connected components, each of which is either a single point or a nondegenerate segment of length strictly less than one. In this case we say that S_i and $g_{ji}(S_j)$ are *slightly overlapping* (cf. Fig.Fig. 4). We observe that S_i and $S_j = g_{ij}(S_i)$ are nonoverlapping, slightly overlapping and equal if and only if $g_{il}(S_i)$ and $g_{jl}(S_j)$ are nonoverlapping, slightly overlapping and equal, respectively, for an arbitrary value of l.

3. Proof of Theorem 1

By Lemma 3 and Remark 3, it is sufficient to prove the theorem for Type 1, Type 2 and Type 3 decompositions of P. In the following, we present the proof for each type in a separate subsection. Throughout the proof, we assume that no D_i contains diametrically opposite points of P, as otherwise the assertion readily follows from Lemma 1.

3.1. Proof for Type 1 decompositions

We choose our notation in such a way that S_1 and S_2 are nondegenerate connected arcs in ∂P whose union is P, and S_3, if it exists, contains at most two points. Observe that in this case n is even, as otherwise S_1 or S_2 contains at least $\frac{n+1}{2}$ vertices of P, including a pair of diametrically opposite points of P. Consider the sets S_1 and $S_1' = g_{21}(S_2)$. By Remark 4, we distinguish three cases.

Case 1, S_1 and S_1' do not overlap. In this case they are nonoverlapping arcs in the boundary of $\operatorname{conv}(D_1)$ whose total length is $\operatorname{perim}(P)$, which implies that $\operatorname{conv}(D_1) = P$ and $\partial(\operatorname{conv} D_1) = S_1 \cup S_1'$. On the other hand, since in this case $S_1 \cup S_1'$ is a simple, closed curve, we have $D_1 = \operatorname{conv}(D_1) = P$, which contradicts the assumption that $k > 1$.

Case 2, S_1 and S_1' slightly overlap. Let L be a sideline of P containing at least an interior point of $S_1' \cap S_1$, and let L' be the supporting line of P parallel to L which is different from L. Let G_1 and G_2 denote the two components of $\operatorname{cl}(\partial P \setminus (L \cup L'))$. Clearly, since L contains a common endpoint of S_1 and S_2, at least one of G_1 and G_2 contains no endpoint of S_1 and S_2 in its interior, and hence, we may assume that e.g. $G_2 \subset S_2$. Then the facts that S_1' and S_1 are slightly overlapping and $S_1' \subset P$ yield that L' also contains a point of S_1', and $G_1 \subset S_1$. Furthermore, since in this case $S_1 \cup S_1'$ form simple closed curve(s) in ∂D_1, we have that $D_1 = \operatorname{conv}(S_1 \cup S_1')$. Let q and q' be the midpoints of the sides of P on L and L', respectively, and observe that the translate G_2' of G_2 whose endpoints are q and q' is contained in D_1. Thus, the area of D_1 is greater than or equal to the area of $\operatorname{conv}(G_1 \cup G_2')$, and the area of the latter region is strictly greater than $\frac{\operatorname{area}(P)}{2}$. This contradicts the fact that the examined tiling of P is monohedral.

Case 3, $S_1 = S_1'$. In this case g_{21} is either the reflection about the line L through the two endpoints of S_1 and S_2, or it is the reflection about o. This implies Theorem 1 for $k = 2$. Furthermore, if $k = 3$, then $D_3 = \operatorname{cl}(P \setminus (D_1 \cup D_2))$ is symmetric to L or o, respectively, and P has an even number of sides.

Assume that $k = 3$ and g_{21} is the reflection about o. Then, since D_1, D_2 and D_3 are all congruent, it follows that both D_1 and D_2 are centrally symmetric. As $D_1, D_2 \subset P$ we also have that the centers of D_1 and D_2 are contained on the line through o parallel to the two sides of P containing the common endpoints of S_1 and S_2. Let these two sides of P be E and E', and let the centers of symmetry of D_1 and D_2 be c_1 and c_2, respectively. From the properties of central symmetry and the fact that $S_1 = S_1'$, we have that for $i = 1, 2$, $E \cap S_i$ and $E' \cap S_i$ are segments of length $1/2$. Furthermore, for $i = 1, 2$ the union of S_i and its reflection about c_i is a simple closed convex curve in ∂D_i, implying that its convex hull is D_i. Thus, D_1 and D_2 overlap; a contradiction.

Finally, assume that $k = 3$ and g_{21} is the reflection about the line L passing through the common endpoints of S_1 and S_2. Since L is a symmetry line of P and no D_i contains diametrically opposite points of P, it follows that L passes through the midpoints of two opposite sides E, E' of P. Furthermore, since D_3 is symmetric to the line L, we obtain that D_1 and D_2 have lines of symmetry, which we denote by L_1 and L_2, respectively. Since both discs are contained in the infinite strip bounded by the two sidelines of P through E and E', L_1 and L_2 are parallel to L, or coincide with the line L^* through o perpendicular to L.

