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Invariant affine subspaces

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Dedicated to Professors Maciej Sablik and László Székelyhidi on the occasion of their 70th birthdays.

Abstract. We study the problem: whether a given operator on a Banach space has an invariant affine subspace. We examine the existence or non-existence of such affine subspaces. Moreover, we give some applications.

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1. Introduction

Does every operator on a Banach space have an invariant subspace? This is the "invariant subspace problem". The problem which we consider in this paper is similar, but not equivalent to the "invariant subspace problem". We introduce and study the modified version of the Invariant Subspace Problem: whether a given operator on a Banach space has a closed invariant affine subspace. We solve this problem for a large class of operators which includes isometries and the shift operators on ℓ_2 and ℓ_1. We apply this to the problem of reflexivity of Banach spaces. Moreover, we prove that every linear mapping preserving trace of matrices has a nonzero eigenvector with an eigenvalue which equals 1.

Let X be a Banach space over the field $\mathbb{R}$. The Banach space of all bounded linear operators from X to X is denoted by $\mathcal{B}(X)$. By X^* we denote all the continuous linear functionals on X. Now, we consider an operator $A \in \mathcal{B}(X)$. If $x^* \in X^*$, then $x^* \circ A\colon X \to \mathbb{R}$ is easily seen to be a continuous linear functional on X. That is, $x^* \circ A \in X^*$. This defines a bounded linear operator $A^* \in \mathcal{B}(X^*)$ by $A^*(x^*) := x^* \circ A$. It is well known that $\|A\| = \|A^*\|$.

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Let M be a subspace of X and let $b\in X$. The set $b+M := \{b+m : m \in M\}$ is called an *affine subspace*. The dimension of $b+M$ is defined by $\dim(b+M) := \dim M$. Similarly, the codimension of $b+M$ is defined by $\operatorname{codim}(b+M) := \operatorname{codim} M$. We say that an affine subspace $b+M$ is *nontrivial*, if $\dim M \geqslant 1$ and $b \notin M$ or, equivalently, $\dim M \geqslant 1$ and $0 \notin b+M$. It means that $b+M$ is not a linear subspace. It is easy to verify that

$$b+M \subseteq c+K \quad \Leftrightarrow \quad b-c \in K \ \text{ and } \ M \subseteq K. \tag{1.1}$$

It is clear that $b+M$ is closed if and only if M is closed. Moreover, $\overline{b+M} = b+\overline{M}$. It is a well-known property of real vector spaces that a non-empty subset $E \subseteq X$ is an affine subspace if and only if $\{tx+(1-t)y : t\in\mathbb{R}\} \subseteq E$ for all $x, y \in E$.

Let $x^* \in X^*$ be a fixed nonzero linear functional and let $\alpha \in \mathbb{R}\setminus\{0\}$ be a fixed real number. Then, the set $\mathcal{K} := \{x \in X : x^*(x) = \alpha\}$ is a nontrivial affine subspace. Indeed, there is a vector $p \in X$ such that $x^*(p) \neq 0$. We can calculate that $\mathcal{K} = \frac{\alpha}{x^*(p)}p + \ker x^*$. The set $\ker x^*$ is a subspace and $\frac{\alpha}{x^*(p)}p \notin \ker x^*$. Therefore $\mathcal{K}$ is a nontrivial affine subspace and $\operatorname{codim} \mathcal{K} = 1$.

If $B \subseteq X$, the *affine hull* of B, denoted by $\operatorname{Aff}(B)$, is the intersection of all the affine subspaces that contain B. The *closed affine hull* of B is the intersection of all closed affine subspaces of X that contain B; it is denoted by $\overline{\operatorname{Aff}}(B)$. Clearly $\operatorname{Aff}(B) \subseteq \operatorname{Lin}(B)$ and $\overline{\operatorname{Aff}}(B) \subseteq \overline{\operatorname{Lin}}(B)$. Moreover, an affine hull of B is an affine subspace. It is known that

$$\operatorname{Aff}(B) = \left\{ \sum_{k=1}^{n} \lambda_k b_k \ \ n\in\mathbb{N},\, b_1,\ldots,b_n \in B,\, \lambda_1,\ldots,\lambda_n \in \mathbb{R}, \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

