Aequat. Math. 97 (2023), 113–120
© The Author(s) 2022
0001-9054/23/010113-8
published online October 28, 2022
https://doi.org/10.1007/s00010-022-00917-y

Aequationes Mathematicae

An alternative equation for generalized monomials

Zoltán Boros and Rayene Menzer

Abstract. In this paper we consider a generalized monomial or polynomial $f : \mathbb{R} \to \mathbb{R}$ that satisfies the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is given by some algebraic condition. In the particular cases when f is a generalized polynomial and there exist non-constant regular polynomials p and q that fulfill

$$D = \{ (p(t), q(t)) \mid t \in \mathbb{R} \}$$

or f is a generalized monomial and there exists a positive rational m fulfilling

$$D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 - my^2 = 1 \},$$

we prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Mathematics Subject Classification. Primary 39B55; Secondary 39B72.

Keywords. Monomial functions, Generalized polynomials, Alternative equation.

1. Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the set of all real numbers, rationals, and positive integers, respectively. We call a function $f : \mathbb{R} \to \mathbb{R}$ *additive* if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in \mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx) = qf(x)$ is fulfilled by every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. As it is also well-known [6, Theorem 5.2.1], if $f : \mathbb{R} \to \mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well.

Research of Z. Boros has been supported by the K-134191 NKFIH Grant and the 2019-2.1.11-TÉT-2019-00049 project. Projects no. 2019-2.1.11-TÉT-2019-00049 and K134191 have been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the TÉT and K_20 funding schemes, respectively.

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We define the following sets:

$$S_0 = \{(x,y) \in \mathbb{R}^2 \,|\, xy = 1\},$$
$$S_1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\},$$
$$S_2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}.$$

Moreover, if p and q are regular, non-constant, real polynomials, while m is a positive real number, we shall also consider the sets

$$R_{p,q} = \{\, (p(t), q(t)) \,|\, t \in \mathbb{R}\,\},$$
$$S_{1,m} = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1\}.$$

Z. Kominek, L. Reich and J. Schwaiger [5] investigated additive real functions that satisfy the additional equation

$$f(x)f(y) = 0 \tag{1}$$

for every $(x,y) \in D$, considering various subsets D of $\mathbb{R}^2$. In several cases, involving $D = R_{p,q}$ and $D = S_2$, they obtained $f(x) = 0$ for every $x \in \mathbb{R}$. Their result for $D = S_2$ was extended by Z. Boros and W. Fechner [1] to the situation when f is a generalized polynomial. On the other hand, P. Kutas [7] has recently established the existence of a non-zero additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling (1) for all $(x,y) \in S_0$. The case of bounded $f(x)f(y)$ on S_2 was investigated by these authors [2].

The purpose of the present paper is to involve the case $D = S_{1,m}$ into this research for every positive rational m. We note that, in some sense, $S_{1,1} = S_1$ is on a half way from S_0 to S_2 , as it is geometrically analogous to S_0 and algebraically analogous to S_2 . Moreover, motivated by [1, Theorem 1], we wish to extend the investigation of the cases $D = R_{p,q}$ and $D = S_{1,m}$ for a generalized polynomial or monomial f , respectively.

2. Concepts and lemmas

Let $n \in \mathbb{N}$. A function $F : \mathbb{R}^n \to \mathbb{R}$ is called *n-additive* if F is additive in each of its variables. Clearly, an n-additive function is also $\mathbb{Q}$-homogeneous in each variable.

Given a function $F : \mathbb{R}^n \to \mathbb{R}$, by the *diagonalization (or trace)* of F we understand the function $f : \mathbb{R} \to \mathbb{R}$ arising from F by putting all the variables (from $\mathbb{R}$) equal:

$$f(x) = F(x, \dots, x) \qquad (x \in \mathbb{R})\,.$$

If, in particular, f is a diagonalization of an n-additive function $F : \mathbb{R}^n \to \mathbb{R}$, we say that f is a *generalized monomial of degree n* . It is convenient to assume that generalized monomials of degree zero are precisely constant mappings. It is also clear that the set of all monomials of degree n is a real linear space with respect to the pointwise operations for any non-negative integer n.

