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Finite dimensional varieties over the Heisenberg group

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Abstract. Spectral analysis and synthesis studies translation invariant function spaces, so-called varieties over topological groups. The basic building blocks are the finite dimensional varieties. In the commutative case finite dimensional varieties are spanned by exponential polynomials. In non-commutative situations no relevant results exist. In this paper we consider finite dimensional left translation invariant linear spaces of continuous complex valued functions over the Heisenberg group. Using basic knowledge about Lie algebra we describe all left varieties of this type. In particular, it turns out that those function spaces are spanned by exponential polynomials as well.

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1. Introduction

Given a commutative topological group G a linear space of continuous complex valued functions on G is called a *variety* if it is closed with respect to uniform convergence on compact sets and with respect to translation. Varieties are the basic building blocks of *spectral analysis* and *spectral synthesis*. Spectral analysis for a variety means that each nonzero subvariety contains a finite dimensional nonzero subvariety, and spectral synthesis means that all finite dimensional subvarieties span a dense subspace in each subvariety. The basics of the theory can be found in [3]. In the non-commutative case, however, no general results are available. In this paper we consider a delicate non-commutative group: the Heisenberg group. We give a complete description of all finite dimensional left invariant closed linear spaces of continuous complex valued functions on this group in Theorem 6. In particular, in Theorem 8 we show that these function spaces are spanned by exponential polynomials. In

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Sects. 6 and 7 we describe the particular form of the generating functions of two and three dimensional varieties. It turns out that nontrivial varieties exist only if the dimension is at least three.

2. The Heisenberg group

The three dimensional Heisenberg group structure is defined on the set $H = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by the following operation: for (x, y, t) and (u, v, s) in H we let

$$(x, y, t) \cdot (u, v, s) = (x + u, y + v, t + s + xv).$$

Then H is a group with identity $(0, 0, 0)$ and the inverse of (x, y, t) is $(-x, -y, -t + xy)$. This group is obviously noncommutative, the commutator of (x, y, t) and (u, v, s) is

$$(x, y, t) \cdot (u, v, s) \cdot (-x, -y, -t + xy) \cdot (-u, -v, -s + uv) = (0, 0, xv - uy).$$

Using the Euclidean topology on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ the Heisenberg group H is a locally compact topological group—in fact, it is a Lie group.

If we identify (x, y, t) with the matrix

$$\begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

then we set up an isomorphism between H and the subgroup of $GL(\mathbb{R}, 3)$ consisting of all matrices of the given type. Indeed,

$$\begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u & s \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + y & t + xv \\ 0 & 1 & y + s \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall denote the Lie group of these matrices with H as well. The Lie algebra of H can be identified with the algebra of matrices of the form

$$\begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

This Lie algebra will be denoted by $\mathfrak{h}$. It is well-known, that the exponential map from $\mathfrak{h}$ onto H is bijective.

The Lie algebra $\mathfrak{h}$ of H has the basis

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

with the commutation relations

$$[A, B] = C, \;\; [A, C] = [B, C] = 0.$$

It follows that $\big[[X,Y],Z\big]=0$ for any three matrices X,Y,Z in $\mathfrak{h}$. By the Campbell–Baker–Hausdorff formula (see e.g. [2, Proposition 1.3.2], p.25), it follows that for any matrices X,Y in $\mathfrak{h}$ we have

$$e^X e^Y e^{-X} e^{-Y} = e^{[X,Y]}, \tag{4}$$

and

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{5}$$

3. Varieties

The space of all continuous complex valued functions on H will be denoted by $\mathcal{C}(H)$, and will be equipped with the topology of compact convergence. Its dual can be identified with the space $\mathcal{M}_c(H)$ of all compactly supported complex Borel measures on H. The space $\mathcal{M}_c(H)$ is equipped with the convolution:

$$\int_H f\,d(\mu*\nu) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x+u,y+v,t+s+xv)\,d\mu(x,y,t)\,d\nu(u,v,s)$$

whenever f is in $\mathcal{C}(H)$ and μ,ν are in $\mathcal{M}_c(H)$. With this convolution—together with the linear operations—$\mathcal{M}_c(H)$ is an algebra. The space $\mathcal{C}(H)$ turns into a left module over $\mathcal{M}_c(H)$ under the action

$$\mu * f(x,y,t) = \int_H f(x+u,y+v,t+s+uy)\,d\mu(u,v,s)$$

corresponding to left translation on H. Closed submodules of this module will be called *varieties.*

