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**Aequationes Mathematicae**

# Vector-valued invariant means

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*Dedicated to Professor Maciej Sablik and Professor László Székelyhidi on the occasion of their 70th birthday.*

**Abstract.** In the paper, we will present some important results from the theory of vector-valued invariant means. Discussing the results contained in the author's paper Badora (Ann Pol Math 58:147–159, 1993), we will add new facts that were previously not published.

**Mathematics Subject Classification.** 43A07.

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## 1. Introduction

Invariant means, closely related to the concept of finitely additive invariant measures, have been studied since the beginning of the last century. At present, we encounter one of the following two equivalent definitions of an invariant mean.

**Definition 1.1.** A linear functional $m : B(S,\mathbb{R}) \to \mathbb{R}$ defined on the space $B(S,\mathbb{R})$ of all real bounded functions on a semigroup $(S,\cdot)$ is called a *left* [*right*] *invariant mean*, if

$$\inf_{x\in S} f(x) \le m(f) \le \sup_{x\in S} f(x) \tag{1.1}$$

and

$$m({}_y f) = m(f) \quad [m(f_y) = m(f)], \tag{1.2}$$

for all $f \in B(S,\mathbb{R})$ and $y \in S$, where by ${}_y f$ $[f_y]$ we denote the shift of the function $f$ by $y$, hence the function defined by the formula

$${}_y f(x) = f(yx) \quad [f_y(x) = f(xy)], \quad x \in S.$$

**Definition 1.2.** A *left* [*right*] *invariant mean* is a linear functional $m : B(S,\mathbb{R}) \to \mathbb{R}$ satisfying the following conditions

$$m(f) \geq 0 \quad \text{if} \quad f \geq 0, \tag{1.3}$$

$$m(\mathbf{1}_S) = 1 \tag{1.4}$$

and

$$m({}_yf) = m(f) \quad [m(f_y) = m(f)], \tag{1.5}$$

for all $f \in B(S,\mathbb{R})$ and $y \in S$, where $\mathbf{1}_S$ denotes the constant function equal to 1 on a semigroup $S$.

To prove the equivalence of these definitions, note that any linear functional $m$ that satisfies condition (1.1) automatically satisfies conditions (1.3) and (1.4). Conversely, if a linear functional $m : B(S,\mathbb{R}) \to \mathbb{R}$ satisfies (1.3) and (1.4), then for $f \in B(S,\mathbb{R})$ we have

$$f - \inf_{x\in S} f(x) \geq 0 \text{ and } \sup_{x\in S} f(x) - f \geq 0.$$

So, from the linearity of $m$ and conditions (1.3) and (1.4) we get

$$0 \leq m\left(f - \inf_{x\in S} f(x)\right) = m(f) - m\left(\inf_{x\in S} f(x)\right) \tag{1.6}$$

$$= m(f) - \inf_{x\in S} f(x) m(\mathbf{1}_S) = m(f) - \inf_{x\in S} f(x) \tag{1.7}$$

and

$$0 \leq m\left(\sup_{x\in S} f(x) - f\right) = m\left(\sup_{x\in S} f(x)\right) - m(f) \tag{1.8}$$

$$= \sup_{x\in S} f(x) m(\mathbf{1}_S) - m(f) = \sup_{x\in S} f(x) - m(f). \tag{1.9}$$

So, $m$ satisfies (1.1), which shows that indeed the two definitions of an invariant mean are equivalent.

A semigroup $S$ which admits a left (right) invariant mean on the space $B(S,\mathbb{R})$ will be termed *left* [*right*] *amenable.*

More on invariant means and amenable semigroups can be found in the monograph Paterson [15]. Here we only stress that every commutative semigroup is amenable (left and right amenable).

In 1985 L. Székelyhidi (see [19]) for the first time used the invariant mean method in the theory of the stability of functional equations. Since then, invariant means have also been used extensively in the theory of functional equations. For many of these applications, it was necessary to extend the concept of invariant means to the case of function spaces of vector-valued mappings.

In the paper, we will present some important results from the theory of vector-valued invariant means, which we will supplement with new facts previously unpublished.

The formulations of the facts as well as their proofs in the case of right-hand invariance are completely analogous to the case of left-hand invariance. Therefore, in the following, in most cases, we limit ourselves to considering left-hand invariance.