If f is a finite sum of generalized monomials, then f is called a *generalized polynomial*.

For more information concerning these notions the reader is referred to the monograph by M. Kuczma [6, Chapter 15.9] as well as to the short introduction in [1].

In order to make use of the already mentioned $\mathbb{Q}$-homogeneity property of n-additive functions, in our arguments we shall repeatedly apply the following observation. If a regular real polynomial $P(s)$ equals zero for infinitely many distinct values of the variable s, then it is identically zero, i.e., the coefficient of s^k equals zero for every non-negative integer k (up to the degree of P). Clearly, it follows from the fact that for a not identically zero polynomial P the equation $P(s) = 0$ is satisfied only by a finite number of distinct values of s. This idea is explicitly stated in [3, Lemma 1]. The application of this idea in the theory of functional equations goes back to the paper by Nishiyama and Horinouchi [8].

We shall also need to verify the following statements.

Lemma 2.1. *If $f : \mathbb{R} \to \mathbb{R}$ is a generalized polynomial, $I \subseteq \mathbb{R}$ is a non-degenerated interval and $f(x) = 0$ for every $x \in I$, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. Due to our assumptions, there exist a positive integer n and k-additive mappings $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 1, \dots, n)$ such that

$$f(x) = \sum_{k=0}^{n} A_k^*(x) \tag{2}$$

for every $x \in \mathbb{R}$, where $A_0^*(x) = A_0 \in \mathbb{R}$ and

$$A_k^*(x) = A_k(x, x, \dots, x) \quad (x \in \mathbb{R},\ k = 1, 2, \dots, n).$$

According to the hypothesis, $I \subseteq \mathbb{R}$ is an interval with positive length. From the density of $\mathbb{Q}$ in $\mathbb{R}$ we can see that for any real number $x \neq 0$ there exist infinitely many $r \in \mathbb{Q}$ such that $rx \in I$ and thus

$$0 = f(rx) = \sum_{k=0}^{n} A_k^*(rx) = \sum_{k=0}^{n} r^k A_k^*(x)\,.$$

We have just obtained a polynomial of degree (at most) n with infinitely many rational zeroes. This implies that the polynomial is identically zero, hence $A_k^*(x) = 0$ for every $k \in \{0, ,1, , \dots, n\}$, which yields $f(x) = 0$. In particular, we have $0 = A_0^*(x) = A_0 = f(0)$. Therefore, f vanishes on $\mathbb{R}$. $\square$

Lemma 2.2. *If $f : \mathbb{R} \to \mathbb{R}$ is a generalized monomial and $p : \mathbb{R} \to \mathbb{R}$ is a regular real polynomial, then $f \circ p$ is a generalized polynomial.*

Proof. Let j be a positive integer such that f is a monomial of degree j, i.e., f is the diagonalization of the j-additive mapping $A : \mathbb{R}^j \to \mathbb{R}$. Moreover, let n be a non-negative integer and $a_k \in \mathbb{R}$ $(k = 0, 1, \ldots .n)$ such that

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

for every $x \in \mathbb{R}$. Then

$$\begin{aligned} f(p(x)) &= f\left(\sum_{k=0}^{n} a_k x^k\right) = A\left(\sum_{k_1=0}^{n} a_{k_1} x^{k_1}, \ldots, \sum_{k_j=0}^{n} a_{k_j} x^{k_j}\right) \\ &= \sum_{k_1=0}^{n} \cdots \sum_{k_j=0}^{n} A\left(a_{k_1} x^{k_1}, \ldots, a_{k_j} x^{k_j}\right). \end{aligned}$$