Assume that one of L_1 and L_2, say L_1, is parallel to L, and let S' denote the reflection of S_1 about L_1. Since $S_1 \cup S'$ is a simple, closed curve in ∂D_1, we have $S_1 \cup S' = \partial D_1$, which yields that $D_1 = \text{conv}(S_1 \cup S')$. On the other hand, as the endpoints of S_1 are midpoints of two opposite sides of P, from this an elementary computation shows that $\text{area}(D_1) > \frac{\text{area}(P)}{3}$, a contradiction. Thus, we have $L_1 = L^*$, and we remark that our argument shows that *any* line of symmetry of D_1 coincides with L^*, and, applying this argument, we obtain the same statement for D_2. On the other hand, if both D_1 and D_2 are symmetric to L^*, then the same holds for D_3. Hence, D_3 is symmetric to both L and L^*, which yields that D_1 (resp., D_2) has a line of symmetry different from L^*, which contradicts our previous observation.

3.2. Proof for Type 2 decompositions

We assume that S_1 is disconnected, and denote the two components of S_1 by S_1' and S_1''. We distinguish three cases.

Case 1, both S_1' and S_1'' contain vertices of P. Without loss of generality, we may assume that the vertices of P in S_1' are $p_1, p_2, \ldots, p_m$ for some $1 \leq m \leq n-1$. First, we show that n is even. Suppose for contradiction that $n = 2t+1$ for some $t \geq 2$. Then, since S_1 contains no diametrically opposite points of P, we have that $p_{t+1}, p_{t+2}, \ldots, p_{t+m+1}$ belong to the same set $S_i \neq S_1$. Without loss of generality, we may assume that they, and also p_{m+1}, belong to S_2. This yields that $p_{m+1}, p_{m+2}, \ldots, p_{m+t+1}$ belong to S_2, and thus, S_2 contains at least $t+1$ vertices of P, which contradicts our assumption that D_2 does not contain diametrically opposite points of P. Thus, we have that n is even.

Let $n = 2t$ for some $t \geq 3$. Similarly like in the previous paragraph, we have that $p_{t+1}, \ldots, p_{t+m}$ are not points of S_1'', and thus, they all belong to S_2 or all belong to S_3. Without loss of generality, assume that they belong to S_2. Thus, $p_{m+1}, \ldots, p_{m+t} \in S_2$. Since no S_i contains diametrically opposite points of P, we also have $m \leq t$ and $p_{m+t+1} \in S_1''$. This implies that $l(S_3) < l(S_2)$. Since neither S_1 nor S_2 contains diametrically opposite vertices of P, we also have that the endpoints of S_2 are interior points of two sides of P. The facts that S_2 contains exactly t vertices of P and S_1 is disconnected yield also that $S_2 \neq g_{12}(S_1)$ and $S_2 \neq g_{32}(S_3)$. Furthermore, $g_{32}(S_3)$ and S_2 are not slightly overlapping, since otherwise S_3 and $g_{23}(S_2)$ are slightly overlapping (cf. Remark 4), which contradicts the fact that in this case the side of P opposite the overlap is contained in S_2. By a similar argument, S_3 and $g_{13}(S_1)$ do not slightly overlap. Thus, we have that either S_1, $g_{21}(S_2)$ and $g_{31}(S_3)$ are mutually nonoverlapping, or S_1 and $g_{21}(S_2)$ slightly overlap.

If S_1, $g_{21}(S_2)$ and $g_{31}(S_3)$ are mutually nonoverlapping, then their total length is equal to $\text{perim}(P)$, implying that $\text{perim}(\text{conv}(D_1)) \geq \text{perim}(P)$. This yields that $D_1 = \text{conv}(D_1) = P$, which contradicts our assumption that $k = 3$

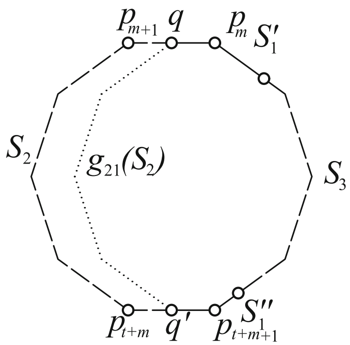

FIGURE 5. An illustration for Case 1 in sect. 3.2, where $t = 4$. In the picture S_1' and S_1'' are drawn with solid, S_2 and S_3 with dashed, and $g_{21}(S_2)$ with dotted lines

for any Type 2 decomposition of P. Thus, the only possibility left is that S_1 and $g_{21}(S_2)$ slightly overlap. Since the endpoints of S_2 are interior points of two opposite sides of P, this yields that $g_{21}(S_2)$ contains at least one endpoint of S_2. On the other hand, since apart from its endpoints, no point of S_1 may belong to D_2, we also have that $g_{21}(S_2)$ contains both endpoints of S_2, and also that it is a translate of S_2. Let the endpoints of S_2 on $[p_m, p_{m+1}]$ and $[p_{m+t}, p_{m+t+1}]$ be q and q', respectively. Then $g_{21}([p_{m+1}, q])$ is either $[p_m, q]$ or $[p_{m+t+1}, q']$, which yields that q is the midpoint of $[p_m, p_{m+1}]$ and q' is the midpoint of $[p_{m+t}, p_{m+t+1}]$ (cf. Fig. 5). On the other hand, since $g_{21}(S_2) \subset \partial D_1$, it separates D_2 from $D_1 \cup D_3$. In other words, D_2 is the region bounded by the union of S_2 and the part of $g_{21}(S_2)$ connecting q and q'. But this and the fact S_3 is not empty yields that D_1 is the translate of D_2 by the vector $q - p_{m+1}$, and hence, $D_3 = \mathrm{cl}(P \backslash (D_1 \cup D_2))$ is not congruent to D_1 and D_2; a contradiction.

