Aequat. Math. 97 (2023), 725–752

0001-9054/23/040725-28
published online May 15, 2023
https://doi.org/10.1007/s00010-023-00950-5

Aequationes Mathematicae

Isometry groups of six-dimensional nilmanifolds

Kornélia Ficzere and Ágota Figula

Abstract. We determine the 6-dimensional nilpotent metric Lie algebras such that the Lie algebra $\mathfrak{n}$ has a descending series of ideals invariant under all automorphisms of $\mathfrak{n}$ and the dimension of the consecutive members of the series decreases by one. We call them metric Lie algebras having a framing determined by ideals. We classify the isometry equivalence classes and determine the isometry groups of connected and simply connected Riemannian nilmanifolds on 6-dimensional nilpotent Lie groups having a Lie algebra $\mathfrak{n}$ as their Lie algebra.

Mathematics Subject Classification. Primary 53C30; Secondary 57M60, 17B30, 22E25.

Keywords. Nilmanifolds, Nilpotent metric Lie algebras, Framing determined by ideals, Isometry groups, Group of orthogonal automorphisms.

1. Introduction

Let $\mathfrak{n}$ be a real nilpotent Lie algebra and N be the connected simply connected Lie group having Lie algebra $\mathfrak{n}$. We call $(\mathfrak{n}, \langle ., . \rangle)$ a metric nilpotent Lie algebra if it is given an Euclidean inner product $\langle ., . \rangle$ on $\mathfrak{n}$. An inner product $\langle ., . \rangle$ on $\mathfrak{n}$ determines a left-invariant metric $\langle ., . \rangle_N$ on N and conversely. Hence $(N, \langle ., . \rangle_N)$ becomes a Riemannian manifold. We denote by $\mathcal{OA}(\mathfrak{n})$ the group of orthogonal automorphisms of the Lie algebra $\mathfrak{n}$ consisting of the automorphisms of $\mathfrak{n}$ which preserve the inner product on $\mathfrak{n}$. A connected Riemannian manifold M which admits a transitive nilpotent Lie group of isometries is called a Riemannian nilmanifold. It is pointed out in [10, Theorem 2(4)], that every Riemannian nilmanifold M can be identified with the unique nilpotent Lie subgroup N of the group $\mathcal{I}(M)$ of isometries of M acting simply transitively on M, equipped with a left-invariant metric. Furthermore, $\mathcal{I}(N)$, the group of isometries of $(N, \langle ., . \rangle_N)$, is the semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$ of the group $\mathcal{OA}(\mathfrak{n})$ and the group N itself. From this observation it follows

This paper was supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K132951.

that the determination of the isometry equivalence classes of connected simply connected nilmanifolds and their isometry groups can be carried out by the investigation of the classes of isometrically isomorphic metric nilpotent Lie algebras. Applying this procedure the isometry equivalence classes and the isometry groups of connected simply connected nilmanifolds of dimension at most 5 are established in [5,7,9]. In the classes of 6-dimensional nilmanifolds the isometry equivalence classes and the isometry groups on two-step nilpotent Lie groups, respectively on five-step nilpotent Lie groups, this means on filiform Lie groups, are accomplished in [3], respectively in [6].

In this paper we deal with 6-dimensional metric Lie algebras having nilpotency class three or four. In [5, Sect. 3], the metric Lie algebras $(\mathfrak{n}, \langle ., . \rangle)$ having a decomposition into an orthogonal direct sum of 1-dimensional subspaces such that each orthogonal automorphism of $(\mathfrak{n}, \langle ., . \rangle)$ preserves this decomposition play an essential role. We say that these metric Lie algebras have a framing. It turns out in [5] that there is a strong connection between a special class $\mathcal{C}$ of framed metric Lie algebras and their ideal structures. Namely the framing of a metric Lie algebra in $\mathcal{C}$ can be constructed in a natural way using a descending series of ideals $\mathfrak{n} = \mathfrak{n}^{(0)} \supset \mathfrak{n}^{(1)} \supset \cdots \supset \mathfrak{n}^{(n-1)} \supset \mathfrak{n}^{(n)} = \{0\}$ invariant under all automorphisms of $\mathfrak{n}$ with $\dim(\mathfrak{n}^{(i)}) - \dim(\mathfrak{n}^{(i+1)}) = 1$, $i = 0, \ldots, n-1$. This type of framings we call *framing determined by ideals*. Every filiform metric Lie algebra of dimension at least four allows a framing determined by ideals (see [5, Theorem 4]).

Applying the classification of 6-dimensional nilpotent Lie algebras given in [4], Sect. 3 is devoted to the thorough study of the ideal structures of these Lie algebras and to the determination of the 6-dimensional nilpotent metric Lie algebras having a framing determined by ideals. We obtain that 6-dimensional indecomposable nilpotent Lie algebras with the exception of six classes possess a suitable series of ideals (cf. Proposition 3.1).

In Sect. 4 we systematically apply the method of classification of the classes of isometrically isomorphic metric Lie algebras given in [5]. We describe the isometry equivalence classes and determine the group of isometries of connected simply connected nilmanifolds on 6-dimensional indecomposable Lie groups such that their Lie algebras have a framing determined by ideals.

Among the classes of nilmanifolds having nilpotency class $n > 2$, the geometric properties of filiform nilmanifolds have been considerably improved. In particular the characterization of totally geodesic subalgebras is given in [1,2,8]. Our results can be utilized for the enquiry of the totally geodesic subalgebras of 6-dimensional nilmanifolds having nilpotency class $n \in \{3, 4\}$.

2. Preliminaries

The *lower central series* of a nilpotent Lie algebra ℓ is $\ell = S^0\ell \supset S^1\ell \supset \cdots \supset S^j\ell \supset S^{j+1}\ell \supset \cdots \supset \{0\}$ such that $S^{j+1}\ell = [\ell, S^j\ell], j \in \mathbb{N}$. A Lie algebra ℓ is called *k-step nilpotent* if $S^k\ell = \{0\}$, but $S^{k-1}\ell \neq \{0\}$ for some $k \in \mathbb{N}$. If an n-dimensional Lie algebra ℓ is $(n-1)$-step nilpotent then it is called *filiform*. The *metric Lie algebra* is a Lie algebra equipped with an inner product, the automorphisms preserving the inner product are called *orthogonal automorphisms*.

Definition 2.1. An orthogonal direct sum decomposition $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ on one-dimensional subspaces $V_1, \ldots, V_n$ of a metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is called a *framing*, if any orthogonal automorphism of $(\mathfrak{n}, \langle ., . \rangle)$ preserves this decomposition. An orthonormal basis $\{G_1, G_2, \ldots, G_n\}$ of $(\mathfrak{n}, \langle ., . \rangle)$ is *adapted* to the framing $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ if $V_i = \mathbb{R}\, G_i$ for $i = 1, \ldots, n$. The metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is called *framed*, if it has a framing.

The following concept originates from the assertion in Lemma 3 in [5].

Definition 2.2. An n-dimensional metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ has a *framing determined by ideals*, if the Lie algebra $\mathfrak{n} = \mathrm{span}(G_1, \ldots, G_n)$ has a descending series of ideals $n^i = \mathrm{span}(G_i, \ldots, G_n)$, $i = 1, \ldots, n$, which is left invariant under all automorphisms of $\mathfrak{n}$.

In this paper we consider 6-dimensional metric nilpotent Lie algebras having a framing determined by ideals.

It is proved in Section 3.1 in [5] that the group $\mathcal{OA}(\mathfrak{n})$ of orthogonal automorphisms of a framed metric nilpotent Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is a subgroup of the group $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where the number of factors is less than or equal to $\dim \mathfrak{n}$. Hence the connected component of the isometry group $\mathcal{I}(N)$ of the connected simply connected Riemannian nilmanifold $(N, \langle ., . \rangle)$ is isomorphic to the Lie group N.

We often use the following (see [5, Lemma 1]).

Lemma 2.3. *Let $(\mathfrak{n}, \langle ., . \rangle)$ and $(\mathfrak{n}^*, \langle ., . \rangle^*)$ be isometrically isomorphic framed metric Lie algebras of dimension n with framings $\mathfrak{n} = \mathbb{R}\, G_1 \oplus \cdots \oplus \mathbb{R}\, G_n$ and $\mathfrak{n}^* = \mathbb{R}\, G_1^* \oplus \cdots \oplus \mathbb{R}\, G_n^*$, where $(G_1, \ldots, G_n)$, respectively $(G_1^*, \ldots, G_n^*)$ are orthonormal bases. If the commutators $[., .]$ of $\mathfrak{n}$ and $[., .]^*$ of $\mathfrak{n}^*$ are of the form*

$$[G_i, G_j] = \sum_{k=1}^{n} a_{i,j}^k G_k \quad \textit{and} \quad [G_i^*, G_j^*]^* = \sum_{k=1}^{n} {a^*}_{i,j}^k G_k^*, \quad i, j, k = 1, \ldots, n,$$

then $a_{i,j}^k = \pm {a^*}_{i,j}^k$ for all $i, j, k = 1, \ldots, n$. Particularly, if $a_{i,j}^k$, ${a^*}_{i,j}^k \geq 0$ then $a_{i,j}^k = {a^*}_{i,j}^k$.

We denote by $\mathbb{E}^6$ a 6-dimensional Euclidean vector space with a distinguished orthonormal basis $\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5, E_6\}$. The classification of metric Lie algebras up to isometric isomorphisms proceeds in the following way given by [5, pp. 371–372]: we apply the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ in the metric Lie algebra $(\mathfrak{l}, \langle .,. \rangle)$ to get an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ expressed by $F_i = \sum_{k=i}^{n} a_{ik} G_k, a_{ik} \in \mathbb{R}$, such that $a_{ii} \geq 0$. After this, we define a Lie bracket on $\mathbb{E}^6$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$ as that of the metric Lie algebra $(\mathfrak{l}, \langle .,. \rangle)$ with respect to its basis F. The obtained metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ on $\mathbb{E}^6$ is isometrically isomorphic to $(\mathfrak{l}, \langle .,. \rangle)$. Finally, we examine under which conditions on the real parameters of metric Lie algebras on $\mathbb{E}^6$ we receive a one-to-one correspondence between the equivalence classes of isometrically isomorphic metric Lie algebras and a family of metric Lie algebras on $\mathbb{E}^6$.

3. Framed metric Lie algebras of dimension 6

In this section we investigate nilpotent Lie algebras of dimension 6 and we wish to determine which Lie algebras in this class have a framing determined by ideals. We deal with Lie algebras which are not direct products of Lie algebras of lower dimensions. According to [4, pp. 646–647], the non-isomorphic Lie algebras in this class are the Lie algebras $L_{6,i}, i = 10, \ldots, 26$, with respect to a basis $\{x_1, x_2, \ldots, x_6\}$. The 6-dimensional filiform nilpotent Lie algebras $L_{6,14}, \cdots, L_{6,18}$ are treated in [6], hence we omit these Lie algebras in our consideration. The 6-dimensional 2-step Lie algebras are the Lie algebras $L^{\varepsilon}_{6,22}$ and $L_{6,26}$. The corresponding Lie algebras do not have a framing determined by ideals, because their characteristic ideal is only the centre. The set of their isometric isomorphism classes are studied in [3]. Therefore our list (3.1) doesn't include these two Lie algebra classes.

For the remaining cases we use the following basis changes: for $L_{6,11}$, $L_{6,12}$: $x_1 \mapsto G_1, x_2 \mapsto G_2, x_3 \mapsto G_4, x_4 \mapsto G_5, x_5 \mapsto G_3, x_6 \mapsto G_6$, for $L_{6,13}$: $x_1 \mapsto G_1, x_2 \mapsto G_3, x_3 \mapsto G_4, x_4 \mapsto G_2, x_5 \mapsto G_5, x_6 \mapsto G_6$, for $L^{\varepsilon}_{6,19}$: $x_1 \mapsto G_2, x_2 \mapsto G_1, x_3 \mapsto G_3, x_4 \mapsto G_4, x_5 \mapsto G_5, x_6 \mapsto G_6$, for $L_{6,23}$, $L_{6,25}$: $x_1 \mapsto G_1, x_2 \mapsto G_2, x_3 \mapsto G_4, x_4 \mapsto G_3, x_5 \mapsto G_6, x_6 \mapsto G_5$, for all other Lie algebras: $x_i \mapsto G_i, i = 1, \ldots 6$, to obtain the ordered bases $(G_6, G_5, G_4, G_3, G_2, G_1)$ as orthonormal basis adapted to the framing of the corresponding metric Lie algebras. After applying the basis changes we obtain Lie algebras $\mathfrak{l}_{6,i}$, $i = 10, \ldots, 13,\ 19, 20, 21,\ 23, 24, 25$, given by the following non-vanishing commutators:

$$\begin{aligned}
&\mathfrak{l}_{6,10} : [G_1, G_2] = G_3, [G_1, G_3] = G_6, [G_4, G_5] = G_6;\\
&\mathfrak{l}_{6,11} : [G_1, G_2] = G_4, [G_1, G_4] = G_5, [G_1, G_5] = G_6, [G_2, G_4] = G_6,\\
&\qquad [G_2, G_3] = G_6;\\
&\mathfrak{l}_{6,12} : [G_1, G_2] = G_4, [G_1, G_4] = G_5, [G_1, G_5] = G_6, [G_2, G_3] = G_6;\\
&\mathfrak{l}_{6,13} : [G_1, G_3] = G_4, [G_1, G_4] = G_5, [G_1, G_5] = G_6, [G_3, G_2] = G_5,\\
&\qquad [G_4, G_2] = G_6;
\end{aligned} \tag{3.1}$$

$\mathfrak{l}^{\varepsilon}_{6,19} : [G_2, G_1] = G_4, [G_2, G_3] = G_5, [G_1, G_4] = G_6, [G_3, G_5] = \varepsilon G_6;$
$\mathfrak{l}_{6,20} : [G_1, G_2] = G_4, [G_1, G_3] = G_5, [G_1, G_5] = G_6, [G_2, G_4] = G_6;$
$\mathfrak{l}^{\varepsilon}_{6,21} : [G_1, G_2] = G_3, [G_1, G_3] = G_4, [G_1, G_4] = G_6, [G_2, G_3] = G_5,$
$[G_2, G_5] = \varepsilon G_6;$
$\mathfrak{l}_{6,23} : [G_1, G_2] = G_4, [G_1, G_3] = G_5, [G_1, G_4] = G_6, [G_2, G_3] = G_6;$
$\mathfrak{l}^{\varepsilon}_{6,24} : [G_1, G_2] = G_3, [G_1, G_3] = G_5, [G_1, G_4] = \varepsilon G_6, [G_2, G_3] = G_6,$
$[G_2, G_4] = G_5;$
$\mathfrak{l}_{6,25} : [G_1, G_2] = G_4, [G_1, G_4] = G_6, [G_1, G_3] = G_5$

such that $\varepsilon \in \{-1, 0, 1\}$.