## 2. Linear lattices

The generalization of invariant means to the case of mappings with values in linear lattices is very natural. Transferring the first definition to this case, we must ensure the existence of an infimum and a supremum. So it makes sense to consider mappings with values in a boundedly complete linear lattice. Recall that a *boundedly complete linear lattice* is an ordered linear space $Y$ with the property that every non-empty upper bounded subset of $Y$ has in $Y$ a supremum, i.e. a least upper bound (which, in fact, is uniquely determined). We obtain an equivalent property if we replace the upper bounded subset of $Y$ by lower bounded ones and the supremum by infimum.

This type of research was initiated by Ramabhadra Sarma in [18] and continued in Rao Chivukula and Ramabhadra Sarma [17]. The final version of the theorem presented below can be found in the paper of Gajda [6] (as a consequence of the more general theorems proved there).

**Theorem 2.1.** *Let $S$ be a left amenable semigroup and let $Y$ be a boundedly complete linear lattice with the order $\preceq$. Then there exists a linear operator $M : B^{\preceq}(S,Y) \to Y$ on the space $B^{\preceq}(S,Y)$ of all ordered bounded functions from $S$ into $Y$ such that*

$$\inf_{x\in S} f(x) \preceq M(f) \preceq \sup_{x\in S} f(x) \tag{2.1}$$

*and*

$$M({}_yf) = M(f), \tag{2.2}$$

*for all $f \in B^{\preceq}(S,Y)$ and $y \in S$.*

As a corollary of Theorem 5.7 proved in Sect. 5 of this paper, we will get a result of this type for boundedly complete locally convex vector lattices.

## 3. Linear-topological spaces

In the case of linear-topological spaces, the condition of the mean being (1.1) is replaced by the condition that the value of the invariant mean belongs to the closure of the convex hull of the set of function values. This approach to transferring the concept of invariant mean to the case of functions with vector values can be found in Dixmier [5]. These studies were further continued by Husain et al. [9], [10], [8], Bombal and Vera [3], Gajda [6] and other mathematicians. Ultimately, we can conclude that

**Theorem 3.1.** *Let $S$ be a left amenable semigroup and let $Y$ be a semi-reflexive locally convex linear-topological space. Then there exists a linear operator $M : B(S,Y) \to Y$ on the space $B(S,Y)$ of all bounded functions from $S$ into $Y$ satisfying*

$$M(f) \in \mathit{clconv}\, f(S) \tag{3.1}$$

*and*

$$M({}_y f) = M(f), \tag{3.2}$$

*for all $f \in B(S,Y)$ and $y \in S$, where $\mathit{clconv}\, f(S)$ denotes the closure of the convex hull of the set of values of the function $f$.*

To provide the necessary terminology we recall that a locally convex linear-topological space $Y$ is termed *semi-reflexive* if and only if the natural embedding of $Y$ into its second conjugate $Y^{**}$ is surjective (but not necessarily homeomorphic).

On the other hand, independently, Bombal and Vera [3] and Tabor [20] proved the following theorem.

**Theorem 3.2.** *Let $S$ be a left amenable semigroup and let $Y$ be a Banach space. Then there exists a linear operator $M : B(S,Y) \to Y$ fulfilling (3.1) and (3.2) if and only if the space $Y$ is reflexive.*

These types of invariant means were investigated further. As an example we will present the theorem from the paper of R. Badora, R. Ger, Zs. Páles [2].

**Theorem 3.3.** *Let $S$ be a left amenable semigroup and let $Y$ be a Hausdorff locally convex linear space. Then there exists a linear operator $M : \mathcal{C}_w(S,Y) \to Y$ on the space $\mathcal{C}_w(S,Y)$ of all $Y$-valued functions on $S$ whose range has a weakly compact closed convex hull satisfying*

$$M(f) \in \overline{co}\, f(S)$$

*and*

$$m({}_y f) = m(f),$$

*for all $f \in \mathcal{C}_w(S,Y)$ and $y \in S$, where $\overline{co}\, f(S)$ denotes the weakly closed convex hull of the range of $f$.*

For example, from this theorem, the existence of an invariant mean on the space of all bounded functions on an amenable semigroup and with values in a semi-reflexive locally convex linear-topological space is obvious.

## 4. Normed spaces

Transferring the concept of invariant mean to the case of function spaces with values in normed spaces, we can use the second definition, Definition 1.2. Such research can be found in the paper by A. Pełczyński [16], and in the paper by R. Ger [7] where we find the following definition.