Proposition 1. *The closed subspace of $\mathcal{C}(H)$ is a variety if and only if it is closed under left translation.*

Proof. Suppose that V is a variety in $\mathcal{C}(H)$, and f is in V, (u,v,s) is in H. If $\delta_{u,v,s}$ denotes the point mass supported at the singleton (u,v,s), then we have

$$\begin{aligned}\delta_{u,v,s} * f(x,y,t) &= \int_H f(x+p,y+q,t+r+py)\,d\delta_{u,v,s}(p,q,r)\\ &= f(x+u,y+v,t+s+uy) = f\big((u,v,s)\cdot(x,y,t)\big),\end{aligned}$$

which is the left translate of f by (u,v,s). As V is a variety, the function $\delta_{u,v,s}*f$ is in V, hence V is left translation invariant. The converse statement follows from the fact, that point masses span a weak*-dense subspace in $\mathcal{M}_c(H)$, hence if convolutions with point masses from the left leave V invariant, then the same holds for their finite linear combinations and their weak*-limits as well, which implies that V is a variety. $\square$

As an illustration we describe all one dimensional varieties in $\mathcal{C}(H)$. If V is one dimensional, then let f be a nonzero function in V. Then for each (u,v,s) in H there exists a complex number $\lambda(u,v,s)$ such that

$$f(x+u,y+v,t+s+uy)=\lambda(u,v,s)f(x,y,t) \tag{6}$$

holds for each (x,y,t). Clearly, $f(0,0,0)\neq 0$, hence we have that $c\lambda=f$ for some nonzero complex number c. In particular, λ is continuous. It follows that

$$\lambda(x+u,y+v,t+s+uy)=\lambda(u,v,s)\lambda(x,y,t) \tag{7}$$

holds for each (x,y,t) and (u,v,s). Putting $u=s=y=0$ we get

$$\lambda(x,v,t)=\lambda(0,v,0)\lambda(x,0,t).$$

On the other hand, from (7) we infer with $y=v=0$ that

$$\lambda(x,0,t)\lambda(u,0,s)=\lambda(x+u,0,t+s),$$

which implies that

$$\lambda(x,0,t)=e^{\mu x+\xi t}$$

holds with some complex numbers μ,ξ. Similarly, we obtain from (7) with $x=u=t=s=0$ that

$$\lambda(0,y+v,0)=\lambda(0,v,0)\lambda(0,y,0),$$

which implies that

$$\lambda(0,y,0)=e^{\nu y}$$

holds with some complex number ν. Finally, we have

$$\lambda(x,y,t)=\lambda(0,y,0)\lambda(x,0,t)=e^{\mu x+\nu y+\xi t}.$$

On the other hand, substitution into (7) yields

$$e^{\mu(x+u)+\nu(y+v)+\xi(t+s+uy)}=e^{\mu u+\nu v+\xi s}\cdot e^{\mu x+\nu y+\xi t},$$

which implies $e^{\xi uy}=1$ for each u,y in $\mathbb{R}$, that is $\xi=0$. Finally, we arrive at $\lambda(x,y,t)=e^{\mu x+\nu y}$. It is easy to check that indeed, such functions span one dimensional varieties in $\mathcal{C}(H)$ for any choice of complex numbers μ,ν, hence we have proved the following result:

Proposition 2. *A variety in $\mathcal{C}(H)$ is one dimensional if and only if it is spanned by a function of the form $(x,y,t)\mapsto e^{\mu x+\nu y}$ with some complex numbers μ,ν.*

4. Matrix functional equations

In this section we go on to the describe finite dimensional varieties with dimension greater than one. Clearly, finite sums of one dimensional varieties of the above type may result in higher dimensional varieties – we shall call them *trivial*. In other words, a variety is called trivial, if it consists of functions independent of the variable t. Trivial finite dimensional varieties can be described easily: they are spanned by finitely many functions of the form

$$(x, y, t) \mapsto e^{\mu x + \nu y},$$

where μ, ν are arbitrary complex numbers. Our main goal will be to describe all nontrivial finite dimensional varieties over the Heisenberg group H.