Proposition 3.1. *Among the* 6*-dimensional indecomposable metric Lie algebras the metric Lie algebras* $(\mathfrak{l}_{6,j}, \langle ., .\rangle)$*,* $j = 11, \dots, 18, 20, 23, 25$*,* $(\mathfrak{l}^{\varepsilon=0}_{6,19}, \langle ., .\rangle)$*,* $(\mathfrak{l}^{\varepsilon=0}_{6,21}, \langle ., .\rangle)$ *have a framing determined by ideals.*

Proof. According to Theorem 1 in [5, p. 5], the filiform metric Lie algebras $L_{6,k}$ for $k = 14, \dots, 18$ have a framing determined by ideals.

In the Lie algebras $\mathfrak{l}_{6,k}$, $k = 11, 12, 13$ the center is $Z(\mathfrak{l}_{6,k}) = \mathrm{span}(G_6)$, the commutator subalgebra is $\mathcal{S}^1(\mathfrak{l}_{6,k}) = \mathrm{span}(G_4, G_5, G_6)$, the second member of the lower central series is $\mathcal{S}^2(\mathfrak{l}_{6,k}) = \mathrm{span}(G_5, G_6)$. In the Lie algebras $\mathfrak{l}_{6,l}$, $l = 11, 13$ the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,l}))$ is $\mathrm{span}(G_3, G_4, G_5, G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^2(\mathfrak{l}_{6,l}))$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$. For the Lie algebra $\mathfrak{l}_{6,12}$ the preimage $\pi^{-1}(Z(\mathfrak{l}_{6,12}/\mathcal{S}^2(\mathfrak{l}_{6,12})))$ of the center of the factor algebra $\mathfrak{l}_{6,12}/\mathcal{S}^2(\mathfrak{l}_{6,12})$ in $\mathfrak{l}_{6,12}$ is $\mathrm{span}(G_3, G_4, G_5, G_6)$ and the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,12}))$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$.

In the Lie algebra $\mathfrak{l}^{\varepsilon=0}_{6,19}$ the center is $Z(\mathfrak{l}^{\varepsilon=0}_{6,19}) = \mathrm{span}(G_5, G_6)$, the commutator subalgebra is $\mathcal{S}^1(\mathfrak{l}^{\varepsilon=0}_{6,19}) = \mathrm{span}(G_4, G_5, G_6)$, the second member of the lower central series is $\mathcal{S}^2(\mathfrak{l}^{\varepsilon=0}_{6,19}) = \mathrm{span}(G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}^{\varepsilon=0}_{6,19}))$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$. The preimage of the center of the factor algebra $\mathfrak{l}^{\varepsilon=0}_{6,19}/Z(\mathfrak{l}^{\varepsilon=0}_{6,19})$ in $\mathfrak{l}^{\varepsilon=0}_{6,19}$ is $\pi^{-1}(Z(\mathfrak{l}^{\varepsilon=0}_{6,19}/Z(\mathfrak{l}^{\varepsilon=0}_{6,19}))) = \mathrm{span}(G_3, G_4, G_5, G_6)$.

In the Lie algebra $\mathfrak{l}_{6,20}$ the centre is $Z(\mathfrak{l}_{6,20}) = \mathrm{span}(G_6)$, the commutator subalgebra is $\mathcal{S}^1(\mathfrak{l}_{6,20}) = \mathrm{span}(G_4, G_5, G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))$ is $\mathrm{span}(G_3, G_4, G_5, G_6)$, the commutator $[\mathfrak{l}_{6,20}, \mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))]$ is $\mathrm{span}(G_5, G_6)$. We denote by $\overline{l_{6,20}}$ the factor Lie algebra $l_{6,20}/Z(l_{6,20}) = \mathrm{span}(\overline{G_1}, \overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5})$ with the Lie brackets $[\overline{G_1}, \overline{G_2}] = \overline{G_4}$, $[\overline{G_1}, \overline{G_3}] = \overline{G_5}$. The factor Lie algebra $\overline{\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))} = \mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))/Z(l_{6,20})$ is the Lie algebra $\mathrm{span}(\overline{G_3}, \overline{G_4}, \overline{G_5})$. The centralizer $\overline{\mathcal{C}}(\overline{\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))})$ of $\overline{\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))}$ in $\overline{\mathfrak{l}_{6,20}}$ is $\mathrm{span}(\overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5})$. The preimage $\pi^{-1}(\overline{\mathcal{C}}(\overline{\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,20}))}))$ in $l_{6,20}$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$.

In the Lie algebra $\mathfrak{l}^{\varepsilon=0}_{6,21}$ the centre is $Z(\mathfrak{l}^{\varepsilon=0}_{6,21}) = \mathrm{span}(G_5, G_6)$, the commutator subalgebra is $\mathcal{S}^1(\mathfrak{l}^{\varepsilon=0}_{6,21}) = \mathrm{span}(G_3, G_4, G_5, G_6)$, the second member of the lower central series is $\mathcal{S}^2(\mathfrak{l}^{\varepsilon=0}_{6,21}) = \mathrm{span}(G_4, G_5, G_6)$, the third member of the lower central series is $\mathcal{S}^3(\mathfrak{l}_{6,21}^{\varepsilon=0}) = \mathrm{span}(G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^2(\mathfrak{l}_{6,21}^{\varepsilon=0}))$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$.

In the Lie algebras $\mathfrak{l}_{6,k}$, $k = 23, 25$ the center is $Z(\mathfrak{l}_{6,k}) = \mathrm{span}(G_5, G_6)$, the commutator subalgebra is $\mathcal{S}^1(\mathfrak{l}_{6,k}) = \mathrm{span}(G_4, G_5, G_6)$, the second member of the lower central series is $\mathcal{S}^2(\mathfrak{l}_{6,k}) = \mathrm{span}(G_6)$ and the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,k}))$ is $\mathrm{span}(G_2, G_3, G_4, G_5, G_6)$. The preimage $\pi^{-1}(Z(\mathfrak{l}_{6,k}/Z(\mathfrak{l}_{6,k})))$ of the center of the factor algebra $\mathfrak{l}_{6,k}/Z(\mathfrak{l}_{6,k})$ in $\mathfrak{l}_{6,k}$ is $\mathrm{span}(G_3, G_4, G_5, G_6)$.

Hence the subspaces $\mathrm{span}(G_i, \cdots, G_6)$, $i = 1, \ldots, 6$, of the Lie algebras $\mathfrak{l}_{6,11}$, $\mathfrak{l}_{6,12}$, $\mathfrak{l}_{6,13}$, $\mathfrak{l}_{6,19}^{\varepsilon=0}$, $\mathfrak{l}_{6,20}$, $\mathfrak{l}_{6,21}^{\varepsilon=0}$, $\mathfrak{l}_{6,23}$, $\mathfrak{l}_{6,25}$ form a descending series of characteristic ideals. Therefore the metric Lie algebras listed in the proposition have a framing determined by ideals (see Lemma 3 in [5]).

The metric Lie algebra belonging to $\mathfrak{l}_{6,10}$ does not have a framing determined by ideals, since the characteristic ideals of $\mathfrak{l}_{6,10}$ are the centre $Z(\mathfrak{l}_{6,10}) = \mathrm{span}(G_6)$, the commutator subalgebra $\mathcal{S}^1(\mathfrak{l}_{6,10}) = \mathrm{span}(G_3, G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,10})) = \mathrm{span}(G_2, G_3, G_4, G_5, G_6)$ and the preimage $\pi^{-1}(Z(\mathfrak{l}_{6,10}/Z(\mathfrak{l}_{6,10}))) = \mathrm{span}(G_3, G_4, G_5, G_6)$ of the centre of the factor Lie algebra $\mathfrak{l}_{6,10}/Z(\mathfrak{l}_{6,10})$ in $\mathfrak{l}_{6,10}$.

A framing determined by ideals does not exist for the metric Lie algebra belonging to $\mathfrak{l}_{6,19}^{\varepsilon}$, $\varepsilon \in \{-1, 1\}$, because the characteristic ideals of $\mathfrak{l}_{6,19}^{\varepsilon}$ are the centre $Z(\mathfrak{l}_{6,19}^{\varepsilon}) = \mathrm{span}(G_6)$, the commutator subalgebra $\mathcal{S}^1(\mathfrak{l}_{6,19}^{\varepsilon}) = \mathrm{span}(G_4, G_5, G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,19}^{\varepsilon})) = \mathrm{span}(G_2, G_4, G_5, G_6)$.

The characteristic ideals of $\mathfrak{l}_{6,21}^{\varepsilon}$, $\varepsilon \in \{-1, 1\}$ are the centre $Z(\mathfrak{l}_{6,21}^{\varepsilon}) = \mathrm{span}(G_6)$, the commutator subalgebra $\mathcal{S}^1(\mathfrak{l}_{6,21}^{\varepsilon}) = \mathrm{span}(G_3, G_4, G_5, G_6)$, the second member of the lower central series $\mathcal{S}^2(\mathfrak{l}_{6,21}^{\varepsilon}) = \mathrm{span}(G_4, G_5, G_6)$. Hence the metric Lie algebra corresponding to $\mathfrak{l}_{6,21}^{\varepsilon}$, $\varepsilon \in \{-1, 1\}$ does not allow a framing determined by ideals.

The metric Lie algebra belonging to $\mathfrak{l}_{6,24}^{\varepsilon}$, $\varepsilon \in \{-1, 0, 1\}$ does not have a framing determined by ideals, because the characteristic ideals of $\mathfrak{l}_{6,24}^{\varepsilon}$ are the centre $Z(\mathfrak{l}_{6,24}^{\varepsilon}) = \mathrm{span}(G_5, G_6)$, the commutator subalgebra $\mathcal{S}^1(\mathfrak{l}_{6,24}^{\varepsilon}) = \mathrm{span}(G_3, G_5, G_6)$, the centralizer $\mathcal{C}(\mathcal{S}^1(\mathfrak{l}_{6,24}^{\varepsilon})) = \mathrm{span}(G_3, G_4, G_5, G_6)$. □

4. Isometry classes of metric Lie algebras

Firstly, we consider the 6-dimensional Lie algebras $\mathfrak{l}_{6,11}$ and $\mathfrak{l}_{6,12}$.

Definition 4.1. Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. We denote by $\mathfrak{n}_{6,11}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, .., 4$, $j = 1, ..., 6$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$\begin{aligned} &[E_1, E_2] = \alpha_1 E_4 + \beta_1 E_5 + \beta_2 E_6, && [E_1, E_4] = \alpha_2 E_5 + \beta_5 E_6, && [E_2, E_3] = \beta_6 E_6, \\ &[E_1, E_3] = \beta_3 E_5 + \beta_4 E_6, && [E_1, E_5] = \alpha_3 E_6, && [E_2, E_4] = \alpha_4 E_6. \end{aligned} \tag{4.1}$$

Let $\mathfrak{n}_{6,12}(\alpha_i,\beta_j), \alpha_i,\beta_j \in \mathbb{R}, i=1,\dots,4, j=1,\dots,5$ with $\alpha_i \neq 0$ be the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$\begin{aligned}&[E_1,E_2]=\alpha_1E_4+\beta_1E_5+\beta_2E_6,\ [E_1,E_4]=\alpha_2E_5+\beta_5E_6,\ [E_2,E_3]=\alpha_4E_6,\\&[E_1,E_3]=\beta_3E_5+\beta_4E_6,\qquad [E_1,E_5]=\alpha_3E_6.\end{aligned}\tag{4.2}$$

The bracket operations (4.1) and (4.2) satisfy the Jacobi identity.