If X, Y are Banach spaces, a function $\varphi : X\times Y \to \mathbb{R}$ is a *bilinear form* if

(i) $\varphi(x_1+x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$ for $x_1, x_2 \in X$ and $y \in Y$,

(ii) $\varphi(x, y_1+y_2) = \varphi(x, y_1) + \varphi(x, y_2)$ for $x \in X$ and $y_1, y_2 \in Y$,

(iii) $\varphi(\alpha x, y) = \alpha\cdot\varphi(x, y)$ and $\varphi(x, \beta y) = \beta\cdot\varphi(x, y)$ for $\alpha, \beta \in \mathbb{R}$.

A bilinear form is *bounded* if there is a constant η such that

(iv) $|\varphi(x, y)| \leqslant \eta\|x\|\cdot\|y\|$ for all $x, y \in X$.

The constant η is called a *bound* for φ.

Consider a Banach space X. Define $\langle\cdot,\cdot\rangle : X\times X^* \to \mathbb{R}$ by $\langle x, y^*\rangle := y^*(x)$. It is clear that $\langle\cdot,\cdot\rangle$ is a bounded bilinear form with bound 1. The next two theorems are known (see [2, pp. 18, 19, 20] and [3, pp. 58, 59, 60]).

Theorem 1.1. *If $\varphi : X \times X^* \to \mathbb{R}$ is a bounded bilinear form with bound η, then there is a unique operator $U \in \mathcal{B}(X^*)$ such that $\varphi(x, y^*) = \langle x, U(y^*)\rangle$ for all x in X and y^* in X^* and $\|U\| \leqslant \eta$.*

Theorem 1.2. *Let X be a reflexive Banach space. If $\varphi : X \times X^* \to \mathbb{R}$ is a bounded bilinear form with bound η, then there is a unique operator $T \in \mathcal{B}(X)$ such that*

$$\varphi(x, y^*) = \langle T(x), y^* \rangle = \langle x, T^*(y^*) \rangle$$

for all x in X and y^* in X^* and $\|T\| \leqslant \eta$.

2. Main results - part I

In this section, we introduce and study affine subspaces that are invariant. Let A be the operator on X. We say that $b+M$ is an *invariant affine subspace* for A if $A(b+M) \subseteq b+M$. It is easy to see that

$$A(b+M) \subseteq b+M \quad \Leftrightarrow \quad b - A(b) \in M \;\text{ and }\; A(M) \subseteq M. \tag{2.1}$$

So, if A has a nontrivial closed invariant affine subspace, then A has a closed invariant subspace. But the converse is not true. Define $T \in \mathcal{B}(\mathbb{R}^2)$ by $T(x, y) := \frac{1}{2}(x, y)$. It follows that the space $\mathrm{Lin}\{\lambda(1,1) : \lambda \in \mathbb{R}\}$ is a closed invariant subspace for T. But there is no nontrivial closed invariant affine subspace for T. Indeed, we have the following result.

Theorem 2.1. *Let X be a Banach space. Let an operator $A \in \mathcal{B}(X)$ satisfy $\|A\| < 1$ and $A \neq 0$. Then, there is no nontrivial closed invariant affine subspace for A.*

Proof. Suppose that $A(b+M) \subseteq b+M$ is true for some b, M such that $0 \notin b+M$. Assume that $b+M$ is closed. Fix $x_o \in b+M$. Then $\|A^k(x_o)\| \leqslant \|A\|^k \cdot \|x_o\| \to 0$, so $A^k(x_o)$ tends to 0. But, $A^k(x_o) \in b+M$, so $0 \in b+M$. This is a contradiction. □

Now we will prove the finite-dimensional case.