For any fixed non-negative integers $k_l \in \{0, ,1, , \ldots, n\}$ $(l = 1, 2, \ldots, j)$, let $s = \sum_{l=1}^{j} k_l$ and

$$G(t_1, \ldots, t_s) = A\left(a_{k_1} t_1 \ldots t_{k_1}\, ,\, a_{k_2} t_{k_1+1} \ldots t_{k_1+k_2}\, ,\, \ldots\, ,\, a_{k_j} t_{s-k_j+1} \ldots t_s\right),$$

where any empty product equals 1 (i.e., for $k_l = 0$ we have only a_{k_l} in the l-th entry of A). Due to the distributivity of multiplication of real numbers and the j-additivity of A, G is s-additive and

$$A\left(a_{k_1} x^{k_1}\, ,\, a_{k_2} x^{k_2}\, ,\, \ldots\, ,\, a_{k_j} x^{k_j}\right) = G(x, x, \ldots, x).$$

Thus $f \circ p$ is a finite sum of generalized monomials, hence it is a generalized polynomial. □

3. Main results

Now we can establish our main theorems. The first one involves two non-constant regular polynomials with possibly different degrees.

Theorem 3.1. *Let p and q be polynomials of degrees at least one. If the generalized polynomial $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation*

$$f(p(x))f(q(x)) = 0 \tag{3}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ identically.

Proof. Since generalized polynomials are obtained as finite sums of generalized monomials, Lemma 2.2 implies that both $f \circ p$ and $f \circ q$ are generalized polynomials. Now we can use a result [4, Theorem 2] by Halter-Koch, Reich and Schwaiger claiming that the set of generalized polynomials is an integral domain. In particular, if the product of two generalized polynomials is identically zero, then one of those generalized polynomials has to be identically zero as well. Therefore the functional equation (3) implies that either $f(p(x)) = 0$ identically or $f(q(x)) = 0$ identically. Due to our assumptions the ranges $p(\mathbb{R})$ and $q(\mathbb{R})$ are unbounded intervals, hence f vanishes on an unbounded interval. Applying Lemma 2.1 we obtain that $f(x) = 0$ for all $x \in \mathbb{R}$. □

Our second theorem involves particular hyperbolas. The major tool in our arguments is obtained by a family of linear transformations that leave such a hyperbola invariant.

Lemma 3.2. *Let m denote a positive real number. Suppose that x, y, α and β are real numbers such that*

$$x^2 - my^2 = 1 \quad \textit{and} \quad \alpha^2 - m\beta^2 = 1\,.$$

Then we have

$$(\alpha x + \beta m y)^2 - m(\beta x + \alpha y)^2 = 1$$

as well.

Proof. It is obtained by a straightforward calculation. □

Remark 3.3. The geometric interpretation of this observation is that, for any $(\alpha, \beta) \in S_{1,m}$, the linear transformation on $\mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} \alpha & m\beta \\ \beta & \alpha \end{pmatrix}$$

leaves the hyperbola $S_{1,m}$ invariant.

Theorem 3.4. *Let m denote a positive rational. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial and $f(x)f(y) = 0$ for all solutions of the equation $x^2 - my^2 = 1$. Then f is identically equal to zero.*

Proof. Given a generalized monomial f, we can associate a positive integer k and a k-additive and symmetric functional $A\colon \mathbb{R}^k \to \mathbb{R}$ with f in such a way that

$$f(x) = A(x, \ldots, x) \tag{4}$$

for all $x \in \mathbb{R}$.

Now, let $x \in \mathbb{R}$ such that $x \geq 1$. Then there exists $0 \leq y \in \mathbb{R}$ such that $x^2 - my^2 = 1$. If α, β are rational numbers such that $\alpha^2 - m\beta^2 = 1$, Lemma 3.2 and our assumptions on f imply

$$f(\alpha x + \beta m y) f(\beta x + \alpha y) = 0\,. \tag{5}$$

Next, let us denote

$$a_l = A(\underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k-l})$$

for $l=0,1,\dots k$. With this notation we can calculate that

$$f(\alpha x+\beta my)=A(\alpha x+\beta my\,,\,\dots\,,\,\alpha x+\beta my)=\sum_{l=0}^{k}\binom{k}{l}(\alpha)^l(\beta m)^{k-l}a_l$$

and

$$f(\beta x+\alpha y)=A(\beta x+\alpha y\,,\,\dots\,,\,\beta x+\alpha y)=\sum_{l=0}^{k}\binom{k}{l}\alpha^{k-l}\beta^l a_l\,.$$

Due to Eq. (5), for every pair of rationals (α,β) fulfilling $\alpha^2-m\beta^2=1$, at least one of the foregoing expressions is equal to zero.