Theorem 4.2. *Let $\langle .,. \rangle$ be an inner product on the 6-dimensional Lie algebra $\mathfrak{l}_{6,11}$, respectively $\mathfrak{l}_{6,12}$.*

1. *There is a unique metric Lie algebra $(\mathfrak{n}_{6,11}(\alpha_i,\beta_j),\langle .,. \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,11},\langle .,. \rangle)$ with $\alpha_i>0$, $i=1,\dots,4$, and such that one of the following cases is satisfied*
   1. *at least two of the elements of the set $\{\beta_1,\beta_3,\beta_4,\beta_5,\beta_6\}$ are positive with the exception of the pairs $\{\beta_1,\beta_5\}$ and $\{\beta_3,\beta_6\}$,*
   2. $\beta_1>0$ *or* $\beta_5>0$, $\beta_3=\beta_4=\beta_6=0$,
   3. $\beta_3>0$ *or* $\beta_6>0$, $\beta_1=\beta_4=\beta_5=0$,
   4. $\beta_4>0$, $\beta_1=\beta_3=\beta_5=\beta_6=0$,
   5. $\beta_1=\beta_3=\beta_4=\beta_5=\beta_6=0$.

   There is a unique metric Lie algebra $(\mathfrak{n}_{6,12}(\alpha_i,\beta_j),\langle .,. \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,12},\langle .,. \rangle)$ with $\alpha_i>0$, $i=1,\dots,4$ and such that one of the above cases 1. − 5. *holds with* $\beta_6=0$.
2. *The group $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,11}(\alpha_i,\beta_j),\langle .,. \rangle)$ is the following group:*
   (a) *in case* 1. *the group $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))$ is trivial,*
   (b) *in case* 2. *one has* $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))=\{TE_i=E_i, i=1,2,4,5,6, TE_3=\varepsilon E_3,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (c) *in case* 3. *one has* $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))=\{TE_2=E_2, TE_5=E_5, TE_i=\varepsilon E_i, i=1,3,4,6,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (d) *in case* 4. *one has* $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))=\{TE_2=E_2, TE_3=E_3, TE_5=E_5, TE_i=\varepsilon E_i, i=1,4,6,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (e) *in case* 5. *one has* $\mathcal{OA}(\mathfrak{n}_{6,11}(\alpha_i,\beta_j))=\{TE_2=E_2, TE_5=E_5, TE_i=\varepsilon_1E_i, i=1,4,6, TE_3=\varepsilon_3E_3,\ \varepsilon_1,\varepsilon_3=\pm1\}\simeq\mathbb{Z}_2\times\mathbb{Z}_2$.
3. *The group $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,12}(\alpha_i,\beta_j),\langle .,. \rangle)$ is the following group:*
   (a) *in case* 1. *the group $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))$ is trivial,*
   (b) *in case* 2. *one has* $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))=\{TE_1=E_1, TE_3=E_3, TE_i=\varepsilon E_i, i=2,4,5,6,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (c) *in case* 3. *one has* $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))=\{TE_2=E_2, TE_5=E_5, TE_i=\varepsilon E_i, i=1,3,4,6,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (d) *in case* 4. *one has* $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))=\{TE_4=E_4, TE_6=E_6, TE_i=\varepsilon E_i, i=1,2,3,5,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,
   (e) *in case* 5. *one has* $\mathcal{OA}(\mathfrak{n}_{6,12}(\alpha_i,\beta_j))=\{TE_i=\varepsilon_1E_i, i=1,3, TE_j=\varepsilon_2E_j, j=2,5, TE_k=\varepsilon_1\varepsilon_2E_k, k=4,6,\ \varepsilon_1,\varepsilon_2=\pm1\}\simeq\mathbb{Z}_2\times\mathbb{Z}_2$.

Proof. According to Proposition 3.1 we apply the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ and we obtain an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,11}$ and $\mathfrak{l}_{6,12}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. Hence the orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$ and $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we receive for $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$ and $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$

$$[F_1, F_2] = \alpha_1 F_4 + \beta_1 F_5 + \beta_2 F_6, \qquad [F_1, F_4] = \alpha_2 F_5 + \beta_5 F_6,$$
$$[F_1, F_3] = \beta_3 F_5 + \beta_4 F_6, \qquad [F_1, F_5] = \alpha_3 F_6, \tag{4.3}$$

and for $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$

$$[F_2, F_3] = \beta_6 F_6, \qquad [F_2, F_4] = \alpha_4 F_6, \tag{4.4}$$

for $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$

$$[F_2, F_3] = \alpha_4 F_6, \tag{4.5}$$

with $\alpha_i > 0$, $i = 1, \ldots, 4$, and $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 6$. Changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = -F_4, \tilde{F_5} = F_5, \tilde{F_6} = -F_6$ we obtain for $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$ and $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} + \beta_2 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_4}] = \alpha_2 \tilde{F_5} - \beta_5 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_3}] = \beta_3 \tilde{F_5} - \beta_4 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_5}] = \alpha_3 \tilde{F_6},$$

and for $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$

$$[\tilde{F_2}, \tilde{F_3}] = \beta_6 \tilde{F_6}, \qquad [\tilde{F_2}, \tilde{F_4}] = \alpha_4 \tilde{F_6},$$

for $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$

$$[\tilde{F_2}, \tilde{F_3}] = \alpha_4 \tilde{F_6}.$$

Similarly, for $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$ the change of the basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = F_5, \tilde{F_6} = -F_6$ yields

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} + \beta_2 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_4}] = \alpha_2 \tilde{F_5} - \beta_5 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_5}] = \alpha_3 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_3}] = -\beta_3 \tilde{F_5} + \beta_4 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_3}] = -\beta_6 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_4}] = \alpha_4 \tilde{F_6},$$

and for $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$ the change of the basis: $\tilde{F_1} = F_1, \tilde{F_2} = -F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ gives

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} + \beta_1 \tilde{F_5} + \beta_2 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_4}] = \alpha_2 \tilde{F_5} + \beta_5 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_5}] = \alpha_3 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_3}] = -\beta_3 \tilde{F_5} - \beta_4 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_3}] = \alpha_4 \tilde{F_6}.$$

Hence there is an orthonormal basis such that in commutators (4.3) and (4.4) as well as (4.3) and (4.5) we have $\alpha_i > 0, i = 1, \ldots, 4$ and one of the cases 1. – 5. in assertion 1. is satisfied. This proves the existence of the metric Lie algebras $(\mathfrak{l}_{6,11}, \langle ., .\rangle)$ and $(\mathfrak{l}_{6,12}, \langle ., .\rangle)$ having properties as in assertion 1.

Let the linear map $T : \mathfrak{n}_{6,k}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,k}(\alpha_i', \beta_j')$, $k = 11, 12$, be an isometric isomorphism. The decomposition $\mathbb{R}\, E_1 \oplus \mathbb{R}\, E_2 \oplus \mathbb{R}\, E_3 \oplus \mathbb{R}\, E_4 \oplus \mathbb{R}\, E_5 \oplus \mathbb{R}\, E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0$, $i = 1, \ldots, 4$. Hence by Lemma 2.3 we have $\alpha_i = \alpha_i'$, $i = 1, \ldots, 4$ and $|\beta_j'| = \beta_j$ for $j = 1, \ldots, 6$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$. Using the commutation relations (4.3), (4.4) and (4.5) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, for $(\mathfrak{l}_{6,11}, \langle ., . \rangle)$ and $(\mathfrak{l}_{6,12}, \langle ., . \rangle)$ the equations

$$\begin{aligned} &\varepsilon_1\varepsilon_2 \left(\alpha_1 E_4 + \beta_1' E_5 + \beta_2' E_6\right) = \alpha_1 \varepsilon_4 E_4 + \beta_1 \varepsilon_5 E_5 + \beta_2 \varepsilon_6 E_6, \\ &\varepsilon_1\varepsilon_3 \left(\beta_3' E_5 + \beta_4' E_6\right) = \beta_3 \varepsilon_5 E_5 + \beta_4 \varepsilon_6 E_6, \quad \varepsilon_1\varepsilon_5 \left(\alpha_3 E_6\right) = \alpha_3 \varepsilon_6 E_6, \\ &\varepsilon_1\varepsilon_4 \left(\alpha_2 E_5 + \beta_5' E_6\right) = \alpha_2 \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6, \end{aligned} \tag{4.6}$$

and for $(\mathfrak{l}_{6,11}, \langle ., . \rangle)$ the equations

$$\varepsilon_2\varepsilon_3 \left(\beta_6' E_6\right) = \beta_6 \varepsilon_6 E_6, \quad \varepsilon_2\varepsilon_4 \left(\alpha_4 E_6\right) = \alpha_4 \varepsilon_6 E_6, \tag{4.7}$$

and for $(\mathfrak{l}_{6,12}, \langle ., . \rangle)$ the equation

$$\varepsilon_2\varepsilon_3 \left(\alpha_4 E_6\right) = \alpha_4 \varepsilon_6 E_6. \tag{4.8}$$

From (4.6) and (4.7) it follows $\varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, $\varepsilon_1\varepsilon_5 = \varepsilon_2\varepsilon_4 = \varepsilon_6$. Then one has $\varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_6$. Therefore one has $\beta_2' = \beta_2$.
If $\beta_1 = \beta_1' > 0$ or $\beta_5 = \beta_5' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_5$ or $\varepsilon_1\varepsilon_4 = \varepsilon_6$, which yields that $\varepsilon_i = 1, i = 1, 2, 4, 5, 6$.
If $\beta_3 = \beta_3' > 0$ or $\beta_6 = \beta_6' > 0$, then we have additionally $\varepsilon_1\varepsilon_3 = \varepsilon_5$ or $\varepsilon_2\varepsilon_3 = \varepsilon_6$. Hence one has $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$.
If $\beta_4 = \beta_4' > 0$, then we get $\varepsilon_1\varepsilon_3 = \varepsilon_6$, which gives that $\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$.
Using these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we have $\varepsilon_i = 1, i = 1, 2, 4, 5, 6$,
in case 3. we get $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$,
in case 4. we have $\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$,
in case 5. we obtain $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$.

From (4.6) and (4.8) it follows $\varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, $\varepsilon_1\varepsilon_5 = \varepsilon_2\varepsilon_3 = \varepsilon_6$. Then one has $\varepsilon_1 = \varepsilon_3$, $\varepsilon_2 = \varepsilon_5$, $\varepsilon_1\varepsilon_2 = \varepsilon_4 = \varepsilon_6$. Using this we have $\varepsilon_1\varepsilon_2 = \varepsilon_6$ and hence one has $\beta_2' = \beta_2$.

If $\beta_1 = \beta_1' > 0$ or $\beta_5 = \beta_5' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_5$ or $\varepsilon_1\varepsilon_4 = \varepsilon_6$, hence in both cases we obtain $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$.

If $\beta_3 = \beta_3' > 0$, then we have in addition $\varepsilon_1\varepsilon_3 = \varepsilon_5$, which gives $\varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$.

If $\beta_4 = \beta_4' > 0$, then we get additionally $\varepsilon_1\varepsilon_3 = \varepsilon_6$, which yields $\varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5$.

Applying these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_i = 1, i = 1, \ldots, 6$,

in case 2. we get $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$,
in case 3. we obtain $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$,
in case 4. we have $\varepsilon_4 = \varepsilon_6 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5$,
in case 5. we get $\varepsilon_1 = \varepsilon_3$, $\varepsilon_2 = \varepsilon_5$ and $\varepsilon_1\varepsilon_2 = \varepsilon_4 = \varepsilon_6$.

Hence the system of Eqs. (4.6) and (4.7) as well as (4.6) and (4.8) are satisfied with $\beta'_j = \beta_j, j = 1, \ldots, 6$, in cases $1.-5.$ of the Theorem. This proves the uniqueness of the Lie algebras $\mathfrak{n}_{6,11}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,12}(\alpha_i, \beta_j)$ in cases $1.-5.$ This yields assertion 1.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}_{6,11}(\alpha_i, \beta_j)$, respectively $\mathfrak{n}_{6,12}(\alpha_i, \beta_j)$ then the system of equations given by (4.6) and (4.7), respectively (4.6) and (4.8) is satisfied with $\alpha_i > 0, i = 1, \ldots, 4, \beta'_j = \beta_j, j = 1, \ldots, 6$. Therefore the conditions for $\varepsilon_i, i = 1, \ldots, 6$, are the same as above. Taking this into account the group of orthogonal automorphisms of $\mathfrak{n}_{6,11}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,12}(\alpha_i, \beta_j)$ in case 1. is trivial, in cases 2. – 5. is isomorphic to the group given by 2b–2e and 3b–3e. This proves assertions 2 and 3. $\square$

Corollary 4.3. *Let* $(\aleph_{6,k}(\alpha_i, \beta_j), \langle .,. \rangle)$, $k = 11, 12$, *be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra* $(\mathfrak{n}_{6,k}(\alpha_i, \beta_j), \langle .,. \rangle)$, $k = 11, 12$. *The isometry group of* $(\aleph_{6,11}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is* $\mathcal{I}(\aleph_{6,11}(\alpha_i, \beta_j))$

$$= \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes \aleph_{6,11}(\alpha_i, \beta_j) & if\ \ \beta_j = 0, j = 1, 3, 4, 5, 6, \\ \mathbb{Z}_2 \ltimes \aleph_{6,11}(\alpha_i, \beta_j) & if\ \ \beta_1 > 0\ or\ \beta_5 > 0, \beta_3 = \beta_4 = \beta_6 = 0, \\ & or\ \ \beta_3 > 0\ or\ \beta_6 > 0, \beta_1 = \beta_4 = \beta_5 = 0, \\ & or\ \ \beta_4 > 0, \beta_j = 0, j = 1, 3, 5, 6, \\ \aleph_{6,11}(\alpha_i, \beta_j) & if\ \ \beta_1 > 0, \beta_3 > 0,\ or\ \beta_1 > 0, \beta_4 > 0, \\ & or\ \ \beta_1 > 0, \beta_6 > 0,\ or\ \beta_3 > 0, \beta_4 > 0, \\ & or\ \ \beta_3 > 0, \beta_5 > 0,\ or\ \beta_4 > 0, \beta_5 > 0, \\ & or\ \ \beta_4 > 0, \beta_6 > 0,\ or\ \beta_5 > 0, \beta_6 > 0. \end{cases}$$

The isometry group of $(\aleph_{6,12}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is*

$$\mathcal{I}(\aleph_{6,12}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes \aleph_{6,12}(\alpha_i, \beta_j) & if\ \ \beta_1 = \beta_3 = \beta_4 = \beta_5 = 0, \\ \mathbb{Z}_2 \ltimes \aleph_{6,12}(\alpha_i, \beta_j) & if\ \ \beta_1 > 0\ or\ \beta_5 > 0, \beta_3 = \beta_4 = 0, \\ & or\ \ \beta_3 > 0, \beta_1 = \beta_4 = \beta_5 = 0, \\ & or\ \ \beta_4 > 0, \beta_1 = \beta_3 = \beta_5 = 0, \\ \aleph_{6,12}(\alpha_i, \beta_j) & if\ \ \beta_1 > 0, \beta_3 > 0,\ or\ \beta_1 > 0, \beta_4 > 0, \\ & or\ \ \beta_3 > 0, \beta_4 > 0,\ or\ \beta_3 > 0, \beta_5 > 0, \\ & or\ \ \beta_4 > 0, \beta_5 > 0. \end{cases}$$

Secondly, we consider the 6-dimensional Lie algebra $\mathfrak{l}_{6,13}$.