**Definition 4.1.** Let $(S, \cdot)$ be a semigroup and let $(Y, \|\cdot\|)$ be a Banach space. Then a continuous linear operator $M : B(S, Y) \to Y$ defined on the space $B(S, Y)$ of all bounded functions from $S$ into $Y$ is called a *left* [*right*] *generalized invariant mean* if

$$M(c_S) = c, \quad c \in Y \tag{4.1}$$

and

$$M({}_yf) = M(f) \quad [M(f_y) = M(f)], \tag{4.2}$$

for all $f \in B(S, Y)$ and $y \in S$.

R. Ger in [7] proved that for a left [right] amenable semigroup there exists a left [right] generalized invariant mean when $Y$ is reflexive or $Y$ has the Hahn–Banach extension property or $Y$ forms a boundedly complete Banach lattice with a strong unit.

In the paper H. Bustos Domecq [4] we find the following facts.

**Theorem 4.2.** *Let $S$ be a left amenable semigroup and $Y$ be a Banach space. Suppose $Y$ is complemented in its second dual by a projection $\pi$. Then $B(S, Y)$ admits a left generalized invariant mean of the norm at most $\|\pi\|$.*

**Theorem 4.3.** *Let $Y$ a Banach space and let $K \geq 1$. Suppose that for every commutative semigroup $S$ there is a generalized invariant mean $M$ on the space $B(S, Y)$ with $\|M\| \leq K$. Then $Y$ is complemented in its bidual by a projection of norm at most $K$.*

However, as observed by Z. Lipecki in his Mathematical Review of the paper H. Bustos Domecq [4] (MR1943762), the proof of Theorem 4.3 contains a gap.

For many years, the problem of presenting a correct proof of Theorem 4.3 remained open. At that time R. Łukasik's paper [13] appeared in which the author proved the following theorem.

**Theorem 4.4.** *Let $S$ be a left amenable semigroup and let $(Y, \|\cdot\|)$ be a Banach space. If there exists a linear, continuous operator $\vartheta : Y^{**} \to Y$ such that*

$$\|\vartheta\| \leq 1 \quad \text{and} \quad \vartheta \circ \kappa = id_Y$$

*then there exists a generalized left invariant mean on the space $B(S, Y)$.*

From this theorem, as conclusions, we obtain many of the previously proven facts.

Finally, Kania in [12] corrected the gap in the paper Bustos Domecq [4] providing the correct proof of Theorem 4.3.

## 5. Invariant means as selections of multifunctions

In this part of the paper, we will look at invariant means as selections of multifunctions. This idea for the generalization of invariant means was proposed by Badora in [1].

**Definition 5.1.** Let $\mathcal{F}$ be a left [right] invariant linear space of functions mapping a semigroup $S$ into a linear space $Y$ (that is ${}_yf \in \mathcal{F}$ [$f_y \in \mathcal{F}$] if $f \in \mathcal{F}$ and $y \in S$), $\mathcal{C}$ be a family of subsets of $Y$ and let $F : \mathcal{F} \to \mathcal{C}$. Then a linear operator $M : \mathcal{F} \to Y$ will be called a *left* [*right*] *invariant* $F$*-mean* if and only if

$$M(f) \in F(f), \quad f \in \mathcal{F} \tag{5.1}$$

and

$$M({}_yf) = M(f) \quad [M(f_y) = M(f)], \quad f \in \mathcal{F}, \quad y \in S. \tag{5.2}$$

In the remainder of the paper, one of the families of sets important for us will be families having the binary intersection property. Let us recall that a collection $\mathcal{C}$ of sets is said to have the *binary intersection property* if and only if every subcollection of $\mathcal{C}$, any two members of which intersect, has a nonempty intersection. For example, the collection of bounded order intervals in a boundedly complete linear lattice has the binary intersection property (see Ioffe [11]) and the collection of closed balls in a normed space with the Hahn–Banach extension property has the binary intersection property (see Nachbin [14]).

In [1] R. Badora proved the following version of Dixmier's theorem from the paper [5].