The problem of describing finite dimensional varieties over H is closely related to the study of finite dimensional representations of the Heisenberg group, that is, to the study of the matrix functional equation

$$F(x, y, t)F(u, v, s) = F(x + u, y + v, t + s + xv), \tag{8}$$

where $F : H \to M(\mathbb{C}^n)$ is a function, and $M(\mathbb{C}^n)$ denotes the algebra of $n \times n$ matrices with complex entries. We note that the functional Eq. (8) provides a method for creating finite dimensional varieties in $\mathcal{C}(H)$. Indeed, if $F : H \to M(\mathbb{C}, n)$ is a continuous solution of (8), and $F(x, y, t) = \big(F_{ij}(x, y, t)\big)_{i,j=1,2,\ldots,n}$, then

$$F_{ij}(x + u, y + v, t + s + uy) = \sum_{k=1}^{n} F_{ik}(u, v, s)F_{kj}(x, y, t)$$

holds for each $i, j = 1, 2, \ldots n$ and for every x, y, t, u, v, s in $\mathbb{R}$, hence every left translate of the functions in the linear space V generated by the functions F_{ij} is in V, hence the functions F_{ij} for $i, j = 1, 2, \ldots, n$ generate a finite dimensional variety. Later, in Theorem 6 we shall see that every finite dimensional variety arises in this way.

Proposition 3. *If $F : H \to M(\mathbb{C}^n)$ is a continuous solution of the functional Eq. (8) such that $F(0, 0, 0)$ is invertible, then $F(x, y, t)$ is invertible for each x, y, t in $\mathbb{R}$, $F(0, 0, 0) = I$, the identity matrix, and F is analytic.*

Proof. We have, by (8)

$$F(0, 0, 0)F(0, 0, 0) = F(0, 0, 0), \tag{9}$$

and multiplying by the inverse of $F(0, 0, 0)$ we have that $F(0, 0, 0) = I$. Now (8) implies

$$F(x, y, t)F(-x, -y, -t + xy) = F(0, 0, 0) = I,$$

hence each $F(x, y, t)$ is invertible. Then the analiticity of F follows from the fact that F is a continuous Lie group homomorphism of H into the Lie group $GL(\mathbb{C}, n)$ (see e.g. [1], p. 50). $\square$

Given F with the above properties $G = F(H)$ is a (linear) Lie group as well: we shall denote its Lie algebra by $\mathfrak{g}$. The following theorem is well-known (see e.g. [2, Theorem 2.6.1], p.78.)

Theorem 4. *The continuous homomorphism $F : H \to G$ of the Lie group H onto the Lie group G induces a Lie algebra homomorphism $\varphi : \mathfrak{h} \to \mathfrak{g}$ onto $\mathfrak{g}$, which is given by the differential of F at the identity $(0, 0, 0)$ of H, denoted by dF. Further we have*

$$F(\exp X) = \exp dF(X) \tag{10}$$

for each X in $\mathfrak{h}$.

The following theorem describes all continuous solutions F of the functional Eq. (8) from H to $M(\mathbb{C}^n)$ satisfying $F(0, 0, 0) = I$.

Theorem 5. *The continuous function $F : H \to GL(\mathbb{C}^n)$ with invertible $F(0, 0, 0)$ is a solution of the functional Eq. (8) if and only if there exist matrices X, Y in $M(\mathbb{C}^n)$ such that in the Lie algebra generated by X, Y and $[X, Y]$ the commutator $[X, Y]$ is central, and for each x, y, t in $\mathbb{R}$*

$$F(x, y, t) = e^{Xx+Yy+[X,Y](t-\frac{xy}{2})} \tag{11}$$

holds.

Proof. First we prove the necessity. Equation (8), together with the continuity of F and $F(0, 0, 0) = I$ (see Proposition 3) implies that $F : H \to G$ is a Lie group homomorphism. Via simple substitutions in (8) we have

$$F(x, y, t) = F(0, y, 0)F(x, 0, 0)F(0, 0, t) \tag{12}$$

for each x, y, t in $\mathbb{R}$.