Definition 4.4. Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,13}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, \dots, 4$, $j = 1, \dots, 7$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$\begin{aligned} &[E_1, E_2] = \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = \frac{\alpha_2 \alpha_3}{\alpha_4} E_4 + \beta_4 E_5 + \beta_5 E_6, \\ &[E_1, E_4] = \alpha_1 E_5 + \beta_6 E_6, && [E_1, E_5] = \alpha_2 E_6, \\ &[E_3, E_2] = \alpha_3 E_5 + \beta_7 E_6, && [E_4, E_2] = \alpha_4 E_6. \end{aligned} \tag{4.9}$$

The bracket operation (4.9) satisfies the Jacobi identity.

Theorem 4.5. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional Lie algebra $\mathfrak{l}_{6,13}$.*

1. *There is a unique metric Lie algebra $(\mathfrak{n}_{6,13}(\alpha_i, \beta_j), \langle ., . \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,13}, \langle ., . \rangle)$ with $\alpha_i > 0$, $i = 1, \dots, 4$ and such that one of the following cases is satisfied*
   1. *at least two of the elements of $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_6, \beta_7\}$ are positive with the exception of the pairs $\{\beta_1, \beta_3\}$, $\{\beta_4, \beta_6\}$, $\{\beta_4, \beta_7\}$, $\{\beta_6, \beta_7\}$,*
   2. *$\beta_1 > 0$ or $\beta_3 > 0$, $\beta_2 = \beta_4 = \beta_6 = \beta_7 = 0$,*
   3. *$\beta_2 > 0$, $\beta_1 = \beta_3 = \beta_4 = \beta_6 = \beta_7 = 0$,*
   4. *$\beta_4 > 0$ or $\beta_6 > 0$ or $\beta_7 > 0$, $\beta_1 = \beta_2 = \beta_3 = 0$,*
   5. *$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_6 = \beta_7 = 0$.*
2. *The group $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,13}(\alpha_i, \beta_j), \langle ., . \rangle)$ is the following group:*
   (a) *in case 1. the group $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j))$ is trivial,*
   (b) *in case 2. one has $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_3 = E_3, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1, 4, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
   (c) *in case 3. one has $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_4 = E_4, TE_6 = E_6, TE_i = \varepsilon E_i, i = 1, 3, 5,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
   (d) in case 4. one has $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_2 = E_2, TE_i = \varepsilon E_i, i = 3, 4, 5, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
   (e) in case 5. one has $\mathcal{OA}(\mathfrak{n}_{6,13}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_3 E_i, i = 3, 5, TE_j = \varepsilon_1 \varepsilon_3 E_j, j = 4, 6,\ \varepsilon_1, \varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. According to Proposition 3.1 we utilize the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ which yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,13}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,13}, \langle ., . \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$[F_1, F_2] = \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, \qquad [F_1, F_3] = \frac{\alpha_2 \alpha_3}{\alpha_4} F_4 + \beta_4 F_5 + \beta_5 F_6,$$

$$[F_1, F_4] = \alpha_1 F_5 + \beta_6 F_6, \qquad [F_1, F_5] = \alpha_2 F_6, \tag{4.10}$$
$$[F_3, F_2] = \alpha_3 F_5 + \beta_7 F_6, \qquad [F_4, F_2] = \alpha_4 F_6$$

with $\alpha_i > 0$, $i = 1, \ldots, 4$ and $\beta_j \in \mathbb{R}$, $j = 1, \ldots, 7$. Changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = F_4, \tilde{F_5} = -F_5, \tilde{F_6} = F_6$ we obtain

$$[\tilde{F_1}, \tilde{F_2}] = -\beta_1 \tilde{F_4} + \beta_2 \tilde{F_5} - \beta_3 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_3}] = \frac{\alpha_2\alpha_3}{\alpha_4} \tilde{F_4} - \beta_4 \tilde{F_5} + \beta_5 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_4}] = \alpha_1 \tilde{F_5} - \beta_6 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_5}] = \alpha_2 \tilde{F_6},$$
$$[\tilde{F_3}, \tilde{F_2}] = \alpha_3 \tilde{F_5} - \beta_7 \tilde{F_6}, \qquad [\tilde{F_4}, \tilde{F_2}] = \alpha_4 \tilde{F_6}.$$

Similarly, the change of the basis: $\tilde{F_1} = F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ gives

$$[\tilde{F_1}, \tilde{F_2}] = -\beta_1 \tilde{F_4} - \beta_2 \tilde{F_5} - \beta_3 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_3}] = \frac{\alpha_2\alpha_3}{\alpha_4} \tilde{F_4} + \beta_4 \tilde{F_5} + \beta_5 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_4}] = \alpha_1 \tilde{F_5} + \beta_6 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_5}] = \alpha_2 \tilde{F_6},$$
$$[\tilde{F_3}, \tilde{F_2}] = \alpha_3 \tilde{F_5} + \beta_7 \tilde{F_6}, \qquad [\tilde{F_4}, \tilde{F_2}] = \alpha_4 \tilde{F_6}.$$

Hence there is an orthonormal basis such that in commutators (4.10) we have $\alpha_i > 0, i = 1, \ldots, 4$ and one of the cases in assertion 1. is satisfied. This proves the existence of $\mathfrak{n}_{6,13}(\alpha_i, \beta_j)$ with the properties in assertion 1.

Let the linear map $T : \mathfrak{n}_{6,13}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,13}(\alpha'_i, \beta'_j)$ be an isometric isomorphism. The decomposition $\mathbb{R}\, E_1 \oplus \mathbb{R}\, E_2 \oplus \mathbb{R}\, E_3 \oplus \mathbb{R}\, E_4 \oplus \mathbb{R}\, E_5 \oplus \mathbb{R}\, E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha'_i > 0$, $i = 1, \ldots, 4$. According to Lemma 2.3 we have $\alpha_i = \alpha'_i$, $i = 1, \ldots, 4$ and $|\beta'_j| = \beta_j$, $j = 1, \ldots, 7$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$. Using the commutation relations (4.10) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, the equations

$$\varepsilon_1\varepsilon_2 \left(\beta'_1 E_4 + \beta'_2 E_5 + \beta'_3 E_6\right) = \beta_1 \varepsilon_4 E_4 + \beta_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6,$$
$$\varepsilon_1\varepsilon_3 \left(\frac{\alpha_2\alpha_3}{\alpha_4} E_4 + \beta'_4 E_5 + \beta'_5 E_6\right) = \frac{\alpha_2\alpha_3}{\alpha_4} \varepsilon_4 E_4 + \beta_4 \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6, \tag{4.11}$$
$$\varepsilon_1\varepsilon_4 \left(\alpha_1 E_5 + \beta'_6 E_6\right) = \alpha_1 \varepsilon_5 E_5 + \beta_6 \varepsilon_6 E_6, \quad \varepsilon_1\varepsilon_5 \left(\alpha_2 E_6\right) = \alpha_2 \varepsilon_6 E_6,$$
$$\varepsilon_3\varepsilon_2 \left(\alpha_3 E_5 + \beta'_7 E_6\right) = \alpha_3 \varepsilon_5 E_5 + \beta_7 \varepsilon_6 E_6, \quad \varepsilon_4\varepsilon_2 \left(\alpha_4 E_6\right) = \alpha_4 \varepsilon_6 E_6.$$

From (4.11) it follows $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_3\varepsilon_2 = \varepsilon_5$, $\varepsilon_1\varepsilon_5 = \varepsilon_4\varepsilon_2 = \varepsilon_6$. Hence one has $\varepsilon_2 = 1$, $\varepsilon_3 = \varepsilon_5$, $\varepsilon_1\varepsilon_3 = \varepsilon_4 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_3 = \varepsilon_6$. Therefore one has $\beta'_5 = \beta_5$.

If $\beta_1 = \beta'_1 > 0$ or $\beta_3 = \beta'_3 > 0$, then we have additionally $\varepsilon_1\varepsilon_2 = \varepsilon_4$ or $\varepsilon_1\varepsilon_2 = \varepsilon_6$. Hence one has $\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$.

If $\beta_2 = \beta'_2 > 0$, then we get in addition $\varepsilon_1\varepsilon_2 = \varepsilon_5$, which gives $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$.

If $\beta_4 = \beta'_4 > 0$ or $\beta_6 = \beta'_6 > 0$ or $\beta_7 = \beta'_7 > 0$, then we get additionally $\varepsilon_1\varepsilon_3 = \varepsilon_5$ or $\varepsilon_1\varepsilon_4 = \varepsilon_6$ or $\varepsilon_3\varepsilon_2 = \varepsilon_6$. Hence in these cases we obtain $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$.

Using these relations in assertion 1. of the Theorem
in case 1. we get $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we obtain $\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_6$,
in case 3. we have $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$,
in case 4. we obtain $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$,
in case 5. we get $\varepsilon_2 = 1$, $\varepsilon_3 = \varepsilon_5$ and $\varepsilon_1\varepsilon_3 = \varepsilon_4 = \varepsilon_6$.
Hence the system of Eq. (4.11) is satisfied with $\beta_j' = \beta_j, j = 1, \ldots, 7$ in cases 1.−5. of the Theorem. Therefore the uniqueness of the Lie algebra $\mathfrak{n}_{6,13}(\alpha_i, \beta_j)$ in cases 1. − 5. is proved. This yields assertion 1.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}_{6,13}(\alpha_i, \beta_j)$, then the system of equations given by (4.11) is satisfied with $\alpha_i > 0, i = 1, \ldots, 4$, $\beta_j' = \beta_j, j = 1, \ldots, 7$. Therefore in cases 1. − 5. the conditions for $\varepsilon_i, i = 1, \ldots, 6$, are given above. Taking this into consideration the group of orthogonal automorphisms of $\mathfrak{n}_{6,13}(\alpha_i, \beta_j)$ in case 1. is trivial, in cases 2. − 5. is isomorphic to the group given by 2b–2e. This proves assertion 2. □

Corollary 4.6. *Let* $(\aleph_{6,13}(\alpha_i, \beta_j), \langle ., . \rangle)$ *be the connected and simply connected Riemannian nilmanifold belonging to* $(\mathfrak{n}_{6,13}(\alpha_i, \beta_j), \langle ., . \rangle)$. *The group of isometries of* $(\aleph_{6,13}(\alpha_i, \beta_j), \langle ., . \rangle)$ *is*

$$\mathcal{I}(\aleph_{6,13}(\alpha_i, \beta_j))$$
$$= \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes \aleph_{6,13}(\alpha_i, \beta_j) & if\ \beta_j = 0, j = 1,2,3,4,6,7, \\ \mathbb{Z}_2 \ltimes \aleph_{6,13}(\alpha_i, \beta_j) & if\ \beta_1 > 0\ or\ \beta_3 > 0, \beta_j = 0, j = 2,4,6,7, \\ & or\ \beta_2 > 0, \beta_j = 0, j = 1,3,4,6,7, \\ & or\ \beta_4 > 0\ or\ \beta_6 > 0\ or\ \beta_7 > 0\ and \\ & \beta_1 = \beta_2 = \beta_3 = 0, \\ \aleph_{6,13}(\alpha_i, \beta_j) & if\ \beta_1 > 0, \beta_2 > 0,\ or\ \beta_1 > 0, \beta_4 > 0, \\ & or\ \beta_1 > 0, \beta_6 > 0,\ or\ \beta_1 > 0, \beta_7 > 0, \\ & or\ \beta_2 > 0, \beta_3 > 0,\ or\ \beta_2 > 0, \beta_4 > 0, \\ & or\ \beta_2 > 0, \beta_6 > 0,\ or\ \beta_2 > 0, \beta_7 > 0, \\ & or\ \beta_3 > 0, \beta_4 > 0,\ or\ \beta_3 > 0, \beta_6 > 0, \\ & or\ \beta_3 > 0, \beta_7 > 0. \end{cases}$$

We treat the 6-dimensional Lie algebra $\mathfrak{l}_{6,19}^{\varepsilon=0}$.