**Theorem 5.2.** *Let $\mathcal{F}$ be a left invariant linear space of functions defined on a semigroup $S$ with values in a real linear space $Y$. Let $\mathcal{C}$ be a translation invariant family of subsets of $Y$ having the binary intersection property and let $F : \mathcal{F} \to \mathcal{C}$ satisfy*

$$F(f+g) \subset F(f) + F(g), \quad f, g \in \mathcal{F}; \tag{5.3}$$

$$F(tf) = tF(f), \quad f \in \mathcal{F}, \quad t \in \mathbb{R} \setminus \{0\}. \tag{5.4}$$

*Then there exists a left invariant $F$-mean on $\mathcal{F}$ if and only if*

$$0 \in F(h), \tag{5.5}$$

*for all* $h$ *of the form* $h=\sum_{k=1}^{n}(f_k -_{y_k} f_k)$, *where* $f_1,\ldots,f_n\in\mathcal{F}$ *and* $y_1,\ldots,y_n \in S$.

As a consequence of this theorem, Badora [1] obtained the following.

**Theorem 5.3.** *Let* $(S,\cdot)$ *be a left amenable semigroup, and let* $Y$ *be a real locally convex space. Let* $\mathcal{F}$ *be a left invariant linear subspace of* $B(S,Y)$. *Let* $\mathcal{C}$ *be a translation invariant collection of closed convex sets in* $Y$ *having the binary intersection property. Assume that the map* $F:\mathcal{F}\to\mathcal{C}$ *satisfies (5.3), (5.4) and*

$$h(S)\subset F(h), \tag{5.6}$$

*for all* $h$ *of the form* $h=\sum_{k=1}^{n}(f_k -_{y_k} f_k)$, *where* $f_1,\ldots,f_n\in\mathcal{F}$, $y_1,\ldots,y_n\in S$. *Then there exists a left invariant* $F$*-mean on the space* $\mathcal{F}$.

In Badora's paper [1] we also find the following two facts.

**Lemma 5.4.** *If the family of sets* $\mathcal{C}$ *has the binary intersection property, then the family* $\widehat{\mathcal{C}}$ *of all non-empty intersections of subfamilies of* $\mathcal{C}$ *also has the binary intersection property.*

**Lemma 5.5.** *Let* $\mathcal{C}$ *be a translation invariant family of subsets of a real linear space* $Y$ *having the binary intersection property and satisfying the condition that* $-C\in\mathcal{C}$ *if* $C\in\mathcal{C}$. *Then, for every two subfamilies* $\{A_i\ :\ i\in I\}$ *and* $\{B_j\ :\ j\in J\}$ *of* $\mathcal{C}$ *such that*

$$\bigcap\{A_i:\ i\in I\}\neq\emptyset \ \ \textit{and}\ \ \bigcap\{B_j:\ j\in J\}\neq\emptyset$$

*we have*

$$\bigcap\{A_i+B_j:\ i\in I,\ \ j\in J\}=\bigcap\{A_i:\ i\in I\}+\bigcap\{B_j:\ j\in J\}.$$

We say that the family $\mathcal{C}$ of subsets of a real linear space is *linearly-invariant* if $\mathcal{C}$ is translation invariant and satisfies

$$A,B\in\mathcal{C},\ \ t\in\mathbb{R}\Longrightarrow A+B\in\mathcal{C},\ \ tA\in\mathcal{C}.$$

Note

**Lemma 5.6.** *If a collection* $\mathcal{C}$ *of subsets of a real linear space* $Y$ *is linearly-invariant and has the binary intersection property, then the family* $\widehat{\mathcal{C}}$ *of all non-empty intersections of subfamilies of* $\mathcal{C}$ *is also linearly-invariant and has the binary intersection property.*

*Proof.* By Lemma 5.4 the collection $\widehat{\mathcal{C}}$ has the binary intersection property. The condition that $tA\in\widehat{\mathcal{C}}$ if $A\in\widehat{\mathcal{C}}$ and $t\in\mathbb{R}$ is easy to show. However, to conclude that $A+B\in\widehat{\mathcal{C}}$ for $A,B\in\widehat{\mathcal{C}}$ we will use Lemma 5.5. Indeed, if $A,B\in\widehat{\mathcal{C}}$, then

$$A=\bigcap\{A_i:\ i\in I\},\ \ B=\bigcap\{B_j:\ j\in J\},$$

for some subfamilies $\{A_i \ : \ i \in I\}$ and $\{B_j \ : \ j \in J\}$ of $\mathcal{C}$. Next, by Lemma 5.5

$$A + B = \bigcap\{A_i : \ i \in I\} + \bigcap\{B_j : \ j \in J\} = \bigcap\{A_i + B_j : \ i \in I \\ ; \ j \in J\}$$

which means that $A + B$ is a non-empty intersection of some subfamily of $\mathcal{C}$. So, $A + B \in \widehat{\mathcal{C}}$. □