Now substituting $y = v = t = s = 0$ in (8) we get that $x \mapsto F(x, 0, 0)$ is a one-parameter subgroup in the Lie group $G = F(H)$. Similarly, by the substitutions $x = u = t = s$, resp. $x = y = u = v$ we obtain that $y \mapsto F(0, y, 0)$, resp. $t \mapsto F(0, 0, t)$ are one-parameter subgroups in G, as well. On the other hand, we have

$$(x, 0, 0) = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp Ax,$$

hence $F(x, 0, 0) = F(\exp Ax) = \exp\big(dF(A)x\big)$, by Theorem 4. Similarly, we have

$$F(0, y, 0) = \exp(dF(B)y),\ \ F(0, 0, t) = \exp(dF(C)t)$$

for each x, y, t, where $C = [A, B]$. Now if we take $X = dF(A), Y = dF(B)$ we infer

$$dF(C) = dF([A, B]) = [dF(A), dF(B)],$$

as $dF : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. In particular, X, Y satisfy the conditions of the theorem.

Now we have, by (5),

$$F(x, y, t) = e^{Yy} e^{Xx} e^{[X,Y]t} = e^{Xx+Yy-\frac{1}{2}xy[X,Y]} e^{[X,Y]t}. \tag{13}$$

Using the fact that $[X, Y]$ is central, we can write (13) in the desired form (11).

For the sufficiency we remark, that the assumption on X and Y implies

$$e^{Xx} e^{Yy} e^{-Xx} e^{-Yy} = e^{[Xx,Yy]},$$

exactly as in (4). Then we can compute as follows, using repeatedly the previous identity:

$$\begin{aligned} F(x, y, t)F(u, v, s) &= e^{Yy} e^{Xx+[X,Y]t} \cdot e^{Yv} e^{Xu+[X,Y]s} \\ &= e^{Yy} e^{Xx} \cdot e^{Yv} e^{Xu} e^{[X,Y](t+s)} = e^{Yy} e^{[Xx,Yv]} e^{Yv} e^{Xx} e^{Xu} e^{[X,Y](t+s)} \\ &= e^{Y(y+v)} e^{X(x+u)} e^{[X,Y](t+s+xv)} = F(x + u, y + v, t + s + xv), \end{aligned}$$

and the proof is complete. $\square$

5. Finite dimensional varieties

Now we are in the position to describe all finite dimensional varieties over the Heisenberg group.

Theorem 6. *Let n be a positive integer, and V be an n dimensional variety in $\mathcal{C}(H)$. Then there exist matrices X, Y in $M(\mathbb{C}^n)$ satisfying the conditions of Theorem 5 such that the elements of the matrix function F given by (11) span V.*

Proof. Let $f_1, f_2, \dots, f_n$ be a basis of V – then there exist continuous functions $\alpha_{ij} : H \to \mathbb{C}$ $(i, j = 1, 2, \dots, n)$ such that

$$f_i(x + u, y + v, t + s + uy) = \sum_{j=1}^{n} \alpha_{ij}(u, v, s) f_j(x, y, t) \;\; i = 1, 2, \dots, n \tag{14}$$

holds for each (x, y, t) and (u, v, s) in H. Indeed, the left side is the left translate of f_i by (u, v, s), which belongs to V, by left invariance. Now we apply the associativity of the group operation in H to get

$$f_i\big([(p, q, r) \cdot (u, v, s)] \cdot (x, y, t)\big) = f_i\big((p, q, r) \cdot [(u, v, s) \cdot (x, y, t)]\big),$$

where the left hand side is

$$\begin{aligned} &f_i(x + u + p, y + v + q, t + s + r + uy + py + pv) \\ &\quad = \sum_{j=1}^{n} \alpha_{ij}(p + u, q + v, r + s + pv) f_j(x, y, t), \quad i = 1, 2, \dots, n, \end{aligned} \tag{15}$$

and the right hand side is

$$\sum_{j=1}^{n} \alpha_{ij}(p,q,r) f_j(x+u, y+v, t+s+uy)$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{ij}(p,q,r)\alpha_{jk}(u,v,s) f_k(x,y,t), \quad i = 1,2,\ldots,n. \tag{16}$$