Theorem 2.2. *Let X be a finite-dimensional real vector space with dim $X \geqslant 2$ or more. If A is an operator on X such that rank $A \geqslant 1$, then it has a nontrivial invariant affine closed subspace if and only if 1 is an eigenvalue for A.*

Proof. For the proof of "$\Rightarrow$", fix an arbitrary closed invariant affine subspace $b+M$ of A and assume that $b \notin M$. First, we show that $(I-A)(b+M) \subseteq M$. Fix $m \in M$. Using (1.1) and (2.1) we get $m - Am \in M$ and $b - Ab \in M$. Thus, $(I-A)(b+m) = b - Ab + m - Am \in M$.

Suppose, for a contradiction, that 1 is not an eigenvalue for A, i.e., $Aw \neq w$ for all $w \in X \backslash \{0\}$. Since $\dim X < \infty$, the operator $I - A$ is invertible. Therefore $\dim(I-A)(b+M) = \dim(b+M) = \dim M$. Since $(I-A)(b+M) \subseteq M$, we obtain $(I-A)(b+M) = M$. Thus, there exists a vector $m_o \in M$ such that $(I-A)(b+m_o) = 0$. Hence $A(b+m_o) = b+m_o$. Since $b \notin M$, we have $b + m_o \neq 0$. This contradicts the fact that $Aw \neq w$ for all $w \in X \setminus \{0\}$.

Now, we prove "$\Leftarrow$". Fix $w \in X \setminus \{0\}$ such that $Aw = w$. This leads to two cases. *Possibility 1.* $Ax = x$ for all $x \in X$. In this case, we may consider any closed affine subspace $\mathcal{K} \subseteq X$ with $0 \notin \mathcal{K}$. Then $A(\mathcal{K}) \subseteq \mathcal{K}$.

Possibility 2. $Ax_o \neq x_o$ for some $x_o \in X$. Then we get $x_o \notin \ker(I - A) \ni w$. Thus we have $\ker(I - A) \subsetneq X$ and $1 \leqslant \dim \ker(I - A)$. Since $\dim X < \infty$, we obtain $\dim(I - A)(X) < \dim X$. Finally, it is easy to check that an inclusion $A\big(w + (I - A)(X)\big) \subseteq w + (I - A)(X)$ holds, and we are done. $\square$

In the next section we will prove a similar result. Namely, we will discuss the infinite-dimensional case. And in the last section we will show that every linear mapping preserving trace of matrices has some nonzero eigenvector.

3. Main results - part II

If ℓ_∞ denotes the normed space of all bounded real sequences then its norm is defined by $\|x\|_\infty := \sup\{|x_n| : n = 1, 2, \ldots\}$ for any bounded real sequence $x = (x_1, x_2, \ldots)$. One can restrict this norm to the subspace c of all convergent real sequences. We need the following well-known theorem for further investigations. The proof can be found, e.g., in [1].

Theorem 3.1. *There is a linear functional $L : \ell_\infty \to \mathbb{R}$ such that*

(BL1) $\|L\| = 1$.
(BL2) *If* $x = (x_1, x_2, \ldots) \in c$, *then* $L(x) = L(x_1, x_2, \ldots) = \lim x_n$.
(BL3) *If* $x = (x_1, x_2, \ldots) \in \ell_\infty$, *then* $L(x_1, x_2, x_3 \ldots) = L(x_2, x_3, x_4 \ldots)$.
(BL4) *If* $x_n \geqslant y_n$ *for all* $n \in \mathbb{N}$, *then* $L(x_1, x_2, \ldots) \geqslant L(y_1, y_2, \ldots)$.

A linear functional of the type described in Theorem 3.1 is called a *Banach limit*.

From now on we assume that the considered Banach spaces are real and their dimensions are not less than 2. The following lemma will be useful in the proof of the main result.