Case 2, exactly one of S_1' and S_1'' contains a vertex of P. Let the vertices of P in S_1' be $p_1, p_2, \ldots, p_m$ for some $1 \leq m < \frac{n+1}{2}$.

First, we consider the case that n is odd, namely that $n = 2t + 1$ for some integer $t \geq 2$. Observe that since the diameter graph of the vertex set of P is an odd cycle, and hence, it cannot be colored with two colors, every S_i contains a vertex of P in its interior. We show that the side of P containing S_1'' is opposite a vertex in S_1'. Indeed, suppose for contradiction that $[p_{t+1}, p_{t+2}], \ldots, [p_{t+m}, p_{t+m+1}]$ are disjoint from S_1''. Then they all belong to the same S_i, and thus, we may assume that $p_{m+1}, \ldots, p_{t+m+1}$ belong to S_2. But then S_2 contains diametrically opposite vertices of P, which contradicts our assumption. Hence, we have that for some $1 \leq i \leq m$, $S_1'' \subset [p_{i+t}, p_{i+t+1}]$.

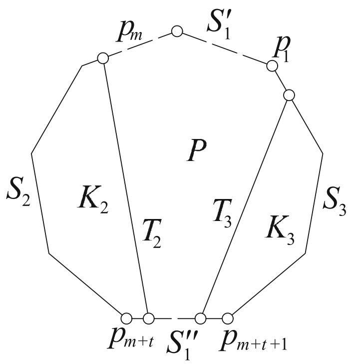

Figure 6. An illustration for Case 2 in Subsection 3.2, where $t = 4$ and $m = 2$. In the picture S_1' and S_1'' are drawn with dashed, and S_2 and S_3 are drawn with solid lines

Note the fact that S_1 is disconnected implies that $g_{1i}(S_1)$ does not coincide with S_i. Assume that, say, $g_{12}(S_1)$ slightly overlaps S_2. Then g_{12} is the composition of a symmetry of P and a (nondenegerate) translation parallel to a side of P, which contradicts the fact that $g_{12}(S_1) \subset P$. Thus, we have that S_1 does not overlap $g_{21}(S_2)$ and $g_{31}(S_3)$. Without loss of generality, we may assume that $l(S_2) \geq l(S_3)$. For $i = 2, 3$, let T_i denote the segment connecting the endpoints of S_i, $K_i = \text{conv}(S_i)$ and $C_i = \text{conv}(D_1) \cap K_i$ (cf. Fig. 6). Recall that $g_{21}(S_2)$ and $g_{31}(S_3)$ are contained in $\partial(\text{conv}(D_1))$ and they do not overlap S_1. Since S_2 and S_3 contain vertices of P, we also have that they do not overlap T_2 and T_3. Thus, in particular, C_2 or C_3 is a plane convex body with perimeter at least $l(T_2) + l(S_2)$ or $l(T_3) + l(S_2)$, respectively. As $\text{perim}(K_i) = l(S_i) + l(T_i)$ for $i = 2, 3$, this yields that $g_{21}(S_2)$ coincides with S_2 or S_3, a contradiction.

In the remaining part of Case 2, we assume that n is even, i.e. $n = 2t$ for some $t \geq 3$. Assume that S_1', S_2, S_1'', S_3 are in counterclockwise direction on ∂P. Note that at least one of S_2 and S_3 contains vertices of P. Furthermore, similarly to the n odd case, the fact that neither S_2 nor S_3 contains diametrically opposite points yields that there are points $q' \in S_1'$ and $q'' \in S_1''$ on opposite sides of P.

Clearly, $g_{12}(S_1) \neq S_2$ as S_1 is disconnected. Assume that they are slightly overlapping. Then $g_{12}(q')$ and $g_{12}(q'')$ lie on opposite sides of P, and hence, they belong to S_2. Thus, S_2 contains t vertices of P, and q', q'' and their images under g_{12} are on the same two sides of P. This yields that g_{12} is either a translation parallel to these sides, or the composition of such a translation with a reflection to a line parallel or perpendicular to these sides, or the origin. Let these two sides be E' and E'' with $q' \in E'$ and $q'' \in E''$. Then $g_{12}((E' \cup E'') \cap S_1) = (E' \cup E'') \cap S_2$. Since $E' \cap S_1$, $E' \cap S_2$, $E'' \cap S_2$ and $E'' \cap S_1$ are in counterclockwise order on ∂P, we have that g_{12} is a translation parallel to E', or its composition with the reflection about the line through o and parallel to E'. But both cases contradict the fact that $E'' \cap (S_1 \cup S_2)$ is strictly shorter than $E' \cap (S_1 \cup S_2) = E'$.