Using the linear independence of the functions f_j we infer from (15) and (16)

$$\alpha_{ij}(p+u, q+v, r+s+pv) = \sum_{k=1}^{n} \alpha_{ik}(p,q,r)\alpha_{kj}(u,v,s), \quad i,j = 1,2,\ldots,n. \tag{17}$$

Let $F : H \mapsto \mathcal{M}(\mathbb{C}^n)$ denote the matrix valued mapping such that the i,j entry $F_{ij}(x,y,t)$ of $F(x,y,t)$ is $\alpha_{ij}(x,y,t)$, then we have from (17) with $x = p, y = q, t = r$ the functional Eq. (8), and, by the definition of F, we have $F(0,0,0) = I$. Hence $F : H \to M(\mathbb{C}^n)$ is a continuous solution of the functional Eq. (8) with $F(0,0,0) = I$. The functions $f_1, f_2, \ldots, f_n$ in the i-th equation of the system (14) are linearly independent, hence there are elements (x_k, y_k, t_k) in H such that the matrix $\big(f_j(x_k, y_k, t_k)\big)_{j,k=1}^{n}$ is regular. Substituting x_k, y_k, t_k in the i-th equation of the system (14) for x, y, t we have a system of linear equations for the unknowns α_{ij} for $j = 1,2,\ldots,n$ with regular matrix. It follows, by Cramer's Rule, that the elements of the matrix F are linear combinations of some left translates of the f_i's, hence they are in V. This means that the matrix elements of F span V and the theorem is proved. $\square$

From this theorem we can see that a variety is trivial if and only if in the representation (11) we have $[X,Y] = 0$, that is, if the matrices X, Y commute. In this case e^X and e^Y commute as well, and F can be written in the form

$$F(x,y,t) = e^{Xx+Yy},$$

where X, Y are commuting matrices. They can be triangularized simultaneously, and from well-known results (see e.g. [3, Lemma 12.8.2] p. 181.) it follows that the elements of $F(x,y,t)$ are exponential polynomials. On the other hand, in the case of nontrivial varieties $[X,Y]$ is nonzero, hence e^X and e^Y cannot be triangularized simultaneously. Nevertheless, as we shall see below, even in those cases the solutions can be described using exponential polynomials.

6. Two dimensional varieties

As an application of our result we show how to describe two and three dimensional varieties in $\mathcal{C}(H)$. In this section we study the case $n = 2$, when the variety V in $\mathcal{C}(H)$ is two dimensional. We may assume that the matrix X

in (11) has Jordan normal form. This means that X has one of the following forms:

$$\text{Case i)}\ \ X = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \text{Case ii)}\ \ X = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix},$$

where a_1, a_2 are arbitrary complex numbers. If Y has the form

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

then in Case i) we have

$$[X,Y] = (a_2 - a_1)\begin{pmatrix} 0 & -b_{12} \\ b_{21} & 0 \end{pmatrix}, \quad \big[X,[X,Y]\big] = (a_1 - a_2)^2\begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix},$$

hence the requirement $\big[X,[X,Y]\big] = 0$ implies either $a_1 = a_2$, or $b_{12} = b_{21} = 0$. In both cases $[X,Y] = 0$, hence V is trivial.

In Case ii) we have

$$[X,Y] = \begin{pmatrix} b_{21} & b_{22} - b_{11} \\ & -b_{21} \end{pmatrix}, \big[X,[X,Y]\big] = \begin{pmatrix} 0 & -2b_{21} \\ 0 & 0 \end{pmatrix},$$

hence $\big[X,[X,Y]\big] = 0$ implies $b_{21} = 0$ and then

$$Y = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}, \ [X,Y] = \begin{pmatrix} 0 & b_{22} - b_{11} \\ 0 & 0 \end{pmatrix}.$$

On the other hand, we have

$$\big[Y,[X,Y]\big] = \begin{pmatrix} 0 & -(b_{11} - b_{22})^2 \\ 0 & 0 \end{pmatrix},$$

hence $\big[Y,[X,Y]\big] = 0$ implies $b_{11} = b_{22}$, $[X,Y] = 0$, consequently V is a trivial variety. It follows that there are no nontrivial varieties on H with dimension less then three.