Lemma 3.2. *Let X be a Banach space and let $A \in \mathcal{B}(X)$. Suppose that $A \neq 0$. If for some $y^* \in X$, $y^* \neq 0$ and for all $x \in X$*

$$\langle Ax, y^* \rangle = \langle x, y^* \rangle, \tag{3.1}$$

then there exists a nontrivial closed invariant affine subspace $\mathcal{K}$ for A such that codim $\mathcal{K} = 1$, *where $\mathcal{K} = \{x \in X : \langle x, y^* \rangle = 1\}$. Moreover, $A^*(y^*) = y^*$.*

Proof. It is easy to see that $\langle \cdot, y^* \rangle \in X^*$ and $\langle \cdot, y^* \rangle \neq 0$. We consider the set $\mathcal{K} := \{x \in X : \langle x, y^* \rangle = 1\}$. As we have observed, the set $\mathcal{K}$ is a nontrivial affine subspace and codim $\mathcal{K} = 1$. We can calculate that $A(\mathcal{K}) \subseteq \mathcal{K}$.

Note, that $\langle Ax, y^* \rangle = y^*(Ax) = (y^* \circ A)(x) = \left(A^*(y^*)\right)(x) = \langle x, A^*(y^*) \rangle$. Now, using (3.1) we can obtain $\forall_{x \in X} \;\; \langle x, A^*(y^*) \rangle = \langle x, y^* \rangle$ and then $\forall_{x \in X} \;\; \left(A^*(y^*)\right)(x) = y^*(x)$. Finally, we have $A^*(y^*) = y^*$. $\square$

Bearing the above proof in mind, we see that the following Lemma can be shown similarly.

Lemma 3.3. *Let $D \in \mathcal{B}(X^*)$. Suppose that $D \neq 0$. If for some $x \in X\backslash\{0\}$ and all $y^* \in X^*$ we have that:*

$$\langle x, Dy^* \rangle = \langle x, y^* \rangle, \tag{3.2}$$

then there exists a nontrivial closed invariant affine subspace $\mathcal{K}$ for D such that $\operatorname{codim} \mathcal{K} = 1$.

From now on we consider the operators $A \in \mathcal{B}(X)$ satisfying the following property: there is $\alpha > 0$ such that $\|A^n\| \leqslant \alpha$ for all $n \in \mathbb{N}$. Then we say that A is a *power bounded operator*. For example, if $\|S\| \leqslant 1$, then S satisfies this property. In particular, every isometry $U \in \mathcal{B}(X)$ satisfies $\|U^n\| \leqslant 1$. Let us consider an operator $W \in \mathcal{B}(\ell_2)$ such that $W(x_1, x_2, x_3, x_4, \ldots) := (\frac{1}{2}x_2, 2x_1, x_3, x_4, \ldots)$. It is easy to check that $\|W\| > 1$, but W is also power bounded.

Now we can state and prove the main results of this paper.

Theorem 3.4. *Let X be a Banach space and let $A \in \mathcal{B}(X)$ be a power bounded operator. Suppose that there are $u \in X$, $w^* \in X^*$ and $\beta > 0$ such that $w^*(A^n(u)) \geqslant \beta$ for all $n \in \mathbb{N}$. Then there exists a nontrivial closed invariant affine subspace $\mathcal{K}$ for A such that* $\operatorname{codim} \mathcal{K} = 1$. *Moreover, there is $d^* \in X^* \backslash \{0\}$ such that $A^*(d^*) = d^*$.*

Proof. Fix a Banach limit L on ℓ_∞. Define the mapping $\varphi : X \times X^* \to \mathbb{R}$ by

$$\varphi(x, y^*) := L\left(y^*(x), y^*(Ax), y^*(A^2x), y^*(A^3x), \ldots\right) \quad \text{for} \quad x \in X, \ y^* \in X^*.$$