We have obtained that $g_{12}(S_1)$ and S_2 do not overlap. It follows similarly that $g_{13}(S_1)$ and S_3 do not overlap, or equivalently, that $g_{21}(S_2)$ and $g_{31}(S_3)$ do not overlap S_1. But in this case we may apply the argument used for n odd.

Case 3, neither S_1' nor S_1'' contains a vertex of P.
Recall that by the definition of Type 2 configuration (cf. Lemma 3), both S_1' and S_1'' are nondegenerate segments.

As we remarked in Case 2, if n is odd, then the diameter graph of the vertex set of P is an odd cycle, which is not 2-colorable. But this contradicts the assumptions that none of S_1, S_2, S_3 contains diametrically opposite vertices of P, and S_1 does not contain a vertex of P. Thus, the condition of Case 3 is satisfied only if n is even, and S_1' and S_1'' lie on opposite sides of P. Let these sides of P be E' and E'' with $S_1' \subset E'$.

Clearly, we have $g_{21}(S_2) \neq S_1$. Consider the case that $g_{21}(S_2)$ slightly overlaps S_1. Then the endpoints of $g_{21}(S_2)$ lie on E' and E'', which yields that either D_2 is the region bounded by $S_2 \cup (g_{21}(S_2) \setminus (E' \cup E''))$, or D_3 is the region bounded by $S_3 \cup (g_{21}(S_2) \setminus g_{21}(E' \cup E''))$. From these two cases we obtain $\text{area}(D_2) < \frac{1}{3}\,\text{area}(P)$ and $\text{area}(D_3) < \frac{1}{3}\,\text{area}(P)$, respectively, which contradicts the fact that the tiling is monohedral. Thus, we are left with the case that $g_{21}(S_2)$ and S_1 do not overlap. Similarly, we obtain that $g_{31}(S_3)$ and S_1 do not overlap. But then S_1, $g_{21}(S_2)$ and $g_{31}(S_3)$ are mutually nonoverlapping arcs in $\partial(\text{conv}(D_1))$ with total length equal to $\text{perim}(P)$, implying that $D_1 = \text{conv}(D_1) = P$; a contradiction.

3.3. Proof for Type 3 decompositions

The proof presented in this subsection roughly follows the structure of the proof in [13] with some of the arguments borrowed from there; in particular, depending on the number of coinciding arcs among $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$, we distinguish three cases.

Case 1, no two of $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$ coincide. If no two of these arcs overlap, then the fact that their total length is equal to $\text{perim}(P)$ yields that $D_1 = \text{conv}(D_1) = P$; a contradiction. Thus, we have that at least one pair among them overlaps. Before proceeding further, we use this observation to show that all of S_1, S_2 and S_3 contain at least one vertex of P different from their endpoints. Indeed, suppose for contradiction that one of them, say S_1, contains no vertex of P in its interior. Then, since D_1, D_2 and D_3 contain no diametrically opposite points of P, we have that n is even, and that both S_2 and S_3 contain a vertex of the side of P that contains S_1 and interior points of the opposite side. Thus, if $g_{21}(S_2)$ and S_1 slightly overlap, then S_1 intersects a pair of opposite edges of P in nondegenerate segments and the configuration is not Type 3; a contradiction. The cases that $g_{31}(S_3)$ slightly overlaps S_1 or $g_{21}(S_2)$ can be eliminated by a similar argument. Thus, we obtain that S_1, $g_{21}(S_2)$ and $g_{31}(S_3)$ do not overlap; a contradiction. Hence, in the remaining part of Case 1 we assume that all of S_1, S_2 and S_3 contain vertices of P different from their endpoints.

Consider the case that there are at least two overlapping pairs among $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$; without loss of generality, we may assume that $g_{21}(S_2)$ and $g_{31}(S_3)$ slightly overlap S_1. Then, for $i = 2, 3$, $g_{1i}(S_1)$ slightly overlaps S_i, which, combining it with the fact that the angles of P are obtuse, implies that two of $\partial D_1, \partial D_2$ and ∂D_3 cross; a contradiction.

We are left with the case that exactly one pair of $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$ overlaps. Without loss of generality, we may assume that S_1 and $g_{21}(S_2)$ overlap. Let us choose our notation such that the vertices of P on S_1 are $p_1, p_2, \ldots, p_m$, and S_1, S_2, S_3 are in counterclockwise order around P. For any $i \neq j$, let q_{ij} be the common endpoint of S_i and S_j. We assume that $q_{23} \in [p_l, p_{l+1}]$ with $q_{23} \neq p_{l+1}$. By our assumption, we have that $g_{21}(q_{12})$ or $g_{21}(q_{23})$ lies in the interior of S_1. Depending on which one of q_{12} and q_{23} lies on which side of S_1, we distinguish four cases.

If $g_{21}(q_{12})$ lies in the interior of S_1 and $g_{21}(q_{12}) \in [p_m, p_{m+1}]$, then q_{12} is the midpoint of $[p_m, p_{m+1}]$ and $g_{21}([q_{12}, p_{m+1}]) = [q_{12}, p_m]$. This implies that ∂D_1 and ∂D_2 cross at q_{12}; a contradiction.