7. Three dimensional varieties

Now we consider three dimensional varieties on H. Again, we assume that X in Theorem 5 has Jordan normal form, and

$$Y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Case (i) In the first case we have three one dimensional Jordan blocks:

$$X = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad [X,Y] = \begin{pmatrix} 0 & b_{12}(a_1 - a_2) & b_{13}(a_1 - a_3) \\ b_{21}(a_2 - a_1) & 0 & b_{23}(a_2 - a_3) \\ b_{31}(a_3 - a_1) & b_{32}(a_3 - a_2) & 0 \end{pmatrix},$$

and

$$\big[X,[X,Y]\big] = \begin{pmatrix} 0 & b_{12}(a_1-a_2)^2 & b_{13}(a_1-a_3)^2 \\ b_{21}(a_1-a_2)^2 & 0 & b_{23}(a_2-a_3)^2 \\ b_{31}(a_1-a_3)^2 & b_{32}(a_2-a_3)^2 & 0 \end{pmatrix}.$$

It is easy to see that $\big[X,[X,Y]\big]=0$ implies $[X,Y]=0$, hence in this case there is no nontrivial variety.

Case (ii) In the second case we have two Jordan blocks—one is two dimensional, and the other is one dimensional:

$$X = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

Then $[X,Y]=$

$$\begin{pmatrix} b_{21} & b_{22}-b_{11} & b_{13}(a_1-a_3)+b_{23} \\ 0 & -b_{21} & b_{23}(a_1-a_3) \\ b_{31}(a_3-a_1) & b_{32}(a_3-a_1)-b_{31} & 0 \end{pmatrix},$$

and $\big[X,[X,Y]\big]=$

$$\begin{pmatrix} 0 & -2b_{21} & (a_1-a_3)\big(b_{13}(a_1-a_3)+2b_{23}\big) \\ 0 & 0 & b_{23}(a_1-a_3)^2 \\ b_{31}(a_1-a_3)^2 & (a_1-a_3)\big(b_{32}(a_1-a_3)+2b_{31}\big) & 0 \end{pmatrix}.$$

The condition $\big[X,[X,Y]\big]=0$ implies $b_{21}=0$, and we have

$$[X,Y] = \begin{pmatrix} 0 & b_{22}-b_{11} & b_{13}(a_1-a_3)+b_{23} \\ 0 & 0 & b_{23}(a_1-a_3) \\ b_{31}(a_3-a_1) & b_{32}(a_3-a_1)-b_{31} & 0 \end{pmatrix},$$

and $\big[X,[X,Y]\big]=$

$$(a_1-a_3)\begin{pmatrix} 0 & 0 & b_{13}(a_1-a_3)+2b_{23} \\ 0 & 0 & b_{23}(a_1-a_3) \\ b_{31}(a_1-a_3) & b_{32}(a_1-a_3)+2b_{31} & 0 \end{pmatrix}.$$

Each entry of $\big[X,[X,Y]\big]$ must be zero. First we assume that $a_1=a_3$, then we have

$$[X,Y] = \begin{pmatrix} 0 & b_{22}-b_{11} & b_{23} \\ 0 & 0 & 0 \\ 0 & -b_{31} & 0 \end{pmatrix},$$

and $\big[X,[X,Y]\big]=0$, further $\big[Y,[X,Y]\big]=$

$$\begin{pmatrix} -b_{23}b_{31} & -b_{11}^2+2b_{11}b_{22}-b_{13}b_{31}-b_{22}^2-b_{23}b_{32} & b_{23}(2b_{11}-b_{22}-b_{33}) \\ 0 & -b_{23}b_{31} & 0 \\ 0 & -b_{31}(b_{11}-2b_{22}+b_{33}) & 2b_{23}b_{31} \end{pmatrix}.$$