By assumption, there is $\alpha > 0$ such that $\|A^n\| \leqslant \alpha$ for all $n \in \mathbb{N}$. Fix $x \in X$ and $y^* \in X^*$. Note, that $|y^*(A^n x)| \leqslant \|y^*\| \cdot \|A^n\| \cdot \|x\| \leqslant \|y^*\| \cdot \alpha \cdot \|x\|$ for all $n \in \mathbb{N}$. Thus $\left(y^*(x), y^*(Ax), y^*(A^2x), y^*(A^3x), \ldots\right) \in \ell_\infty$. So, φ is a well-defined function. It is easy to check that the above mapping is a bounded bilinear form. Furthermore, for all $x \in X$ and $y^* \in X^*$, we have

$$\begin{aligned} \varphi(x, y^*) &= L\left(y^*(x), y^*(Ax), y^*(A^2x), \ldots\right) \\ &\overset{\text{(BL3)}}{=} L\left(y^*(Ax), y^*(A^2x), y^*(A^3x), \ldots\right) \\ &= \varphi(Ax, y^*). \end{aligned} \tag{3.3}$$

By Theorem 1.1 there is an operator $U \in \mathcal{B}(X^*)$ such that

$$\varphi(x, y^*) = \langle x, U(y^*) \rangle$$

for all $x \in X$ and $y^* \in X^*$. Now the condition (3.3) becomes

$$\langle Ax, U(y^*) \rangle = \langle x, U(y^*) \rangle. \tag{3.4}$$

Note that $\varphi(u, w^*) = L(w^*(u), w^*(Au), w^*(A^2u), \ldots) \overset{\text{(BL4)}}{\geqslant} L(\beta, \beta, \ldots) \overset{\text{(BL2)}}{=} \beta > 0$. This proves that $\varphi \neq 0$. Thus $U \neq 0$; hence there is $z^* \in X^*$ such that $U(z^*) \neq 0$. From (3.4) we get

$$\forall_{x \in X} \quad \langle Ax, U(z^*) \rangle = \langle x, U(z^*) \rangle. \tag{3.5}$$

Applying Lemma 3.2 and (3.5) we obtain the assertion. □

We prove that the invariant affine subspace can be extended to the greatest invariant affine subspace.

Theorem 3.5. *Let X be a Banach space and let $A \in \mathcal{B}(X)$ be a power bounded operator. Suppose that there exists a nontrivial closed invariant affine subspace $b+M$. Then there exists a nontrivial closed invariant affine subspace $\mathcal{K}$ for A such that $b+M \subseteq \mathcal{K}$ and* $\operatorname{codim} \mathcal{K}=1$.

Proof. There exists $z^* \in X^* \setminus \{0\}$ such that $z^*(b) \neq 0$ and $M \subseteq \ker z^*$. Let us define $\beta := z^*(b) \neq 0$. Therefore

$$b+M \subseteq b+\ker z^*. \tag{3.6}$$

Fix a Banach limit L on ℓ_∞. We define the mapping $\varphi : X \times X^* \to \mathbb{R}$ by

$$\varphi(x,y^*) := L\left(y^*(x), y^*(Ax), y^*(A^2x), y^*(A^3x), \ldots\right) \quad \text{for} \quad x \in X,\ y^* \in X^*.$$

In a similar way as in the proof of Theorem 3.4 we get the continuity of φ. Using again (BL3), we get

$$\varphi(Ax,z^*) = \varphi(x,z^*) \quad \text{for all} \quad x \in X. \tag{3.7}$$

Let us now define the closed affine subspace $\mathcal{K} := \{x \in X : \varphi(x,z^*) = \beta\}$. By (3.7) we have $A(\mathcal{K}) \subseteq \mathcal{K}$. It is easy to see that $\operatorname{codim} \mathcal{K} = 1$. Moreover, it is clear that $0 \notin \mathcal{K}$; so $\mathcal{K}$ is nontrivial.

We show that $b+M \subseteq \mathcal{K}$. Fix $v \in b+M$. Combining the inclusion (3.6) and the inclusion $A(b+M) \subseteq b+M$ we have

$$\forall_{x \in b+M} \forall_{n=0,1,2,\ldots} \quad A^n(x) \in b+\ker z^*.$$

Therefore

$$\forall_{x \in b+M} \forall_{n=0,1,2,\ldots} \quad z^*(A^n(x)) = \beta, \tag{3.8}$$

whence

$$\varphi(v,z^*) = L\left(z^*(v), z^*(Av), z^*(A^2v), \ldots\right) \overset{(3.8)}{=} L(\beta,\beta,\beta,\ldots) \overset{(\mathrm{BL2})}{=} \beta,$$

so $v \in \mathcal{K}$. This means $b+M \subseteq \mathcal{K}$. □

Theorem 3.6. *Let X be a Banach space and let $A \in \mathcal{B}(X)$ be a power bounded operator. Then the following conditions are equivalent:*

(a) *there exists a nontrivial closed invariant affine subspace for A;*
(b) *there exists a nontrivial closed invariant affine subspace K for A such that* $\operatorname{co\,dim} K = 1$.