Definition 4.7. Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1,2,3, j = 1,...,5$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$[E_2, E_1] = \alpha_1 E_4 + \beta_1 E_5 + \beta_2 E_6, \qquad [E_1, E_3] = \beta_3 E_5 + \beta_4 E_6, \tag{4.12}$$
$$[E_1, E_4] = \alpha_2 E_6, \qquad [E_2, E_3] = \alpha_3 E_5 + \beta_5 E_6.$$

The bracket operation (4.12) satisfies the Jacobi identity.

Theorem 4.8. *Let* $\langle .,. \rangle$ *be an inner product on the* 6*-dimensional Lie algebra* $\mathfrak{l}_{6,19}^{\varepsilon=0}$.

1. *There is a unique metric Lie algebra* $(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j), \langle .,. \rangle)$ *which is isometrically isomorphic to the metric Lie algebra* $(\mathfrak{l}_{6,19}^{\varepsilon=0}, \langle .,. \rangle)$ *with* $\alpha_i > 0$, $i = 1, 2, 3$, *and such that one of the following cases is satisfied*
    1. *at least three of the elements of the set* $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ *are positive with the exception of the triples* $\{\beta_1, \beta_2, \beta_5\}$ *and* $\{\beta_3, \beta_4, \beta_5\}$,
    2. *at least two of the elements of the set* $\{\beta_1, \beta_2, \beta_5\}$ *are positive and* $\beta_3 = \beta_4 = 0$,
    3. *at least two of the elements of the set* $\{\beta_3, \beta_4, \beta_5\}$ *are positive and* $\beta_1 = \beta_2 = 0$,
    4. $\beta_1 > 0,\ \beta_3 > 0,\ \beta_2 = \beta_4 = \beta_5 = 0$,
    5. $\beta_1 > 0,\ \beta_4 > 0,\ \beta_2 = \beta_3 = \beta_5 = 0$,
    6. $\beta_2 > 0,\ \beta_3 > 0,\ \beta_1 = \beta_4 = \beta_5 = 0$,
    7. $\beta_2 > 0,\ \beta_4 > 0,\ \beta_1 = \beta_3 = \beta_5 = 0$,
    8. $\beta_1 > 0,\ \beta_j = 0,\ j = 2, 3, 4, 5$,
    9. $\beta_2 > 0,\ \beta_j = 0,\ j = 1, 3, 4, 5$,
    10. $\beta_3 > 0,\ \beta_j = 0,\ j = 1, 2, 4, 5$,
    11. $\beta_4 > 0,\ \beta_j = 0,\ j = 1, 2, 3, 5$,
    12. $\beta_5 > 0,\ \beta_j = 0,\ j = 1, 2, 3, 4$,
    13. $\beta_j = 0,\ j = 1, 2, 3, 4, 5$.
2. *The group* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j))$ *of orthogonal automorphisms of the metric Lie algebra* $(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is the following group:*
    (a) *in case* 1. *the group* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j))$ *is trivial,*
    (b) *in case* 2. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_3 = E_3, TE_i = \varepsilon E_i, i = 2, 4, 5, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (c) *in case* 3. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_3 = E_3, TE_4 = E_4, TE_i = \varepsilon E_i, i = 1, 2, 5, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (d) *in case* 4. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_4 = E_4, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1, 2, 3, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (e) *in case* 5. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_6 = E_6, TE_i = \varepsilon E_i, i = 1, 3, 4, 5,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (f) *in case* 6. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_2 = E_2, TE_4 = E_4, TE_6 = E_6, TE_i = \varepsilon E_i, i = 3, 5,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (g) *in case* 7. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_5 = E_5, TE_i = \varepsilon E_i, i = 2, 3, 4, 6,\ \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
    (h) *in case* 8. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_i = \varepsilon_1 E_i, i = 1, 3, TE_j = \varepsilon_2 E_j, j = 2, 6, TE_k = \varepsilon_1 \varepsilon_2 E_k, k = 4, 5,\ \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,
    (i) *in case* 9. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_i = \varepsilon_2 E_i, i = 2, 4, 6, TE_3 = \varepsilon_3 E_3, TE_5 = \varepsilon_2 \varepsilon_3 E_5,\ \varepsilon_2, \varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

(j) *in case* 10. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_4 = E_4, TE_i = \varepsilon_1 E_i, i = 1,2,6, TE_3 = \varepsilon_3 E_3, TE_5 = \varepsilon_1\varepsilon_3 E_5,\ \varepsilon_1,\varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

(k) *in case* 11. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_i = \varepsilon_1 E_i, i = 1,5, TE_j = \varepsilon_2 E_j, j = 2,6, TE_k = \varepsilon_1\varepsilon_2 E_k, k = 3,4,\ \varepsilon_1,\varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

(l) *in case* 12. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_3 = E_3, TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i, i = 2,5,6, TE_4 = \varepsilon_1\varepsilon_2 E_4,\ \varepsilon_1,\varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,

(m) *in case* 13. *one has* $\mathcal{OA}(\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i, i = 2,6, TE_3 = \varepsilon_3 E_3, TE_4 = \varepsilon_1\varepsilon_2 E_4, TE_5 = \varepsilon_2\varepsilon_3 E_5,\ \varepsilon_1,\varepsilon_2, \varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. According to Proposition 3.1 the application of the Gram–Schmidt process to the ordered basis $\{G_6,G_5,G_4,G_3,G_2,G_1\}$ yields an orthonormal basis $\{F_1,F_2,F_3,F_4,F_5,F_6\}$ of $\mathfrak{l}_{6,19}^{\varepsilon=0}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j>i)$ and orthogonal to span $(G_j; j>i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,19}^{\varepsilon=0}, \langle .,. \rangle)$ and the vectors of the new basis can be written into the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$. Hence we receive

$$[F_2,F_1] = \alpha_1 F_4 + \beta_1 F_5 + \beta_2 F_6, \qquad [F_1,F_3] = \beta_3 F_5 + \beta_4 F_6, \tag{4.13}$$
$$[F_1,F_4] = \alpha_2 F_6, \qquad [F_2,F_3] = \alpha_3 F_5 + \beta_5 F_6$$

with $\alpha_i > 0$, $i = 1,2,3$ and $\beta_j \in \mathbb{R}$, $j = 1,\ldots,5$. The changes of the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = F_5, \tilde{F_6} = F_6$, respectively $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = F_6$, respectively $\tilde{F_1} = -F_1, \tilde{F_2} = -F_2, \tilde{F_3} = F_3, \tilde{F_4} = F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ give

$$[\tilde{F_2},\tilde{F_1}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \qquad [\tilde{F_1},\tilde{F_3}] = -\beta_3 \tilde{F_5} - \beta_4 \tilde{F_6},$$
$$[\tilde{F_1},\tilde{F_4}] = \alpha_2 \tilde{F_6}, \qquad [\tilde{F_2},\tilde{F_3}] = \alpha_3 \tilde{F_5} + \beta_5 \tilde{F_6},$$

respectively

$$[\tilde{F_2},\tilde{F_1}] = \alpha_1 \tilde{F_4} + \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \qquad [\tilde{F_1},\tilde{F_3}] = -\beta_3 \tilde{F_5} + \beta_4 \tilde{F_6},$$
$$[\tilde{F_1},\tilde{F_4}] = \alpha_2 \tilde{F_6}, \qquad [\tilde{F_2},\tilde{F_3}] = \alpha_3 \tilde{F_5} - \beta_5 \tilde{F_6},$$

respectively

$$[\tilde{F_2},\tilde{F_1}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \qquad [\tilde{F_1},\tilde{F_3}] = \beta_3 \tilde{F_5} + \beta_4 \tilde{F_6},$$
$$[\tilde{F_1},\tilde{F_4}] = \alpha_2 \tilde{F_6}, \qquad [\tilde{F_2},\tilde{F_3}] = \alpha_3 \tilde{F_5} + \beta_5 \tilde{F_6}.$$

Hence there is an orthonormal basis such that in commutators (4.13) one has $\alpha_i > 0$, $i = 1,2,3$, and one of the cases in assertion 1. holds. This proves the existence of $\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j)$ having properties as in assertion 1.

Let the linear map $T : \mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i,\beta_j) \to \mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i',\beta_j')$ be an isometric isomorphism. The decomposition $\mathbb{R}\,E_1 \oplus \mathbb{R}\,E_2 \oplus \mathbb{R}\,E_3 \oplus \mathbb{R}\,E_4 \oplus \mathbb{R}\,E_5 \oplus \mathbb{R}\,E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0$, $i = 1,2,3$. Hence by Lemma 2.3 we have $\alpha_i = \alpha_i'$, $i = 1,2,3$ and $|\beta_j'| = \beta_j$, $j = 1,\ldots,5$. Let $T(E_i) = \varepsilon_i E_i$,

$\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$. Using the commutation relations (4.13) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, the equations

$$\begin{aligned} &\varepsilon_2\varepsilon_1 (\alpha_1 E_4 + \beta_1' E_5 + \beta_2' E_6) = \alpha_1 \varepsilon_4 E_4 + \beta_1 \varepsilon_5 E_5 + \beta_2 \varepsilon_6 E_6, \\ &\varepsilon_1\varepsilon_3 (\beta_3' E_5 + \beta_4' E_6) = \beta_3 \varepsilon_5 E_5 + \beta_4 \varepsilon_6 E_6, \quad \varepsilon_1\varepsilon_4 (\alpha_2 E_6) = \alpha_2 \varepsilon_6 E_6, \\ &\varepsilon_2\varepsilon_3 (\alpha_3 E_5 + \beta_5' E_6) = \alpha_3 \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6. \end{aligned} \tag{4.14}$$

It follows $\varepsilon_2\varepsilon_1 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_6$, $\varepsilon_2\varepsilon_3 = \varepsilon_5$. Hence one has $\varepsilon_2 = \varepsilon_6$.
If $\beta_1 = \beta_1' > 0$, then we get additionally $\varepsilon_2\varepsilon_1 = \varepsilon_5$, which yields $\varepsilon_1 = \varepsilon_3$, $\varepsilon_2 = \varepsilon_6$ and $\varepsilon_4 = \varepsilon_5$.
If $\beta_2 = \beta_2' > 0$, then we have additionally $\varepsilon_2\varepsilon_1 = \varepsilon_6$, which gives $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$.
If $\beta_3 = \beta_3' > 0$, then one has in addition $\varepsilon_1\varepsilon_3 = \varepsilon_5$, which yields $\varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_6$.
If $\beta_4 = \beta_4' > 0$, then we get additionally $\varepsilon_1\varepsilon_3 = \varepsilon_6$. Hence one has $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_6$, $\varepsilon_3 = \varepsilon_4$.
If $\beta_5 = \beta_5' > 0$, then we have additionally $\varepsilon_2\varepsilon_3 = \varepsilon_6$, which gives $\varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_5 = \varepsilon_6$.
Applying these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we get $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$,
in case 3. we have $\varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = \varepsilon_6$,
in case 4. we get $\varepsilon_4 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_6$,
in case 5. we obtain $\varepsilon_2 = \varepsilon_6 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5$,
in case 6. we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$ and $\varepsilon_3 = \varepsilon_5$,
in case 7. we receive $\varepsilon_1 = \varepsilon_5 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$,
in case 8. we get $\varepsilon_1 = \varepsilon_3$, $\varepsilon_2 = \varepsilon_6$ and $\varepsilon_1\varepsilon_2 = \varepsilon_4 = \varepsilon_5$,
in case 9. we have $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$, $\varepsilon_2\varepsilon_3 = \varepsilon_5$,
in case 10. we obtain $\varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_6$, $\varepsilon_2\varepsilon_3 = \varepsilon_1\varepsilon_3 = \varepsilon_5$,
in case 11. we have $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_6$ and $\varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_4$,
in case 12. we get $\varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon_5 = \varepsilon_6$, $\varepsilon_1\varepsilon_2 = \varepsilon_4$,
in case 13. we obtain $\varepsilon_2 = \varepsilon_6$, $\varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_2\varepsilon_3 = \varepsilon_5$.
Hence the system of Eq. (4.14) is satisfied with $\beta_j' = \beta_j, j = 1, \ldots, 5$ in cases 1. – 13. of the Theorem, which proves the uniqueness of the Lie algebra $\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)$ in assertion 1.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)$, then the system of equations given by (4.14) is satisfied with $\alpha_i > 0, i = 1, 2, 3$, $\beta_j' = \beta_j, j = 1, \ldots, 5$. Therefore in cases 1. – 13. we obtain the above conditions for $\varepsilon_i, i = 1, \ldots, 6$. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{6,19}^{\varepsilon=0}(\alpha_i, \beta_j)$ in case 1. is trivial, in cases 2. – 13. is isomorphic to the group given by 2b–2 m, which proves assertion 2. □