Assume that $g_{21}(q_{12})$ lies in the interior of S_1 and $g_{21}(q_{12}) \in [p_n, p_1]$. Then $g_{21}(q_{12}) = p_1$, $g_{21}(p_{m+1}) = q_{13}$ (or equivalently, $g_{12}(p_1) = q_{12}$ and $g_{12}(q_{13}) = p_{m+1}$), and q_{12} and q_{13} are interior points of $[p_m, p_{m+1}]$ and $[p_n, p_1]$, respectively. Note that $\mathrm{conv}(D_2) \subset P$ implies that $\mathrm{conv}(D_1) \subset g_{21}(P)$. On the other hand, $g_{21}(P)$ is the translate of P by the vector $q_{13} - p_n$. Let $P_0 = P \cap (q_{12} - p_n + P)$ (cf. Fig. 7). Then P_0 is a convex polygon such that each one of its sides is parallel to some side of P. Let $C = S_1 \cup g_{21}(S_2)$, and observe that C and $g_{31}(S_3)$ are non-overlapping curves in $\partial(\mathrm{conv}(D_1))$. If q_{23} is not a vertex of P, then the total turning angle along the curves C and $g_{32}(S_3)$ is 2π, which implies that $\partial(\mathrm{conv}(D_1))$ is the union of C and $g_{31}(S_3)$ and possibly two segments such that the lines through them contain segments from both C and $g_{32}(S_3)$; this contradicts the fact that $\mathrm{conv}(D_1)$ is contained in P_0. If q_{23} is a vertex of P, we may apply the same argument after observing that $\mathrm{conv}(C \cup g_{31}(S_3))$ is a convex polygon, and the fact that $\mathrm{conv}(C \cup g_{31}(S_3)) \subset \mathrm{conv}(D_1) \subset P_0$ implies that the turning angle of $\mathrm{conv}(C \cup g_{31}(S_3))$ at $g_{21}(q_{23})$ is at least $\frac{2\pi}{n}$.

Finally, in the remaining cases, if $g_{21}(q_{23})$ lies in the interior of S_1 with $g_{21}(q_{23}) \in [p_m, p_{m+1}]$ or $g_{21}(q_{23}) \in [p_n, p_1]$, we can apply the argument in the previous case.

Case 2, exactly two of $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$ coincide. Without loss of generality, we may assume that $S_1 = g_{21}(S_2)$. In the consideration in this

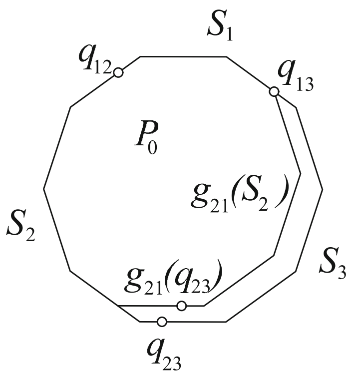

FIGURE 7. An illustration for Case 1 in Subsection 3.3

case, for brevity, for any $i \neq j$ we let q_{ij} denote the intersection point of S_i and S_j, let q denote the unique point in $D_1 \cap D_2 \cap D_3$, and set $C_{ij} = D_i \cap D_j$. By Lemma 3, we have that C_{ij} is a simple (possibly degenerate) curve connecting q_{ij} and q.

Note that since S_3 cannot contain more than $\frac{n}{2}$ vertices of P, we have that each of S_1 and S_2 contains at least $\frac{n}{4}$ and at most $\frac{n}{2}$ vertices of P. This and $n \geq 5$ implies that $g_{21}(P) = P$; that is, g_{21} is an isometry of P. In particular, g_{21} (and consequently g_{12}) is either a reflection about a symmetry line of P, or a rotation around o with angle $\alpha = \frac{2m\pi}{n}$ for some integer $1 \leq m \leq n$. Depending on the type of g_{21}, we distinguish two subcases.

Subcase 2.1, g_{21} is a rotation around o. Then the angle of rotation is $\alpha = \frac{2m\pi}{n}$ for some integer $\frac{n}{4} \leq m < \frac{n}{2}$, which implies, in particular, that $l(S_1) = l(S_2) = m$, and $l(S_3) = n - 2m$.

Observe that since o is a fixed point of g_{21}, we have either $o \in D_1 \cap D_2$ or $o \in \operatorname{int}(D_3)$. First, assume that $o \in D_1 \cap D_2$. If $o = q$, then $g_{21}(q_{12}) = q_{13}$, $g_{21}(q_{23}) = q_{12}$, and S_1 and S_3 are congruent, yielding that the tiling is rotationally generated. If $o \neq q$, then o has a closed circular neighborhood B disjoint from D_3. Let $t \mapsto C(t)$ be a continuous parametrization of the curve C_{12} with $C(0) = o$, and let $t^+ = \sup\{t : C([0,t]) \subset B\}$, and $t^- = \inf\{t : C([t,0]) \subset B\}$. Then $g_{21}(C(t^{\pm})) = C(t^{\mp})$, implying that g_{21} is a reflection about o, which contradicts the condition that the configuration is Type 3. Thus, we have $o \in \operatorname{int}(D_3)$.