Moreover, each of the above conditions implies

(c) *there is $d^* \in X^* \setminus \{0\}$ such that $A^*(d^*) = d^*$.*

Proof. Clearly (b)⇒(a). By Theorem 3.5 we have (a)⇒(b). For the proof of (a)⇒(c) fix arbitrarily a nontrivial closed affine subspace $b + M$ such that

$$A(b + M) \subseteq b + M. \tag{3.9}$$

The set $\{0\}$ is compact and convex. The set $b + M$ is closed and convex. Moreover, we have $\{0\} \cap b + M = \emptyset$. From the Separation Theorem, there are $w^* \in X^*$ and $\beta > 0$ such that

$$w^*(b + M) \subseteq [\beta, +\infty). \tag{3.10}$$

Fix $u \in b+M$. Using (3.9) and (3.10) we obtain $w^*(A^n(u)) \geqslant \beta$ for all $n \in \mathbb{N}$. Now by Theorem 3.4 we obtain (c). □

In order to summarize our considerations, we formulate below theorem.

Theorem 3.7. *Let X be a reflexive Banach space and let $A \in \mathcal{B}(X)$ be a power bounded operator. Then the following conditions are equivalent:*

(a1) *there are $u \in X$, $w^* \in X^*$ and $\beta > 0$ such that $w^*(A^n(u)) \geqslant \beta$ for all $n \in \mathbb{N}$;*
(a2) *there are $u \in X$, $w^* \in X^*$ and $\beta > 0$ such that $(A^{*^n} w^*)(u) \geqslant \beta$ for all $n \in \mathbb{N}$;*
(b1) *there is a nontrivial closed invariant affine subspace for A;*
(b2) *there is a nontrivial closed invariant affine subspace K for A such that* $\operatorname{codim} K = 1$*;*
(b3) *there is a nontrivial closed invariant affine subspace for A^*;*
(b4) *there is a nontrivial closed invariant affine subspace K for A^* such that* $\operatorname{codim} K = 1$*;*
(c1) *there is $c \in X \setminus \{0\}$ such that $Ac = c$;*
(c2) *there is $d^* \in X^* \setminus \{0\}$ such that $A^*(d^*) = d^*$.*

Since all the equivalences are either obvious or proved before, we omit the proof here. However, we will show that the reflexivity is necessary. So, as an illustration of the applications of Theorem 3.7 we obtain the following result.

Theorem 3.8. *Let X be a Banach space and let $A \in \mathcal{B}(X)$ be a bounded operator. Assume that there is a nontrivial closed invariant affine subspace for A. Assume that $Ax \neq x$ for all $x \in X \setminus \{0\}$. Then the space X is not reflexive.*

Proof. Suppose that the space X is reflexive. Applying (b1)⇔(c1) (see Theorem 3.7), we get $Ac = c$ for some $c \in X \setminus \{0\}$. This contradiction shows that X is not reflexive. □

We have obtained a new method: to show that a given space is not reflexive, it suffices to define a suitable operator. As a corollary we prove here a well known property of the space ℓ_1. The next result may be known to the reader, but it is proved here in a new manner.

Remark 3.9. The space ℓ_1 is not reflexive.

In order to prove this fact, we define an operator $A\colon \ell_1 \to \ell_1$ by $A(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. It is easy to check that A is linear isometry. Therefore, we have $\|A^n\| \leqslant 1$ for all $n \in \mathbb{N}$. Let us consider $e_1 := (1, 0, 0, \ldots)$, $e_2 := (0, 1, 0, 0, \ldots), \ldots \in \ell_1$. Define $E := \{e_k : k \in \mathbb{N}\}$.