Corollary 4.9. *Let $(\aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to $(\mathfrak{n}^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j),\langle .,.\rangle)$. The isometry group of $(\aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j),\langle .,.\rangle)$ is*

$$\mathcal{I}(\aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 \ltimes \aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j) & if\ \beta_j=0, j=1,2,3,4,5,\\ \mathbb{Z}_2\times\mathbb{Z}_2 \ltimes \aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_j=0, j=2,3,4,5,\\ & or\ \beta_2>0,\beta_j=0, j=1,3,4,5,\\ & or\ \beta_3>0,\beta_j=0, j=1,2,4,5,\\ & or\ \beta_4>0,\beta_j=0, j=1,2,3,5,\\ & or\ \beta_5>0,\beta_j=0, j=1,2,3,4,\\ \mathbb{Z}_2 \ltimes \aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2>0,\beta_3=\beta_4=0,\\ & or\ \beta_1>0,\beta_5>0,\beta_3=\beta_4=0,\\ & or\ \beta_2>0,\beta_5>0,\beta_3=\beta_4=0,\\ & or\ \beta_3>0,\beta_4>0,\beta_1=\beta_2=0,\\ & or\ \beta_3>0,\beta_5>0,\beta_1=\beta_2=0,\\ & or\ \beta_4>0,\beta_5>0,\beta_1=\beta_2=0,\\ & or\ \beta_1>0,\beta_3>0,\beta_2=\beta_4=\beta_5=0,\\ & or\ \beta_1>0,\beta_4>0,\beta_2=\beta_3=\beta_5=0,\\ & or\ \beta_2>0,\beta_3>0,\beta_1=\beta_4=\beta_5=0,\\ & or\ \beta_2>0,\beta_4>0,\beta_1=\beta_3=\beta_5=0,\\ \aleph^{\varepsilon=0}_{6,19}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2>0,\beta_3>0,\\ & or\ \beta_1>0,\beta_2>0,\beta_4>0,\\ & or\ \beta_1>0,\beta_3>0,\beta_4>0,\\ & or\ \beta_1>0,\beta_3>0,\beta_5>0,\\ & or\ \beta_1>0,\beta_4>0,\beta_5>0,\\ & or\ \beta_2>0,\beta_3>0,\beta_4>0,\\ & or\ \beta_2>0,\beta_3>0,\beta_5>0,\\ & or\ \beta_2>0,\beta_4>0,\beta_5>0. \end{cases}$$

We consider the 6-dimensional Lie algebra $\mathfrak{l}_{6,20}$.

Definition 4.10. Let $\{E_1,E_2,E_3,E_4,E_5,E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,20}(\alpha_i,\beta_j),\alpha_i,\beta_j\in\mathbb{R}, i=1,\dots,4$, $j=1,\dots,5$ with $\alpha_i\neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$\begin{aligned} &[E_1,E_2]=\alpha_1E_4+\beta_1E_5+\beta_2E_6, && [E_1,E_4]=\beta_4E_6, && [E_2,E_3]=\beta_5E_6,\\ &[E_1,E_3]=\alpha_2E_5+\beta_3E_6, && [E_1,E_5]=\alpha_3E_6, && [E_2,E_4]=\alpha_4E_6. \end{aligned} \tag{4.15}$$

The bracket operation (4.15) satisfies the Jacobi identity.

Theorem 4.11. *Let $\langle .,. \rangle$ be an inner product on the* 6*-dimensional Lie algebra* $\mathfrak{l}_{6,20}$.

1. *There is a unique metric Lie algebra* $(\mathfrak{n}_{6,20}(\alpha_i, \beta_j), \langle .,. \rangle)$ *which is isometrically isomorphic to the metric Lie algebra* $(\mathfrak{l}_{6,20}, \langle .,. \rangle)$ *with* $\alpha_i > 0$, $i = 1, \dots, 4$ *and such that one of the following cases is satisfied*
   1. *at least two of the elements of the set* $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ *are positive with the exception of the pairs* $\{\beta_1, \beta_4\}$ *and* $\{\beta_2, \beta_5\}$,
   2. $\beta_1 > 0$ *or* $\beta_4 > 0$, $\beta_2 = \beta_3 = \beta_5 = 0$,
   3. $\beta_2 > 0$ *or* $\beta_5 > 0$, $\beta_1 = \beta_3 = \beta_4 = 0$,
   4. $\beta_3 > 0$, $\beta_1 = \beta_2 = \beta_4 = \beta_5 = 0$,
   5. $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$.
2. *The group* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j))$ *of orthogonal automorphisms of the metric Lie algebra* $(\mathfrak{n}_{6,20}(\alpha_i, \beta_j), \langle .,. \rangle)$ *is the following group:*
   (a) *in case* 1. *the group* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j))$ *is trivial,*
   (b) *in case* 2. *one has* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j)) = \{TE_4 = E_4, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1, 2, 3, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
   (c) *in case* 3. *one has* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1, 3, 4, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
   (d) *in case* 4. *one has* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_3 = E_3, TE_5 = E_5, TE_6 = E_6, TE_i = \varepsilon E_i, i = 2, 4, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
   (e) *in case* 5. *one has* $\mathcal{OA}(\mathfrak{n}_{6,20}(\alpha_i, \beta_j)) = \{TE_5 = E_5, TE_i = \varepsilon_1 E_i, i = 1, 3, 6, TE_2 = \varepsilon_2 E_2, TE_4 = \varepsilon_1 \varepsilon_2 E_4, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. According to Proposition 3.1 we apply the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ and obtain an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,20}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,20}, \langle .,. \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$\begin{aligned} &[F_1, F_2] = \alpha_1 F_4 + \beta_1 F_5 + \beta_2 F_6, \quad [F_1, F_4] = \beta_4 F_6, \quad [F_2, F_3] = \beta_5 F_6, \\ &[F_1, F_3] = \alpha_2 F_5 + \beta_3 F_6, \quad [F_1, F_5] = \alpha_3 F_6, \quad [F_2, F_4] = \alpha_4 F_6 \end{aligned} \tag{4.16}$$

with $\alpha_i > 0$, $i = 1, \dots, 4$ and $\beta_j \in \mathbb{R}$, $j = 1, \dots, 5$. The change of the orthonormal basis: $\tilde{F_1} = F_1, \tilde{F_2} = -F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = F_5, \tilde{F_6} = F_6$ gives

$$\begin{aligned} &[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_4}] = -\beta_4 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_3}] = -\beta_5 \tilde{F_6}, \\ &[\tilde{F_1}, \tilde{F_3}] = \alpha_2 \tilde{F_5} + \beta_3 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_5}] = \alpha_3 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_4}] = \alpha_4 \tilde{F_6}. \end{aligned}$$

Similarly, changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = -F_4, \tilde{F_5} = F_5, \tilde{F_6} = -F_6$ we obtain

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} + \beta_2 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_4}] = -\beta_4 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_3}] = \beta_5 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_3}] = \alpha_2 \tilde{F_5} - \beta_3 \tilde{F_6}, \quad [\tilde{F_1}, \tilde{F_5}] = \alpha_3 \tilde{F_6}, \quad [\tilde{F_2}, \tilde{F_4}] = \alpha_4 \tilde{F_6}.$$

Hence there exists an orthonormal basis such that in commutators (4.16) we have $\alpha_i > 0, i = 1, \ldots, 4$ and one of the cases in assertion 1. is satisfied. This proves the existence of $\mathfrak{n}_{6,20}(\alpha_i, \beta_j)$ with the properties in assertion 1.

Let the linear map $T : \mathfrak{n}_{6,20}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,20}(\alpha_i', \beta_j')$ be an isometric isomorphism. The decomposition $\mathbb{R}\, E_1 \oplus \mathbb{R}\, E_2 \oplus \mathbb{R}\, E_3 \oplus \mathbb{R}\, E_4 \oplus \mathbb{R}\, E_5 \oplus \mathbb{R}\, E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0$, $i = 1, \ldots, 4$. Hence by Lemma 2.3 we have $\alpha_i = \alpha_i'$, $i = 1, \ldots, 4$ and $|\beta_j'| = \beta_j$, $j = 1, \ldots, 5$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, using the commutation relations (4.16) the equations

$$\begin{aligned} &\varepsilon_1 \varepsilon_2 \left(\alpha_1 E_4 + \beta_1' E_5 + \beta_2' E_6\right) = \alpha_1 \varepsilon_4 E_4 + \beta_1 \varepsilon_5 E_5 + \beta_2 \varepsilon_6 E_6, \\ &\varepsilon_1 \varepsilon_3 \left(\alpha_2 E_5 + \beta_3' E_6\right) = \alpha_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6, \quad \varepsilon_1 \varepsilon_4 \left(\beta_4' E_6\right) = \beta_4 \varepsilon_6 E_6, \\ &\varepsilon_1 \varepsilon_5 \left(\alpha_3 E_6\right) = \alpha_3 \varepsilon_6 E_6, \quad \varepsilon_2 \varepsilon_3 \left(\beta_5' E_6\right) = \beta_5 \varepsilon_6 E_6, \quad \varepsilon_2 \varepsilon_4 \left(\alpha_4 E_6\right) = \alpha_4 \varepsilon_6 E_6. \end{aligned} \tag{4.17}$$

From (4.17) it follows $\varepsilon_1 \varepsilon_2 = \varepsilon_4$, $\varepsilon_1 \varepsilon_3 = \varepsilon_5$, $\varepsilon_1 \varepsilon_5 = \varepsilon_2 \varepsilon_4 = \varepsilon_6$, which yields $\varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_6$.
If $\beta_1 = \beta_1' > 0$ or $\beta_4 = \beta_4' > 0$, then we have additionally $\varepsilon_1 \varepsilon_2 = \varepsilon_5$ or $\varepsilon_1 \varepsilon_4 = \varepsilon_6$, which gives that $\varepsilon_4 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_6$.
If $\beta_2 = \beta_2' > 0$ or $\beta_5 = \beta_5' > 0$, then we get additionally $\varepsilon_1 \varepsilon_2 = \varepsilon_6$ or $\varepsilon_2 \varepsilon_3 = \varepsilon_6$. Hence one has $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$.
If $\beta_3 = \beta_3' > 0$, then we have $\varepsilon_1 \varepsilon_3 = \varepsilon_6$, which yields that $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6 = 1$ and $\varepsilon_2 = \varepsilon_4$.
Using these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we have $\varepsilon_4 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_6$,
in case 3. we get $\varepsilon_2 = \varepsilon_5 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$,
in case 4. we obtain $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6 = 1$ and $\varepsilon_2 = \varepsilon_4$,
in case 5. we get $\varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_6$ and $\varepsilon_1 \varepsilon_2 = \varepsilon_4$.
Hence the system of Eq. (4.17) is satisfied with $\beta_j' = \beta_j, j = 1, \ldots, 5$ in cases 1. – 5. of the Theorem, which proves the uniqueness of the Lie algebra $\mathfrak{n}_{6,20}(\alpha_i, \beta_j)$. This shows assertion 1.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}_{6,20}(\alpha_i, \beta_j)$, then the system of equations given by (4.17) is satisfied with $\alpha_i > 0, i = 1, \ldots, 4$, $\beta_j' = \beta_j, j = 1, \ldots, 5$. Therefore in cases 1. – 5. the conditions for $\varepsilon_i, i = 1, \ldots, 6$, are given above. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{6,20}(\alpha_i, \beta_j)$ in case 1. is trivial, in cases 2. – 5. it is isomorphic to the group given by 2b–2e and assertion 2 is proved. □

Corollary 4.12. *Let $(\aleph_{6,20}(\alpha_i,\beta_j),\langle .,.\rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to $(\mathfrak{n}_{6,20}(\alpha_i,\beta_j),\langle .,.\rangle)$. The isometry group of $(\aleph_{6,20}(\alpha_i,\beta_j),\langle .,.\rangle)$ is*

$$\mathcal{I}(\aleph_{6,20}(\alpha_i,\beta_j)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2\ltimes\aleph_{6,20}(\alpha_i,\beta_j) & if\ \beta_j=0, j=1,2,3,4,5,\\ \mathbb{Z}_2\ltimes\aleph_{6,20}(\alpha_i,\beta_j) & if\ \beta_1>0\ or\ \beta_4>0, \beta_2=\beta_3=\beta_5=0,\\ & or\ \beta_2>0\ or\ \beta_5>0, \beta_1=\beta_3=\beta_4=0,\\ & or\ \beta_3>0, \beta_1=\beta_2=\beta_4=\beta_5=0,\\ \aleph_{6,20}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2>0,\ or\ \beta_1>0,\beta_3>0,\\ & or\ \beta_1>0,\beta_5>0,\ or\ \beta_2>0,\beta_3>0,\\ & or\ \beta_2>0,\beta_4>0,\ or\ \beta_3>0,\beta_4>0,\\ & or\ \beta_3>0,\beta_5>0,\ or\ \beta_4>0,\beta_5>0. \end{cases}$$

We consider the 6-dimensional Lie algebra $\mathfrak{l}_{6,21}^{\varepsilon=0}$.

Definition 4.13. Let $\{E_1,E_2,E_3,E_4,E_5,E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j), \alpha_i,\beta_j\in\mathbb{R}, i=1,\dots,4$, $j=1,\dots,6$ with $\alpha_i\neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$\begin{aligned} &[E_1,E_2]=\alpha_1E_3+\beta_1E_4+\beta_2E_5+\beta_3E_6, && [E_1,E_4]=\alpha_3E_6,\\ &[E_1,E_3]=\alpha_2E_4+\beta_4E_5+\beta_5E_6, && [E_2,E_3]=\alpha_4E_5+\beta_6E_6. \end{aligned} \tag{4.18}$$

The bracket operation (4.18) satisfies the Jacobi identity.