What is more, we can find infinitely many distinct pairs (α_j,β_j) such that $\alpha_j^2-m\beta_j^2=1$ and both α_j and β_j are rationals, so let us take

$$\alpha_j=\frac{mj^2+1}{mj^2-1}\quad\text{and}\quad\beta_j=\frac{2j}{mj^2-1}\tag{6}$$

for $j\in\mathbb{N}$ such that $mj^2\neq1$.

Thus, for every $j\in N_m\doteq\mathbb{N}\setminus\{1/\sqrt{m}\}$, we have either

$$0=\sum_{l=0}^{k}\binom{k}{l}\left(\frac{mj^2+1}{mj^2-1}\right)^l\left(\frac{2mj}{mj^2-1}\right)^{k-l}a_l$$

or

$$0=\sum_{l=0}^{k}\binom{k}{l}\left(\frac{mj^2+1}{mj^2-1}\right)^{k-l}\left(\frac{2j}{mj^2-1}\right)^{l}a_l\,.$$

Multiplying both equations by $(mj^2-1)^k$ and introducing the functions

$$P(j)=\sum_{l=0}^{k}\binom{k}{l}\left(mj^2+1\right)^l(2mj)^{k-l}a_l\,,$$
$$\tilde{P}(j)=\sum_{l=0}^{k}\binom{k}{l}\left(mj^2+1\right)^{k-l}(2j)^{l}a_l\,,$$

we have $P(j)=0$ or $\tilde{P}(j)=0$ for each integer $j\in N_m$. Hence either P or $\tilde{P}$ has infinitely many zeros. On the other hand, both P and $\tilde{P}$ are polynomials of degree not greater than $2k$. Therefore, one of them has to be identically equal to 0. So either

$$0=P(0)=a_k=A_k(x\,,\,x\,,\,\dots\,,\,x)=f(x)$$

or

$$0=\tilde{P}\left(\frac{i}{\sqrt{m}}\right)=\left(\frac{2i}{\sqrt{m}}\right)^k a_k=\left(\frac{2i}{\sqrt{m}}\right)^k A_k(x\,,\,x\,,\,\dots\,,\,x)=\left(\frac{2i}{\sqrt{m}}\right)^k f(x)\,,$$

i.e., $f(x) = 0$ (here i denotes the imaginary unit as polynomials with real coefficients can be considered as polynomials over the complex number field as well).

We have thus proved $f(x) = 0$ for every real number $x \geq 1$. Hence, applying Lemma 2.1, we obtain that $f(x) = 0$ for all $x \in \mathbb{R}$. □

Corollary 3.5. *Let a and b denote positive real numbers such that $\frac{a^2}{b^2}$ is rational. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized monomial and $f(x)f(y) = 0$ for all solutions of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then f is identically equal to zero.*

Proof. Let u and w be real numbers fulfilling the condition $u^2 - \frac{a^2}{b^2} w^2 = 1$. Moreover, let $g(t) = f(at)$ for all $t \in \mathbb{R}$. Clearly, then g is a generalized monomial as well. For $x = au$ and $y = aw$ we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = u^2 - \frac{a^2}{b^2} w^2 = 1\,,$$

hence our assumption yields $g(u)g(w) = f(au)f(aw) = f(x)f(y) = 0$. Therefore g satisfies the assumptions in Theorem 3.4 with $m = \frac{a^2}{b^2}$, hence g is identically equal to zero, which yields $f(x) = g(x/a) = 0$ for every $x \in \mathbb{R}$ as well. □

Funding Information Open access funding provided by University of Debrecen.

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Zoltán Boros and Rayene Menzer
Institute of Mathematics
University of Debrecen
P. O. Box: 400Debrecen H–4002
Hungary
e-mail: zboros@science.unideb.hu

Rayene Menzer
e-mail: rayene.menzer@science.unideb.hu;
riene1997@gmail.com

Received: December 10, 2021
Revised: September 8, 2022
Accepted: October 3, 2022