It is easy to prove that $A(\mathrm{Aff}(E)) \subseteq \mathrm{Aff}(E)$. The continuity of A asserts that $A(\overline{\mathrm{Aff}}(E)) \subseteq \overline{\mathrm{Aff}}(E)$. We show that the affine subspace $\overline{\mathrm{Aff}}(E)$ is nontrivial. Fix an arbitrary vector $x \in \mathrm{Aff}(E)$. Then, $x = \sum_{k=1}^{n} \lambda_k e_k$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{k=1}^{n} \lambda_k = 1$. Thus we have

$$\|x\|_1 = \left\| \sum_{k=1}^{n} \lambda_k e_k \right\|_1 = \sum_{k=1}^{n} |\lambda_k| \geqslant \sum_{k=1}^{n} \lambda_k = 1,$$

and hence $\mathrm{dist}(0, \mathrm{Aff}(E)) \geqslant 1$. It follows that $\mathrm{dist}(0, \overline{\mathrm{Aff}}(E)) \geqslant 1$. Since $0 \notin \overline{\mathrm{Aff}}(E)$, the invariant affine subspace $\overline{\mathrm{Aff}}(E)$ is nontrivial.

Moreover, it is easy to check that $A(u) \neq u$ for any $u \in \ell_1 \backslash \{0\}$. Applying Theorem 3.8, ℓ_1 is not reflexive.

Remark 3.10. It follows from Remark 3.9 that reflexivity of X is necessary to obtain Theorem 3.7. Indeed, the implications (b1)$\Rightarrow$(c1) and (b2)$\Rightarrow$(c1) do not hold for the operator $A(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$, where $A\colon \ell_1 \to \ell_1$.

4. Applications

We are interested in the applications of Theorems 2.2, 3.4, 3.6 and 3.7. Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ matrices with real entries. Recall that given an $n \times n$ matrix $M = [a_{ij}]$, its *trace* is a number $\mathrm{tr}M := \sum\limits_{k=1}^{n} a_{kk}$.

Linear maps of $M_n(\mathbb{R})$ into itself, which preserve trace, are connected with the known from the literature *preservers problem.* There are neither precise nor useful characterizations of such maps. Perhaps, it can be explained, if we see the three following examples.

Example 4.1. Let us define the first linear mapping $A\colon M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by the following formula

$$A\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) := \begin{bmatrix} 2a_{33} & 0 & a_{13} - a_{22} \\ 0 & -a_{33} & 0 \\ 0 & 5a_{32} + a_{31} & a_{11} + a_{22} \end{bmatrix}$$

for all $M = [a_{ij}] \in M_3(\mathbb{R})$. Clearly, we get $\mathrm{tr}M = \mathrm{tr}A(M)$ for all $M \in M_n(\mathbb{R})$.

Example 4.2. Let us consider the second linear mapping $B\colon M_3(\mathbb{R}) \to M_3(\mathbb{R})$ given by the following formula

$$B\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) := \begin{bmatrix} \mathrm{tr}M & 0 & a_{32} + 4a_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\mathrm{tr}M = \mathrm{tr}B(M)$ for all $M \in M_n(\mathbb{R})$.

Example 4.3. Finally, let us define the third linear mapping $C\colon M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by the following formula

$$C\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) := \begin{bmatrix} -2\mathrm{tr}M & 0 & 0 \\ 0 & 4\mathrm{tr}M & a_{33} \\ a_{12} + \mathrm{tr}M & 0 & -\mathrm{tr}M \end{bmatrix}.$$

Similarly, it is obvious again that $\mathrm{tr}M = \mathrm{tr}C(M)$ for all $M \in M_n(\mathbb{R})$.

So, the above examples showed that it may be hard to find a convenient characterization of such mappings. However, we prove in this section that every linear mapping preserving trace of matrices has some eigenvector.