Theorem 4.14. *Let $\langle .,.\rangle$ be an inner product on the 6-dimensional Lie algebra $\mathfrak{l}_{6,21}^{\varepsilon=0}$.*

1. *There is a unique metric Lie algebra $(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j),\langle .,.\rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,21}^{\varepsilon=0},\langle .,.\rangle)$ with $\alpha_i>0$, $i=1,\dots,4$ and such that one of the following cases is satisfied*
    1. *at least two of the elements of the set $\{\beta_1,\beta_2,\beta_4,\beta_5,\beta_6\}$ are positive with the exception of the pairs $\{\beta_1,\beta_5\}$ and $\{\beta_2,\beta_6\}$,*
    2. *$\beta_1>0$ or $\beta_5>0$, $\beta_2=\beta_4=\beta_6=0$,*
    3. *$\beta_2>0$ or $\beta_6>0$, $\beta_1=\beta_4=\beta_5=0$,*
    4. *$\beta_4>0$, $\beta_1=\beta_2=\beta_5=\beta_6=0$,*
    5. *$\beta_1=\beta_2=\beta_4=\beta_5=\beta_6=0$.*
2. *The group $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j),\langle .,.\rangle)$ is the following group:*
    (a) *in case 1. the group $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j))$ is trivial,*
    (b) *in case 2. one has $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j))=\{TE_1=E_1, TE_5=E_5, TE_i=\varepsilon E_i, i=2,3,4,6,\ \varepsilon=\pm1\}\simeq\mathbb{Z}_2$,*

(c) *in case* 3. *one has* $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_2 = E_2, TE_4 = E_4, TE_i = \varepsilon E_i, i = 1,3,5,6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
(d) *in case* 4. *one has* $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_3 = E_3, TE_6 = E_6, TE_i = \varepsilon E_i, i = 1,2,4,5, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,
(e) *in case* 5. *one has* $\mathcal{OA}(\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j)) = \{TE_i = \varepsilon_1 E_i, i = 1,5, TE_j = \varepsilon_2 E_j, j = 2,4, TE_k = \varepsilon_1\varepsilon_2 E_k, k = 3,6, \varepsilon_1,\varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Invoking Proposition 3.1, we apply the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ and we receive an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,21}^{\varepsilon=0}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,21}^{\varepsilon=0}, \langle .,. \rangle)$ and the vectors of the new basis has the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$. Using this we have

$$\begin{aligned} &[F_1,F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_1,F_4] = \alpha_3 F_6, \\ &[F_1,F_3] = \alpha_2 F_4 + \beta_4 F_5 + \beta_5 F_6, && [F_2,F_3] = \alpha_4 F_5 + \beta_6 F_6 \end{aligned} \tag{4.19}$$

with $\alpha_i > 0$, $i = 1,\ldots,4$ and $\beta_j \in \mathbb{R}$, $j = 1,\ldots,6$. Changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ we obtain

$$\begin{aligned} &[\tilde{F_1},\tilde{F_2}] = \alpha_1 \tilde{F_3} - \beta_1 \tilde{F_4} + \beta_2 \tilde{F_5} + \beta_3 \tilde{F_6}, && [\tilde{F_1},\tilde{F_4}] = \alpha_3 \tilde{F_6}, \\ &[\tilde{F_1},\tilde{F_3}] = \alpha_2 \tilde{F_4} - \beta_4 \tilde{F_5} - \beta_5 \tilde{F_6}, && [\tilde{F_2},\tilde{F_3}] = \alpha_4 \tilde{F_5} + \beta_6 \tilde{F_6}. \end{aligned}$$

Similarly, the change of the basis: $\tilde{F_1} = -F_1, \tilde{F_2} = -F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = F_6$ gives

$$\begin{aligned} &[\tilde{F_1},\tilde{F_2}] = \alpha_1 \tilde{F_3} - \beta_1 \tilde{F_4} - \beta_2 \tilde{F_5} + \beta_3 \tilde{F_6}, && [\tilde{F_1},\tilde{F_4}] = \alpha_3 \tilde{F_6}, \\ &[\tilde{F_1},\tilde{F_3}] = \alpha_2 \tilde{F_4} + \beta_4 \tilde{F_5} - \beta_5 \tilde{F_6}, && [\tilde{F_2},\tilde{F_3}] = \alpha_4 \tilde{F_5} - \beta_6 \tilde{F_6}. \end{aligned}$$

Hence there is an orthonormal basis such that in commutators (4.19) we have $\alpha_i > 0, i = 1,\ldots,4$ and one of the cases in assertion 1. holds. Therefore the existence of $\mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j)$ with properties given by assertion 1. follows.

Let the linear map $T : \mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i,\beta_j) \to \mathfrak{n}_{6,21}^{\varepsilon=0}(\alpha_i',\beta_j')$ be an isometric isomorphism. The decomposition $\mathbb{R}\,E_1 \oplus \mathbb{R}\,E_2 \oplus \mathbb{R}\,E_3 \oplus \mathbb{R}\,E_4 \oplus \mathbb{R}\,E_5 \oplus \mathbb{R}\,E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0$, $i = 1,\ldots,4$. Hence by Lemma 2.3 we have $\alpha_i = \alpha_i'$, $i = 1,\ldots,4$ and $|\beta_j'| = \beta_j$, $j = 1,\ldots,6$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1,\ldots,6$, then we obtain from $[TE_i, TE_j]' = T[E_i,E_j]$, $i,j = 1,\ldots,6$, using the commutation relations (4.19) the equations

$$\begin{aligned} &\varepsilon_1\varepsilon_2\,(\alpha_1 E_3 + \beta_1' E_4 + \beta_2' E_5 + \beta_3' E_6) = \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6, \\ &\varepsilon_1\varepsilon_3\,(\alpha_2 E_4 + \beta_4' E_5 + \beta_5' E_6) = \alpha_2\varepsilon_4 E_4 + \beta_4\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6, \\ &\varepsilon_1\varepsilon_4\,(\alpha_3 E_6) = \alpha_3\varepsilon_6 E_6, \quad \varepsilon_2\varepsilon_3\,(\alpha_4 E_5 + \beta_6' E_6) = \alpha_4\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6. \end{aligned} \tag{4.20}$$

Hence we obtain $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_6, \varepsilon_2\varepsilon_3 = \varepsilon_5$, which yields $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_6$. Therefore one has $\beta_3' = \beta_3$.
If $\beta_1 = \beta_1' > 0$ or $\beta_5 = \beta_5' > 0$, then we have additionally $\varepsilon_1\varepsilon_2 = \varepsilon_4$ or $\varepsilon_1\varepsilon_3 = \varepsilon_6$, which yields that $\varepsilon_1 = \varepsilon_5 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$.
If $\beta_2 = \beta_2' > 0$ or $\beta_6 = \beta_6' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_5$ or $\varepsilon_2\varepsilon_3 = \varepsilon_6$. Hence one has $\varepsilon_2 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$.
If $\beta_4 = \beta_4' > 0$, then one has in addition $\varepsilon_1\varepsilon_3 = \varepsilon_5$, which gives $\varepsilon_3 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_5$.
Applying these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we have $\varepsilon_1 = \varepsilon_5 = 1$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6$,
in case 3. we receive $\varepsilon_2 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$,
in case 4. we obtain $\varepsilon_3 = \varepsilon_6 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_5$,
in case 5. we get that $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_6$.
Therefore the system of Eq. (4.20) is satisfied with $\beta_j' = \beta_j, j = 1, \ldots, 6$ in cases 1.−5. of the Theorem and the uniqueness of the metric Lie algebra $\mathfrak{n}^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j)$ with properties given by assertion 1. follows. The proof of assertion 1. is done.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j)$, then the system of equations given by (4.20) is satisfied with $\alpha_i > 0, i = 1, \ldots, 4$, $\beta_j' = \beta_j, j = 1, \ldots, 6$. Hence for $\varepsilon_i, i = 1, \ldots, 6$, we have the same conditions as above. Taking this into account the group of orthogonal automorphisms of $\mathfrak{n}^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j)$ in case 1. is trivial, in cases 2. − 5. it is isomorphic to the group given by 2b-2e. This proves assertion 2. □

Corollary 4.15. *Let $(\aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to $(\mathfrak{n}^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(\aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}(\aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes \aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j) & if\ \beta_j = 0, j = 1, 2, 4, 5, 6, \\ \mathbb{Z}_2 \ltimes \aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j) & if\ \beta_1 > 0\ or\ \beta_5 > 0, \beta_2 = \beta_4 = \beta_6 = 0, \\ & or\ \beta_2 > 0\ or\ \beta_6 > 0, \beta_1 = \beta_4 = \beta_5 = 0, \\ & or\ \beta_4 > 0, \beta_1 = \beta_2 = \beta_5 = \beta_6 = 0, \\ \aleph^{\varepsilon=0}_{6,21}(\alpha_i, \beta_j) & if\ \beta_1 > 0, \beta_2 > 0,\ or\ \beta_1 > 0, \beta_4 > 0, \\ & or\ \beta_1 > 0, \beta_6 > 0,\ or\ \beta_2 > 0, \beta_4 > 0, \\ & or\ \beta_2 > 0, \beta_5 > 0,\ or\ \beta_4 > 0, \beta_5 > 0, \\ & or\ \beta_4 > 0, \beta_6 > 0,\ or\ \beta_5 > 0, \beta_6 > 0. \end{cases}$$

Finally we deal with the 6-dimensional Lie algebras $\mathfrak{l}_{6,23}$ and $\mathfrak{l}_{6,25}$.

Definition 4.16. Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. We denote by $\mathfrak{n}_{6,25}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, 3,$ $j = 1, 2, 3$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators

$$[E_1, E_2] = \alpha_1 E_4 + \beta_1 E_5 + \beta_2 E_6,\ [E_1, E_3] = \alpha_2 E_5 + \beta_3 E_6,\ [E_1, E_4] = \alpha_3 E_6. \tag{4.21}$$

Denote by $\mathfrak{n}_{6,23}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, \ldots, 4, j = 1, 2, 3$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by (4.21) and by the additional commutator

$$[E_2, E_3] = \alpha_4 E_6. \tag{4.22}$$

The bracket operations (4.21) as well as (4.21) and (4.22) satisfy the Jacobi identity.

Theorem 4.17. *Let $\langle .,. \rangle$ be an inner product on the 6-dimensional Lie algebra $\mathfrak{l}_{6,23}$, respectively $\mathfrak{l}_{6,25}$.*

1. *There is a unique metric Lie algebra $(\mathfrak{n}_{6,25}(\alpha_i, \beta_j), \langle .,. \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,25}, \langle .,. \rangle)$ with $\alpha_i > 0$, $i = 1, 2, 3$, such that one of the following cases is satisfied*
   1. *at least two of the elements of the set $\{\beta_1, \beta_2, \beta_3\}$ are positive,*
   2. $\beta_1 > 0, \beta_2 = \beta_3 = 0$,
   3. $\beta_2 > 0, \beta_1 = \beta_3 = 0$,
   4. $\beta_3 > 0, \beta_1 = \beta_2 = 0$,
   5. $\beta_1 = \beta_2 = \beta_3 = 0$.

   There is a unique metric Lie algebra $(\mathfrak{n}_{6,23}(\alpha_i, \beta_j), \langle .,. \rangle)$ which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{6,23}, \langle .,. \rangle)$ with $\alpha_i > 0, i = 1, 2, 3, 4$, and such that one of the above cases 1. –5. is satisfied.
2. *The group $\mathcal{OA}\ (\mathfrak{n}_{6,23}(\alpha_i, \beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,23}(\alpha_i, \beta_j), \langle .,. \rangle)$ is the following group:*
   - (a) *in case 1. the group $\mathcal{OA}(\mathfrak{n}_{6,23}(\alpha_i, \beta_j))$ is trivial,*
   - (b) *in case 2. one has $\mathcal{OA}(\mathfrak{n}_{6,23}(\alpha_i, \beta_j)) = \{TE_2 = E_2, TE_3 = E_3, TE_6 = E_6, TE_i = \varepsilon E_i, i = 1, 4, 5, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
   - (c) *in case 3. one has $\mathcal{OA}(\mathfrak{n}_{6,23}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_3 = E_3, TE_5 = E_5, TE_i = \varepsilon E_i, i = 2, 4, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
   - (d) *in case 4. one has $\mathcal{OA}(\mathfrak{n}_{6,23}(\alpha_i, \beta_j)) = \{TE_3 = E_3, TE_4 = E_4, TE_i = \varepsilon E_i, i = 1, 2, 5, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$,*
   - (e) *in case 5. one has $\mathcal{OA}(\mathfrak{n}_{6,23}(\alpha_i, \beta_j)) = \{TE_3 = E_3, TE_i = \varepsilon_1 E_i, i = 1, 5, TE_j = \varepsilon_2 E_j, j = 2, 6, TE_4 = \varepsilon_1 \varepsilon_2 E_4, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*
3. *The group $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i, \beta_j))$ of orthogonal automorphisms of the metric Lie algebra $(\mathfrak{n}_{6,25}(\alpha_i, \beta_j), \langle .,. \rangle)$ is the following group:*
   - (a) *in case 1. one has $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i, \beta_j)) = \{TE_1 = E_1, TE_i = \varepsilon E_i, i = 2, 3, 4, 5, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.*