Let $(S, \cdot)$ be a semigroup and let $\mathcal{C}$ be a linearly-invariant family having the binary intersection property of non-empty, closed, bounded, and convex subsets of a real, locally convex linear-topological space $Y$. For a function $f : S \to Y$, the symbol $C(f)$ denotes the set of sets $A$ from the family $\mathcal{C}$ such that $f(S) \subset A$. Hence

$$\mathcal{C}(f) = \{A \in \mathcal{C} \mid f(S) \subset A\}.$$

The space of all functions $f : S \to Y$ for which the family $C(f)$ is non-empty is denoted by $B^{\mathcal{C}}(S, Y)$. Therefore

$$B^{\mathcal{C}}(S, Y) = \{f : S \to Y \mid \mathcal{C}(f) \neq \emptyset\}.$$

The space $B^{\mathcal{C}}(S, Y)$ is a real linear subspace of $B(S, Y)$. Moreover, this space is both left and right invariant. Indeed, if for a function $f : S \to Y$ there exists some set $A \in \mathcal{C}$ such that $f(S) \subset A$, then for every $y \in S$ we get

$${}_yf(S) = f(yS) \subset f(S) \subset A,$$

which means that ${}_yf \in B^{\mathcal{C}}(S, Y)$.

As a consequence of Theorem 5.3 we obtain the following generalization of Theorem 4 of [1].

**Theorem 5.7.** *Let $(S, \cdot)$ be a left amenable semigroup and let $\mathcal{C}$ be a linearly-invariant family having the binary intersection property of non-empty, closed, bounded and convex subsets of a real, locally convex linear-topological space $Y$. Then there exists a linear operator $M : B^{\mathcal{C}}(S, Y) \to Y$ such that*

$$M(f) \in \bigcap C(f) \tag{5.7}$$

*and*

$$M({}_yf) = M(f), \tag{5.8}$$

*for all $f \in B^{\mathcal{C}}(S, Y)$ and $y \in S$.*

*Proof.* To prove this theorem, it is sufficient to show that the multifunction $F : B^{\mathcal{C}}(S, Y) \to \widehat{\mathcal{C}}$ given by the formula

$$F(f) = \bigcap C(f), \quad f \in B^{\mathcal{C}}(S, Y)$$

satisfies conditions (5.3), (5.4) and (5.6) of Theorem 5.3. For the proof of condition (5.3), let us establish $f, g \in B^{\mathcal{C}}(S, Y)$ and sets $A \in C(f)$, $B \in C(g)$. Then

$$(f+g)(S) \subset f(S) + g(S) \subset A + B$$

which allows us to conclude that

$$F(f+g)(S) \subset A + B.$$

By Lemma 5.5 we get

$$F(f+g) \in \bigcap (C(f) + C(g)) = \bigcap C(f) + \bigcap C(g) = F(f) + F(g)$$

what needed to be shown. Further, for $f \in B^{\mathcal{C}}(S, Y)$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} tF(f) &= t\bigcap\{A \in \mathcal{C} : \ f(S) \subset A\} = \bigcap\{tA \in \mathcal{C} : \ f(S) \subset t^{-1}tA\} \\ &= \bigcap\{B \in \mathcal{C} : \ (tf)(S) \subset B\} = F(tf), \end{aligned}$$

hence condition (5.4) is proved. Moreover, condition (5.6) is a direct consequence of the definition of the multifunction $F$ and the proof of the theorem is complete. □

Observe that if $Y$ is a boundedly complete locally convex vector lattice (a topological boundedly complete vector lattice that is also a locally convex space) and $\mathcal{C}$ is the collection of all bounded order intervals in $Y$ then Theorem 5.7 reduces to Theorem 2.1 (with the space $Y$ being a boundedly complete locally convex vector lattice).

Theorems presented here show that the binary intersection property is a sufficient property for the existence of a generalized invariant mean. However, looking at Theorem 3.3 (the family of compact sets has the finite intersection property but does not have to have the binary intersection property) the binary intersection property is a sufficient property for the existence of a generalized invariant mean, but it is not a necessary condition.