Let $q_2 = g_{12}(q)$ and $q_1 = g_{21}(q)$. Then $q_2 \in \partial D_2$, $q_1 \in \partial D_1$ and $q_1, q_2 \notin \partial(P)$. Let $P_0 \subset D_3$ be the homothetic copy of P of maximum homothety ratio centered at o. Then P_0 touches at least one of the curves C_{13} and C_{23}, say, at a point $x_2 \in C_{23}$. Let $x_1 = g_{21}(x_2)$. By the definition of P_0, we have $x_1 \in D_3$, and by $x_2 \in D_2$, we have $x_1 \in D_1$. From this it follows that $x_1 \in C_{13}$, implying that $C_{13} \cap g_{21}(C_{23}) \neq \emptyset$. As $C_{12} \cup C_{13}$ is a connected curve from q_{12} to q_{13},

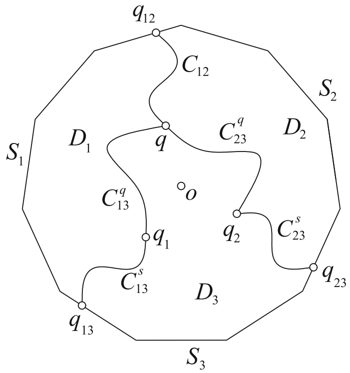

Figure 8. An illustration for Subcase 2.1 in Subsection 3.3 with $n = 11$ and $m = 4$

and $g_{21}(C_{23})$ is a connected curve in $C_{12} \cup C_{13}$ from q_{12} to q_1, the relation $C_{13} \cap g_{21}(C_{23}) \neq \emptyset$ implies that q_1 is an interior point of C_{13}, from which $q_2 \in int(C_{23})$ also follows. For $i = 1, 2$, let C^s_{i3} and C^q_{i3} denote the closed arcs of C_{i3} from q_{i3} to q_i, and from q_i to q, respectively (cf. Fig. 8). Then $g_{21}(C^s_{23}) = C_{12} = g_{12}(C^s_{13})$ and $g_{21}(C^q_{23}) = C^q_{13}$ yield that the corresponding arcs are congruent. Thus, by Lemma 6, as $\partial D_1 = S_1 \cup C^q_{13} \cup C^s_{13} \cup C_{12}$ and $\partial D_3 = S_3 \cup C^q_{13} \cup C^s_{13} \cup C^q_{23} \cup C^s_{23}$, the equidecomposability of ∂D_1 and ∂D_3 yields that $S_3 \cup C^q_{13}$ and S_1 are equidecomposable, implying that C^q_{13} (and also C^q_{23}) is a polygonal curve of length $l(S_1) - l(S_3)$. This yields, in particular, that $l(S_1) > l(S_3)$, $\alpha = \frac{2m\pi}{n} > \frac{2\pi}{3}$, $m > \frac{n}{3}$, and that S_1 contains at least one side of P.

Note that as g_{21} is a rotation around o, either all the q_{ij} are interior points of some edges of P, or all are vertices. Furthermore, $l(S_1) = l(S_2) > l(S_3)$, and $l(C^q_{13}) = l(C^q_{23})$ are positive integers, and if the q_{ij} are vertices of P, then $l(S_3) \geq 2$. We prove the assertion under the assumption that all the q_{ij} are interior points of some edges of P, as in the opposite case a slight modification of our argument can be applied.

Let us call a *copy* of S_i in ∂D_j a subset S of ∂D_j congruent to S_i such that S is not a proper subset of some connected curve $S' \subset \partial D_j$ with the property that the unique congruent copy of ∂P containing S also contains S', and observe that any two copies of S_i are either nonoverlapping or slightly overlapping. Recall that by Lemma 4, for any values of i and j, ∂D_j contains finitely many copies of S_i.

Consider the case that q is an interior point of a copy S of S_1 in ∂D_3. Let S' and S'' denote the parts of S in C^s_{13} and in C^s_{23}, respectively, and assume that q_1, q_2 are not interior points of S. Then $l(C^q_{13}) = l(S_1) - l(S_3) \leq m - 1 = l(S_1) - 1$ implies that S' and $g_{21}(S'')$ have a common vertex in their interiors.

Thus, they belong to the boundary of the same congruent copy P' of P. On the other hand, from this we obtain that g_{21} is a symmetry of P', and hence, P' is centered at o, a contradiction. If exactly one of q_1 or q_2, say q_2 is an interior point of S, then $S' \subseteq C^s_{13} = g_{21}(S'')$, which yields that S' contains a vertex or it is a segment of length strictly less than one. In the first case we can apply the previous consideration, and in the second case, since by the properties of rotation S'' contains a segment of length $l(S')$ that ends at q_2, we reach a contradiction with the fact that $l(C^s_{23}) = l(S_1) - l(S_3)$ is a positive integer. Finally, if both q_1 and q_2 are interior points of S, then, by the properties of rotation, $C^s_{13} \cup C^s_{23} \subset S$ yields that the unique regular n-gon that contains S in its boundary is P; a contradiction. Thus, we have that q does not belong to the interior of any copy of S_1 in D_3.