(b) in case 2. one has $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i,\beta_j)) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i,$ $\mathrm{i} = 2,3,6, TE_j = \varepsilon_1\varepsilon_2 E_j, j = 4,5, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) in case 3. one has $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i,\beta_j)) = \{TE_1 = E_1, TE_i = \varepsilon_2 E_i,$ $\mathrm{i} = 2,4,6, TE_j = \varepsilon_3 E_j, j = 3,5, \varepsilon_2, \varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) in case 4. one has $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i,\beta_j)) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i,$ $\mathrm{i} = 2,5,6, TE_j = \varepsilon_1\varepsilon_2 E_j, j = 3,4, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(e) in case 5. one has $\mathcal{OA}(\mathfrak{n}_{6,25}(\alpha_i,\beta_j)) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i,$ $i = 2,6, TE_3 = \varepsilon_3 E_3, TE_4 = \varepsilon_1\varepsilon_2 E_4, TE_5 = \varepsilon_1\varepsilon_3 E_5, \varepsilon_1, \varepsilon_2,$ $\varepsilon_3 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. According to Proposition 3.1 the application of the Gram–Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\mathfrak{l}_{6,23}$, respectively $\mathfrak{l}_{6,25}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\mathfrak{l}_{6,23}, \langle .,. \rangle)$, respectively $(\mathfrak{l}_{6,25}, \langle .,. \rangle)$. The vectors of the new basis have the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$. We get for the metric Lie algebras $(\mathfrak{l}_{6,23}, \langle .,. \rangle)$ and $(\mathfrak{l}_{6,25}, \langle .,. \rangle)$

$$[F_1, F_2] = \alpha_1 F_4 + \beta_1 F_5 + \beta_2 F_6,\ [F_1, F_3] = \alpha_2 F_5 + \beta_3 F_6,\ [F_1, F_4] = \alpha_3 F_6, \tag{4.23}$$

and for $(\mathfrak{l}_{6,23}, \langle .,. \rangle)$ in addition

$$[F_2, F_3] = \alpha_4 F_6, \tag{4.24}$$

where $\alpha_i > 0$, $i = 1,2,3,4$ and $\beta_j \in \mathbb{R}$, $j = 1,2,3$. Changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = -F_2, \tilde{F_3} = F_3, \tilde{F_4} = F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ we obtain

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} - \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_3}] = \alpha_2 \tilde{F_5} + \beta_3 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_4}] = \alpha_3 \tilde{F_6}, \qquad [\tilde{F_2}, \tilde{F_3}] = \alpha_4 \tilde{F_6}.$$

Similarly, the change of the basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = F_6$ yields

$$[\tilde{F_1}, \tilde{F_2}] = \alpha_1 \tilde{F_4} + \beta_1 \tilde{F_5} - \beta_2 \tilde{F_6}, \qquad [\tilde{F_1}, \tilde{F_3}] = \alpha_2 \tilde{F_5} - \beta_3 \tilde{F_6},$$
$$[\tilde{F_1}, \tilde{F_4}] = \alpha_3 \tilde{F_6}, \qquad [\tilde{F_2}, \tilde{F_3}] = \alpha_4 \tilde{F_6}.$$

Hence there is an orthonormal basis such that in commutators (4.23) and (4.24) we have $\alpha_i > 0$, $i = 1, \ldots, 4$ and one of the cases in assertion 1. is satisfied. Consequently the existence of $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$, respectively $\mathfrak{n}_{6,25}(\alpha_i, \beta_j)$ with the properties in assertion 1. is proved.

Let the linear map $T : \mathfrak{n}_{6,k}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,k}(\alpha'_i, \beta'_j)$, $k = 23, 25$, be an isometric isomorphism. The decomposition $\mathbb{R}\, E_1 \oplus \mathbb{R}\, E_2 \oplus \mathbb{R}\, E_3 \oplus \mathbb{R}\, E_4 \oplus \mathbb{R}\, E_5 \oplus \mathbb{R}\, E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha'_i > 0$, $i = 1, \ldots, 4$. Hence by Lemma 2.3 we have $\alpha_i = \alpha'_i$, $i = 1, \ldots, 4$ and $|\beta'_j| = \beta_j$ for all $j = 1,2,3$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$. Using the commutation relations

(4.23) and (4.24) we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, for $\mathfrak{n}_{6,25}(\alpha_i, \beta_j)$ and $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$ the equations

$$\varepsilon_1\varepsilon_2\,(\alpha_1 E_4 + \beta_1' E_5 + \beta_2' E_6) = \alpha_1\varepsilon_4 E_4 + \beta_1\varepsilon_5 E_5 + \beta_2\varepsilon_6 E_6, \qquad (4.25)$$
$$\varepsilon_1\varepsilon_3\,(\alpha_2 E_5 + \beta_3' E_6) = \alpha_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6, \quad \varepsilon_1\varepsilon_4\,(\alpha_3 E_6) = \alpha_3\varepsilon_6 E_6,$$

and in addition for $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$ the equation

$$\varepsilon_2\varepsilon_3\,(\alpha_4 E_6) = \alpha_4\varepsilon_6 E_6. \qquad (4.26)$$

From (4.25) and (4.26) for the metric Lie algebra $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$ we get $\varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_3 = \varepsilon_5$, $\varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3 = \varepsilon_6$. Then one has $\varepsilon_3 = 1$, $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_6$.
If $\beta_1 = \beta_1' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_5$, which gives $\varepsilon_2 = \varepsilon_3 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_4 = \varepsilon_5$.
If $\beta_2 = \beta_2' > 0$, then we obtain $\varepsilon_1\varepsilon_2 = \varepsilon_6$. Hence one has $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = 1$, $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$.
If $\beta_3 = \beta_3' > 0$, then we get additionally $\varepsilon_1\varepsilon_3 = \varepsilon_6$, which yields $\varepsilon_3 = \varepsilon_4 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = \varepsilon_6$.
Using the conditions for $\beta_j, j = 1, 2, 3$ given in assertion 1. of the Theorem
in case 1. we get $\varepsilon_i = 1, i = 1, \ldots, 6$,
in case 2. we obtain $\varepsilon_2 = \varepsilon_3 = \varepsilon_6 = 1$ and $\varepsilon_1 = \varepsilon_4 = \varepsilon_5$,
in case 3. we have $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = 1$ and $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$,
in case 4. we obtain $\varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_5 = \varepsilon_6$,
in case 5. we get $\varepsilon_3 = 1$, $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_6$ and $\varepsilon_1\varepsilon_2 = \varepsilon_4$.

For the metric Lie algebra $\mathfrak{n}_{6,25}(\alpha_i, \beta_j)$ from (4.25) it follows $\varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_1\varepsilon_3 = \varepsilon_5$, $\varepsilon_1\varepsilon_4 = \varepsilon_6$. Then one has $\varepsilon_2 = \varepsilon_6$.
If $\beta_1 = \beta_1' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_5$. Hence one has $\varepsilon_2 = \varepsilon_3 = \varepsilon_6$ and $\varepsilon_4 = \varepsilon_5$.
If $\beta_2 = \beta_2' > 0$, then we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_6$, which gives $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$, $\varepsilon_3 = \varepsilon_5$.
If $\beta_3 = \beta_3' > 0$, then we get additionally $\varepsilon_1\varepsilon_3 = \varepsilon_6$, which yields that $\varepsilon_2 = \varepsilon_5 = \varepsilon_6$ and $\varepsilon_3 = \varepsilon_4$.
Applying these relations in assertion 1. of the Theorem
in case 1. we obtain $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6$,
in case 2. we get $\varepsilon_2 = \varepsilon_3 = \varepsilon_6$, $\varepsilon_4 = \varepsilon_5$ and $\varepsilon_1\varepsilon_2 = \varepsilon_4$,
in case 3. we have $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_4 = \varepsilon_6$ and $\varepsilon_3 = \varepsilon_5$,
in case 4. we get $\varepsilon_2 = \varepsilon_5 = \varepsilon_6$, $\varepsilon_3 = \varepsilon_4$ and $\varepsilon_1\varepsilon_2 = \varepsilon_4$,
in case 5. we obtain $\varepsilon_2 = \varepsilon_6$, $\varepsilon_1\varepsilon_2 = \varepsilon_4$ and $\varepsilon_1\varepsilon_3 = \varepsilon_5$.

Hence in both metric Lie algebras the system of Eq. (4.25) is satisfied with $\beta_j' = \beta_j, i = 1, 2, 3$, in cases 1.−5. This proves the uniqueness of the Lie algebra $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$, respectively $\mathfrak{n}_{6,25}(\alpha_i, \beta_j)$ in cases 1. − 5., which gives assertion 1.

If the map $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, is an orthogonal automorphism of $\mathfrak{n}_{6,25}(\alpha_i, \beta_j)$, respectively $\mathfrak{n}_{6,23}(\alpha_i, \beta_j)$, then the system of equations given by (4.25), respectively (4.25) and (4.26) is satisfied with $\beta_j' = \beta_j, j = 1, 2, 3$. Therefore in cases 1. − 5. for $\varepsilon_i, i = 1, \ldots, 6$ we have the conditions as above. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{6,23}(\alpha_i,\beta_j)$, respectively $\mathfrak{n}_{6,25}(\alpha_i,\beta_j)$ in cases 1. – 5. is isomorphic to the group given by 2a–2e, respectively 3a–3e. This proves assertions 2 and 3. □

Corollary 4.18. *Let* $(\aleph_{6,k}(\alpha_i,\beta_j),\langle .,.\rangle)$, $k=23,25$, *be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra* $(\mathfrak{n}_{6,k}(\alpha_i,\beta_j),\langle .,.\rangle)$, $k=23,25$. *The isometry group of* $(\aleph_{6,23}(\alpha_i,\beta_j),\langle .,.\rangle)$ *is*

$$\mathcal{I}(\aleph_{6,23}(\alpha_i,\beta_j)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2\ltimes\aleph_{6,23}(\alpha_i,\beta_j) & if\ \beta_1=\beta_2=\beta_3=0,\\ \mathbb{Z}_2\ltimes\aleph_{6,23}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2=\beta_3=0,\\ & or\ \beta_2>0,\beta_1=\beta_3=0,\\ & or\ \beta_3>0,\beta_1=\beta_2=0,\\ \aleph_{6,23}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2>0, or\ \beta_1>0,\beta_3>0,\\ & or\ \beta_2>0,\beta_3>0. \end{cases}$$

The isometry group of $(\aleph_{6,25}(\alpha_i,\beta_j),\langle .,.\rangle)$ *is*

$$\mathcal{I}(\aleph_{6,25}(\alpha_i,\beta_j)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\ltimes\aleph_{6,25}(\alpha_i,\beta_j) & if\ \beta_1=\beta_2=\beta_3=0,\\ \mathbb{Z}_2\times\mathbb{Z}_2\ltimes\aleph_{6,25}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2=\beta_3=0,\\ & or\ \beta_2>0,\beta_1=\beta_3=0,\\ & or\ \beta_3>0,\beta_1=\beta_2=0,\\ \mathbb{Z}_2\ltimes\aleph_{6,25}(\alpha_i,\beta_j) & if\ \beta_1>0,\beta_2>0, or\ \beta_1>0,\beta_3>0,\\ & or\ \beta_2>0,\beta_3>0. \end{cases}$$

Author contributions Kornélia Ficzere and Ágota Figula wrote the main manuscript text. Both authors reviewed the manuscript.

Funding Open access funding provided by University of Debrecen. Open access funding provided by University of Debrecen. This paper was supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K132951.

Availability of data and materials Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

[1] Cairns, G., Galić, A.H., Nikolayevsky, Y.: Totally geodesic subalgebras of nilpotent Lie algebras. J. Lie Theory **23**(4), 1023–1049 (2013)

[2] Cairns, G., Galić, A.H., Nikolayevsky, Y.: Totally geodesic subalgebras of filiform nilpotent Lie algebras. J. Lie Theory **23**(4), 1051–1074 (2013)

[3] Console, S., Fino, A., Samiou, E.: The moduli space of six-dimensional two-step nilpotent Lie algebras. Ann. Glob. Anal. Geom. **27**(1), 17–32 (2005)

[4] De Graaf, W.A.: Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. J. Algebra **309**(2), 640–653 (2007)

[5] Figula, Á., Nagy, P.T.: Isometry classes of simply connected nilmanifolds. J. Geom. Phys. **132**, 370–381 (2018)

[6] Figula, Á., Abbas, S.A.: Isometry groups of six-dimensional filiform nilmanifolds. Int. J. Group Theory **12**(2), 67–80 (2023)

[7] Homolya, S., Kowalski, O.: Simply connected two-step homogeneous nilmanifolds of dimension 5. Note Mat. **26**(1), 69–77 (2006)

[8] Kerr, M.M., Payne, T.L.: The geometry of filiform nilpotent Lie groups. Rocky Mt. J. Math. **40**(5), 1587–1610 (2010)

[9] Lauret, J.: Homogeneous nilmanifolds of dimension 3 and 4. Geom. Dedic. **68**(2), 145–155 (1997)

[10] Wilson, E.N.: Isometry groups on homogeneous nilmanifolds. Geom. Dedic. **12**(3), 337–346 (1982)

Kornélia Ficzere
Doctoral School of Mathematical and Computational Sciences
University of Debrecen
P.O. Box 400 4002 Debrecen
Hungary
e-mail: ficzere.kornelia@science.unideb.hu

Ágota Figula
Institute of Mathematics
University of Debrecen
P.O. Box 400 4002 Debrecen
Hungary
e-mail: figula@science.unideb.hu

Received: July 29, 2022
Revised: March 19, 2023
Accepted: March 23, 2023