Theorem 4.4. *Let $A\colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be linear. Suppose that A preserves trace. Then there is a nonzero matrix $[w_{ij}] \in M_n(\mathbb{R})$ such that $A\big([w_{ij}]\big) = [w_{ij}]$.*

Proof. It is easy to see that a subset $\mathcal{K} := \big\{W \in M_n(\mathbb{R}) : \mathrm{tr}(W) = 1\big\}$ is a closed affine subspace and $[0] \notin \mathcal{K}$. By assumption, we have $\mathrm{tr}A\big(U\big) = \mathrm{tr}U$. From this we get $A(\mathcal{K}) \subseteq \mathcal{K}$. Now we can apply Theorem 2.2, and the proof is complete. □

The next result is an easy observation. If $N \in \mathcal{B}(X)$ is a nilpotent operator, then N may have a closed invariant subspace. But it has no closed nontrivial affine subspace. Indeed, we have the following result.

Corollary 4.5. *Let $N \in \mathcal{B}(X)$ be nilpotent, i.e., $N^k = 0$ for some $k \in \mathbb{N}$. Then there is no nontrivial invariant affine subspace for N.*

Proof. Assume, contrary to our claim, that there is a nontrivial closed invariant affine subspace for N. Then there is a $d^* \in X^* \backslash \{0\}$ such that $N^*(d^*) = d^*$ (see Theorem 3.6). On the other hand it is easy to verify that $(N^*)^k = 0$. Therefore, $0 = (N^*)^k(d^*) = d^* \neq 0$ and we obtain a contradiction. □

The next theorem shows that every orbit is close, in a certain sense, to the zero vector.

Theorem 4.6. *Let X be a reflexive Banach space and let $A \in \mathcal{B}(X)$ be a power bounded operator. Assume that $Ax \neq x$ for all $x \in X$. Then, for an arbitrary subset $F \subseteq X$ we have $0 \in \overline{\mathrm{conv}}\,\{A^n(F) : n = 1, 2, \ldots\}$.*

Proof. Assume, contrary to our claim, that $0 \notin \overline{\mathrm{conv}}\,\{A^n(F) : n = 1, 2, \ldots\}$ for some $F \subseteq X$. Then there is some $w^* \neq 0$ and some $\beta > 0$ (using the Separation Theorem) such that

$$\overline{\mathrm{conv}}\,\{A^n(F) : n = 1, 2, \ldots\} \subseteq \{x \in X : w^*(x) \geqslant \beta\}. \tag{4.1}$$

Fix $u \in F$. Using (4.1) we obtain $w^*(A^n(u)) \geqslant \beta$ for all $n \in \mathbb{N}$. By again applying Theorem 3.7 we get $Ax_o = x_o$ for some $x_o \neq 0$, which is a contradiction. □

Let us consider the well known unilateral shift on ℓ_2. We show that this operator does not have a closed nontrivial affine invariant subspace.

Theorem 4.7. *If $S\colon \ell_2 \to \ell_2$ is defined by $S(y_1, y_2, \ldots) := (0, y_1, y_2, \ldots)$, then there is no nontrivial closed invariant affine subspace for S.*

Proof. We know that S is an isometry. Therefore, we have $\|S^n\| \leqslant 1$ for all $n \in \mathbb{N}$. It is easy to check that $S(u) = u \Leftrightarrow u = 0$. In particular, from implication (b1)⇒(c1) we derive the result. □

Corollary 4.8. *If $V\colon \ell_2 \to \ell_2$ is defined by $V(y_1, y_2, \ldots) := (y_2, y_3, y_4, \ldots)$, then there is no nontrivial closed invariant affine subspace for V.*

Proof. We know that $S^* = V$. Now by applying Theorems 3.7 and 4.7 we arrive at the desired assertion. □

Remark 4.9. Let us consider the shifts $S\colon \ell_2 \to \ell_2$ and $A\colon \ell_1 \to \ell_1$. The operators S and A seem to be similar. It is amazing that the first one does not have a nontrivial closed invariant affine subspace but the second one has such a subspace.

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