Note that the numbers of copies of S_1 in ∂D_1 and in ∂D_3 are equal. Furthermore, the number of copies of S_1 in ∂D_1 in $C^s_{13} \cup C_{12}$ is equal to this number in ∂D_3 in $C^s_{13} \cup C^s_{23}$. On the other hand, since $l(C^q_{13}) = l(C^q_{23}) < l(S_1)$ and q does not belong to the interior of a copy of S_1 in ∂D_3, it follows that the number of the copies of S_1 in ∂D_1 containing an element of $Q_1 = \{q_{12}, q_{13}, q, q_1\}$ in their interiors is one less than the number of copies in ∂D_3 containing an element of $Q_3 = \{q_{13}, q_{23}, q_1, q_2\}$ in their interiors.

Observe that for $i = 1, 3$ and any element of Q_i, there is at most one copy of S_1 in ∂D_i containing it in its interior. Furthermore, if the interior of every copy of S_1 in ∂D_3 contains at most one element of Q_3, then the number of copies of S_1 in ∂D_3 whose interiors intersect Q_3 is not larger than the number of copies of S_1 in ∂D_1 whose interiors intersect Q_1. Thus, we may assume that there is a copy S in ∂D_3 containing both q_{13} and q_1, or both q_{23} and q_2. In both cases, we have that S slightly overlaps S_3. Since the internal angle of D_3 at q_{13} or q_{23}, respectively, is obtuse, this yields that both conditions cannot be satisfied simultaneously. Hence, ∂D_3 contains exactly one copy of S_1, which slightly overlaps S_3. Then $g_{13}(S_1)$ coincides with this copy of S_1. Furthermore, ∂D_i is a closed polygonal curve for every value of i.

Assume that $q_{13} \in g_{13}(S_1)$, and let q' denote the vertex of P in S_1 closest to q_{12}. Since q is not an interior point of $g_{13}(S_1)$ and $l(C^q_{13}) = l(S_1) - l(S_3)$, we have that $l(C_{12}) = l(C^s_{13}) \geq l(S_3) - |q' - q_{12}|$. Thus, q is not an interior point of $g_{31}(S_3)$. On the other hand, since $l(S_1) - l(S_3) \geq 1$ and by the definition of a copy, the endpoint of $g_{31}(S_3)$ in ∂D_1, different from q', is not an interior point of C_{12}. Thus, it follows that this endpoint of C_{12} is q. Thus, $l(C_{12}) = l(C^s_{13}) = l(S_3) - |q' - q_{12}|$. This yields that $l(C^s_{13}) + l(C^q_{13}) = l(S_1) - |q' - q_{12}|$, and q is an endpoint of $g_{13}(S_1)$. Now we have completely described the boundaries of the D_i, in particular each consists of segments of unit length and some strictly shorter segments. A simple counting shows that ∂D_1 contains exactly 4 segments of length strictly smaller than 1 (2 in S_1, 1 in C_{12} and 1 in C^q_{13}), and ∂D_3 contains exactly 6 such segments (2 in S_3, 1 in C^q_{13}, 2 in C^q_{23} and 1 in C^s_{23}). This contradicts the fact that D_1 and D_3 are congruent.

If $q_{23} \in g_{13}(S_1)$, a similar consideration proves the assertion.

Subcase 2.2, g_{21} is a reflection about a symmetry line L of P. Without loss of generality, we assume that L is the y-axis, and the common point of S_1 and S_2 lies on the positive half of L. Note that $g_{12} = g_{21}$, and $D_3 = \mathrm{cl}(P \setminus (D_1 \cup D_2))$ is symmetric to L.

Clearly, by Remark 2, L separates D_1 and D_2, and from this it readily follows that $D_1 \cap D_2 = [q_{12}, q]$. Furthermore, we have that D_1 and D_2 are the closures of the subsets of $P \setminus D_3$ contained in the two closed half planes bounded by L.

Let $Y = g_{13}(S_1)$. Assume that Y slightly overlaps S_3. Then Y crosses L. By the symmetry of D_3, the reflected copy Y' of Y about L also belongs to ∂D_3 and it also crosses L. Thus, Y and Y' overlap, and $Y \cup Y'$ intersects L at a right angle. From this we have that $D_3 = \operatorname{conv} D_3$ is the convex region bounded by $S_3 \cup Y \cup Y'$, and an elementary computation yields that $\operatorname{area}(D_3) > \frac{\operatorname{area}(P)}{3}$; a contradiction. Thus, we have that Y does not overlap S_3.

Let Y' be the reflected copy of Y about L. If Y does not contain q in its interior, then the facts that $l(Y) + l(Y') + l(S_3) = l(P)$ and that Y, Y', S_3 are subsets of $\partial(\operatorname{conv}(D_3))$ yield that $D_3 = \operatorname{conv}(D_3) = P$, which contradicts our assumptions. Thus, Y and Y' overlap. If $Y \cap Y'$ contains a vertex in its interior, then $Y \cup Y'$ belongs to the boundary of a regular n-gon, implying that $g_{31}(Y \cup Y') \subset \partial P$. Hence, Y and Y' either slightly overlap, or they coincide.