**Theorem 5.8.** *Let $\mathcal{C}$ be a family of subsets of a real linear space $Y$. For a semigroup $(S, \cdot)$ by $B^{\mathcal{C}}(S, Y)$, as before, we denote the space of all functions $f : S \to Y$ such that $f(S) \subset A$ for some $A \in \mathcal{C}$. Then the space $B^{\mathcal{C}}(S, Y)$ is translation invariant.*

*Moreover, if for every Abelian semigroup $(S, +)$ there exists a map $\mu : B^{\mathcal{C}}(S, Y) \to Y$ on the space $B^{\mathcal{C}}(S, Y)$ satisfying for every $f \in B^{\mathcal{C}}(S, Y)$ and every $A \in \mathcal{C}$*

$$f(S) \subset A \Longrightarrow \mu(f) \in A \tag{5.9}$$

*and for every $f \in B^{\mathcal{C}}(S, Y)$ and every $y \in S$*

$$\mu({}_y f) = \mu(f), \tag{5.10}$$

*then the family $\mathcal{C}$ has the finite intersection property.*

*Proof.* First let us note that if $f \in B^{\mathcal{C}}(S,Y)$, then $f(S) \subset A$ for some $A \in \mathcal{C}$ and for every $y \in S$ we get

$$ {}_yf(S) \subset f(y \cdot S) \subset f(S) \subset A. $$

So, ${}_yf \in B^{\mathcal{C}}(S,Y)$ and the space $B^{\mathcal{C}}(S,Y)$ is translation invariant.

To show that the family $\mathcal{C}$ has the finite intersection property we set such a subfamily $\{C_i \in \mathcal{C} \ : \ i \in I\}$ that for every $n \in \mathbb{N}$ and any $i_1, \ldots, i_n \in I$

$$ C_{i_1} \cap \ldots \cap C_{i_n} \neq \emptyset. $$

We will prove that $\bigcap_{i\in I} C_i \neq \emptyset$.

We fix $i_0 \in I$ and consider a semigroup

$$ S = \{\ x : I \to [0,+\infty) : \ x(i_0) = 0 \ \ \wedge \ \ \mathrm{card}\{\ i \in I : \ x(i) > 0\ \} < \infty\ \} $$

with the operation defined as follows: for $x, y \in S$ let $x + y$ be a map defined by

$$ (x+y)(i) = x(i) + y(i), \quad i \in I. $$

Then $S$ with this operation is an Abelian semigroup.

For a non-empty and finite subset $J$ of the set $I \backslash \{i_0\}$ we choose

$$ c_J \in \bigcap_{j\in J} C_j \cap C_{i_0} $$

(this intersection is non-empty by the assumption on the family $\{C_i \in \mathcal{C} \ : \ i \in I\}$ - we have an intersection of its finite subfamily).

Define the function $f : S \to Y$ as follows

$$ f(x) = c_{\{i\in I:\, x(i)>0\}}, \quad x \in S. $$

Because each $c_J \in C_{i_0}$, we have $f(S) \subset C_{i_0}$, which means, that $f \in B^{\mathcal{C}}(S,Y)$ and, by the assumption, $\mu(f) \in C_{i_0}$.

Now we will show that $\mu(f) \in C_k$, for $k \in I$. Let $k \in I$ be fixed.

If $k = i_0$, then $\mu(f) \in C_{i_0} = C_k$.

If $k \neq i_0$, then let $y : I \to [0,+\infty)$ be defined as follows

$$ y(i) = \begin{cases} 1, \, i = k \\ 0, \, i \neq k. \end{cases} $$

Then $y \in S$ and moreover, for every $x \in S$,

$$ (x+y)(k) = x(k) + y(k) > 0. $$

Therefore, for every $x \in S$

$$ k \in \{i \in I : \ (x+y)(i) > 0\}, $$

which leads to

$$ {}_yf(x) = f(y+x) = f(x+y) = c_{\{i\in I:\, (x+y)(i)>0\}} \in C_k. $$

Whence

$$_yf(S) \subset C_k$$

and, by our assumption, $\mu(_yf) \in C_k$. But we also assume that $\mu(_yf) = \mu(f)$, $y \in S$. So, $\mu(f) \in C_k$.

We have shown that for every $k \in I$ $\mu(f) \in C_k$ holds. Therefore

$$\mu(f) \in \bigcap_{k\in I} C_k \neq \emptyset$$

which ends the proof. □

Thus the finite intersection property is a necessary condition for the existence of the generalized invariant mean proposed above.

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