If $b_{23} \neq 0$, then $b_{31} = 0$, and

$$\big[Y,[X,Y]\big] = \begin{pmatrix} 0 & -(b_{11}-b_{22})^2 - b_{23}b_{32} & b_{23}(2b_{11}-b_{22}-b_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As $b_{23} \neq 0$, we must have $b_{22} = 2b_{11} - b_{33}$, and $b_{32} = -\frac{(b_{11}-b_{22})^2}{b_{23}}$. With this choice we have

$$X = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \ Y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 2b_{11}-b_{33} & b_{23} \\ 0 & -\frac{(b_{33}-b_{11})^2}{b_{23}} & b_{33} \end{pmatrix},$$

$$[X,Y] = \begin{pmatrix} 0 & b_{11}-b_{33} & b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the corresponding variety is nontrivial. We note that with the simple choice $a_1 = b_{11} = b_{12} = b_{13} = b_{33} = 0$ and $b_{23} = 1$ we have the generators of the Lie algebra $\mathfrak{h}$ of the Heisenberg group H.

If $b_{31} \neq 0$, then $b_{23} = 0$ and we must have $b_{32} = 2b_{22} - b_{11}$, further $b_{13} = -\frac{(b_{11}-b_{22})^2}{b_{31}}$. With this choice we have

$$X = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \ Y = \begin{pmatrix} b_{11} & b_{12} & -\frac{(b_{11}-b_{22})^2}{b_{31}} \\ 0 & b_{22} & 0 \\ 0 & 2b_{22}-b_{11} & b_{33} \end{pmatrix},$$

$$[X,Y] = \begin{pmatrix} 0 & b_{22}-b_{11} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and again, the corresponding variety is nontrivial.

Now we consider the case when $a_1 \neq a_3$. In this case $\big[X,[X,Y]\big] = 0$ implies that $b_{13} = b_{21} = b_{23} = b_{31} = b_{32} = 0$. We have

$$X = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \ Y = \begin{pmatrix} b_{11} & b_{12} & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, [X,Y] = \begin{pmatrix} 0 & b_{22}-b_{11} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\big[X,[X,Y]\big] = 0$. On the other hand,

$$\big[Y,[X,Y]\big] = \begin{pmatrix} 0 & -(b_{22}-b_{11})^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence the requirement $\big[Y,[X,Y]\big] = 0$ implies $b_{11} = b_{22}$, and $[X,Y] = 0$, hence in this case there is no nontrivial variety.

Case (iii) Let

$$X = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 1 \\ 0 & 0 & a_1 \end{pmatrix},$$

then

$$[X, Y] = \begin{pmatrix} b_{21} & b_{22} - b_{11} & b_{23} - b_{12} \\ b_{31} & b_{32} - b_{21} & b_{33} - b_{22} \\ 0 & -b_{31} & -b_{32} \end{pmatrix},$$

and

$$\big[X, [X, Y]\big] = \begin{pmatrix} b_{31} & b_{32} - 2b_{21} & b_{11} + b_{33} - 2b_{22} \\ 0 & -2b_{31} & b_{21} - 2b_{32} \\ 0 & 0 & b_{31} \end{pmatrix}.$$

Now $\big[X, [X, Y]\big] = 0$ implies $b_{21} = b_{31} = b_{32} = 0$, hence

$$Y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{21} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix}, \ [X, Y] = \begin{pmatrix} 0 & b_{22} - b_{11} & b_{23} - b_{12} \\ 0 & 0 & b_{33} - b_{22} \\ 0 & 0 & 0 \end{pmatrix},$$

and, again $\big[X, [X, Y]\big] = 0$ implies $b_{11} + b_{33} = 2b_{22}$. On the other hand, in this case we have

$$\big[Y, [X, Y]\big] = \begin{pmatrix} 0 & -(b_{11} - b_{22})^2 & -3(b_{11} - b_{22})(b_{12} - b_{23}) \\ 0 & 0 & -(b_{11} - b_{22})^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The requirement $\big[Y, [X, Y]\big] = 0$ implies $b_{11} = b_{22}$, and then also $b_{33} = b_{22}$, and we conclude

$$Y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{11} & b_{23} \\ 0 & 0 & b_{11} \end{pmatrix}, \ [X, Y] = \begin{pmatrix} 0 & 0 & b_{23} - b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence the corresponding variety is nontrivial. It can be checked that this variety is spanned by the functions

$$\varphi_1(x, y, t) = e^{a_1 x + b_{11} y},$$
$$\varphi_2(x, y, t) = y e^{a_1 x + b_{11} y},$$
$$\varphi_3(x, y, t) = e^{a_1 x + b_{11} y}\big((b_{33} - b_{11})^2 b_{13} y^2 + 2b_{23}(2(b_{33} - b_{11})t + b_{12} y^2 - 2x)\big).$$