Consider the case that Y and Y' slightly overlap, and let $E = Y \cap Y'$. Then $g_{31}(E) \subset g_{31}(Y) = S_1$ lies on the edge containing q_{12}, or the edge containing q_{13}. Since in the first case ∂D_1 and ∂D_2 cross, we have that $g_{31}(E)$ lies on the edge containing q_{13}; we remark the property that S_1 and $g_{31}(Y')$ slightly overlap with $q_{13} \in S_1 \cap g_{31}(Y')$ implies also that q_{13} is not a vertex of P.

Let L_1 be the supporting line of P parallel to L such that the infinite strip bounded by L and L_1 contains S_1. If $L_1 \cap P$ is disjoint from S_1, then D_3 contains diametrically opposite points of P, contradicting our assumptions. Thus, either $q_{13} \in L_1 \cap P$ or $L_1 \cap P$ belongs to the interior of S_1. If $q_{13} \in L_1 \cap P$, then $L_1 \cap P$ is a side of P, and both endpoints of $g_{31}(Y \cup Y')$ lie on L, implying that $D_1 = \operatorname{conv}(D_1) = \operatorname{conv}(g_{31}(Y \cup Y'))$, and thus, ∂D_1 does not contain a part congruent to S_3; a contradiction. Hence, we are left with the case that $L_1 \cap P$ belongs to the interior of S_1. Let $q' = g_{31}(q)$ and let L' be the line intersecting ∂P orthogonally at q' (cf. Fig. 9). Then q_{12} is a unique point in D_1 farthest from q'. On the other hand, by symmetry, the distances of the two endpoints of $g_{31}(Y \cup Y')$ from q' are the same. Since one of these endpoints is q_{12}, it follows that the two endpoints coincide, implying that $g_{31}(Y \cup Y')$ is a simple, closed, convex curve in ∂D_1. Thus, $D_1 = \operatorname{conv}(g_{31}(Y \cup Y'))$, or equivalently, $D_3 = \operatorname{conv}(Y \cup Y')$, which contradicts the assumption that $S_3 \subset \partial D_3$.

Finally, we consider the case that $Y = Y'$. Then Y is symmetric to L, yielding that S_1 is symmetric to the line $L' = g_{31}(L)$. Thus, L' is the bisector of either an edge or an angle of P, which implies that $o \in L'$. Since D_3 is

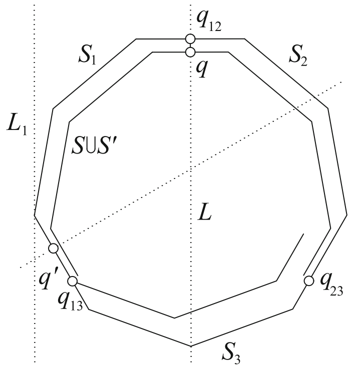

FIGURE 9. An illustration for Subcase 2.2 in Subsection 3.3

symmetric to L, we also have that D_1 is symmetric to L'. Let g be the reflection about L', and consider the transformations $g'_{12}, g'_{13}, g'_{23}$ defined by $g'_{12}(\cdot) = g_{12}(g(\cdot))$; $g'_{13} = g_{13}$ and $g'_{23}(\cdot) = g'_{13}(g'^{-1}_{12}(\cdot))$. Clearly, the transformation g'_{ij} is an isometry mapping D_i into D_j, and g'_{12} is a rotation around o. Thus, in this case we can apply the consideration in Subcase 2.1.

Case 3, all of $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$ coincide.

Since this yields that all of $S_1, g_{21}(S_2)$ and $g_{31}(S_3)$ contain vertices of P, it follows that g_{12} and g_{13} are symmetries of P. This implies that o is a fixed point of both g_{21} and g_{31}, and thus, the unique common point of D_1, D_2 and D_3 is o. If both g_{21} and g_{31} are rotations about o, then the tiling is clearly rotationally generated, and we are done. Assume that, e.g. g_{21} is a reflection about a symmetry line L of P. Then Remark 2 implies that L separates D_1 and D_2, from which we obtain that the curve $D_1 \cap D_2$ is a straight line segment connecting the common point of S_1 and S_2 to o. By the properties of rotations, from this we also have that for any $i \neq j$, $D_i \cap D_j$ is a straight line segment connecting the common point of S_i and S_j to o. This, combined with the fact that $l(S_1) = l(S_2) = l(S_3)$, readily yields that the tiling is rotationally generated.

4. Concluding remarks and open questions

Remark 5. A simplified version of the proof of Theorem 1 can be applied to prove the same statement for monohedral tilings of a regular triangle with at most three tiles.

The authors have found no results about monohedral tilings of convex polygons in spherical or hyperbolic planes. This is our motivation to state the following problem. Before doing so, we note that the symmetry group of an equiangular convex quadrilateral in spherical or hyperbolic planes contains the symmetry group of a Euclidean rectangle as a subgroup.

Problem 1. *Let* $\mathbb{M}^2$ *denote the spherical plane* $\mathbb{S}^2$ *or the hyperbolic plane* $\mathbb{H}^2$*, and let* $P \subset \mathbb{M}^2$*. Characterize the monohedral tilings of* P *with at most three discs if* P *is a*

(i) circular disc;
(ii) a regular polygon;
(iii) an equiangular convex quadrilateral.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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