8. Exponential polynomials

It is well-known that if G is a commutative group, then every finite dimensional variety in $\mathcal{C}(G)$ consists of exponential polynomials. In fact, exponential polynomials can be characterized by the property that they are exactly those functions which are included in some finite dimensional variety. On the real line complex exponential polynomials are the elements of the function algebra generated by the complex exponential functions and complex polynomials. In the non-commutative case the situation is not so simple. Nevertheless, we have seen above that trivial finite dimensional varieties consist of exponential polynomials, and even in the nontrivial cases in three dimensional varieties the matrix elements of the function $F(x, y, t)$ are exponential polynomials. Now we will show that this is the case for every finite dimensional variety over the Heisenberg group. For this we shall use the following result.

Theorem 7. *Every matrix in $M(\mathbb{C}^n)$ is the sum of a semisimple and a nilpotent matrix, which commute.*

We recall that a matrix is called *semisimple*, if it is diagonaziable, that is, similar to a diagonal matrix, and it is called *nilpotent*, if some power of it is zero. Clearly, being semisimple or nilpotent is similarity invariant. If the matrix A is the sum of A_s and A_n where A_s is semisimple and A_n is nilpotent, then $A = A_s + A_n$ is called the *Chevalley–Jordan decomposition* of A.

Theorem 8. *Every finite dimensional variety over the Heisenberg group consists of exponential polynomials.*

Proof. Let V be an n-dimensional variety in $\mathcal{C}(H)$ and let $F : H \mapsto GL(\mathbb{C}, n)$ be of the form (11) so that the matrix elements of F span V. It is enough to show that the matrix elements of F are exponential polynomials. We have

$$F(x, y, t) = e^{Yy}e^{Xx+[X,Y]t}$$

for each x, y, t in $\mathbb{R}$. Here the matrices X, Y satisfy the conditions of Theorem 6. Clearly, it is enough to show that the matrix elements of $y \mapsto e^{Yy}$, the matrix elements of $x \mapsto e^{Xx}$, and also the matrix elements of $t \mapsto e^{[X,Y]t}$ are exponential polynomials. We show this for $y \mapsto e^{Yy}$, the proof is similar in the other cases.

Let $Y = Y_s + Y_n$ be the Chevalley–Jordan decomposition of Y, where Y_s is diagonizable and Y_n is nilpotent, moreover $Y_sY_n = Y_nY_s$. Let P be an invertible matrix such that $PY_sP^{-1} = D$ is diagonal, and we write $N = P^{-1}Y_nP$. We have

$$P^{-1}e^{Yy}P = P^{-1}e^{PY_sP^{-1}y}e^{PY_nP^{-1}y}P = P^{-1}e^{Dy}e^{Ny}P.$$

In general, the matrix elements of Z are the linear combinations of the matrix elements of $P^{-1}ZP$, hence it is enough to show that the matrix elements of e^{Dy} and the matrix elements of e^{Ny} are exponential polynomials.

If the diagonal elements of D are ν_{ii} $(i = 1, 2, \dots, n)$, then the matrix e^{Dy} is a diagonal matrix with diagonal elements $e^{\nu_{ii}y}$ $(i = 1, 2, \dots, n)$. As N is nilpotent, we have $N^n = 0$, and

$$e^{Ny} = I + Ny + \frac{1}{2}N^2y^2 + \cdots + \frac{1}{(n-1)!}N^{n-1}y^{n-1},$$

that is, the matrix elements of e^{Ny} are polynomials of y of degree at most $n-1$. It is clear, that the matrix elements of $e^{Dy} \cdot e^{Ny}$ are linear combinations of the exponential functions $e^{\nu_{ii} y}$ with polynomial coefficients of degree at most $n-1$. The proof is complete. $\square$

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