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Aequationes Mathematicae

Translativity of beta-type functions

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Abstract. Translativity of the beta-type function $B_f : I^2 \to (0,\infty)$,

$$B_f(x,y) := \frac{f(x)\,f(y)}{f(x+y)},$$

where f is a single variable function defined either on $I = \mathbb{R}$ or $I = [0,\infty)$, or $I = (0,\infty)$, is considered. In each of these three cases a complete solution is given.

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1. Introduction

For a positive function f on an interval I that is closed under addition, the bivariable function $B_f : I^2 \to (0,\infty)$ defined by

$$B_f(x,y) := \frac{f(x)\,f(y)}{f(x+y)},$$

is called a beta-type function and f is called its generator.

Some basic properties of beta-type functions, like homogeneity or their relation with means, were studied in [3,4], respectively. In this note we examine the conditions under which the beta-type function is (positively) translative, i.e. that

$$B_f(x+t, y+t) = B_f(x,y) + \alpha(t)$$

for a function $\alpha : (0,\infty) \to \mathbb{R}$ and all $x, y \in I$ and $t \in (0,\infty)$ (Definition 2). Theorem 1 in section 3 gives a full description of such functions in the simplest case when $I = \mathbb{R}$. In a little more difficult case $I = [0,\infty)$, considered in Sect. 4, Theorem 2 gives the solution. The case $I = (0,\infty)$, considered in Sect. 5, turned out to be the most challenging; in particular, the methods used in the previous cases do not work. A complete solution (Theorem 3) required some more sophisticated arguments.

Trying to find a simpler and more universal approach, we prove, in particular, that every translative function acting on $(0,\infty)^2$ has a unique translative extension acting on $\mathbb{R}^2$ (Theorem 4). We observe that the extension of the translative beta-type function is also of this type if and only if $\alpha \equiv 0$.

2. Preliminaries

Definition 1. Let $I \subset \mathbb{R}$ be an interval that is closed under addition, and $f : I \to (0,\infty)$ be an arbitrary function. Then $B_f : I^2 \to (0,\infty)$ given by

$$B_f(x,y) := \frac{f(x)\,f(y)}{f(x+y)}, \quad x,y \in I,$$

is called a *beta-type function* in I, and f is referred to as its generator [2].

For an interval $I \subset \mathbb{R}$ and $t \in \mathbb{R}$ put $t+I := \{t+x : x \in I\}$ and $-I := \{-x : x \in I\}$.

Definition 2. Let $I \subset \mathbb{R}$ be an interval such that $t+I \subset I$ for every $t>0$. A function $F : I^2 \to \mathbb{R}$ is said to be *positively translative*, if there is a function $\alpha : (0,\infty) \to \mathbb{R}$ such that

$$F(x+t,y+t) = F(x,y) + \alpha(t), \quad x,y \in I, \quad t>0. \tag{2.1}$$

Definition 3. Let $I \subset \mathbb{R}$ be an interval such that $t+I \subset I$ for every $t<0$. A function $F : I^2 \to \mathbb{R}$ is said to be *negatively translative*, if there is a function $\alpha : (-\infty,0) \to \mathbb{R}$ such that

$$F(x+t,y+t) = F(x,y) + \alpha(t), \quad x,y \in I, \quad t<0.$$

Remark 1. A function $F : I^2 \to \mathbb{R}$ is negatively translative if, and only if, the function $\bar{F} : (-I)^2 \to \mathbb{R}$ defined by $\bar{F}(x,y) := F(-x,-y)$ is positively translative.

Indeed, if F is negatively translative with a function $\alpha : (-\infty,0) \to \mathbb{R}$ then, setting $\bar{\alpha} : (0,\infty) \to \mathbb{R}$ by $\bar{\alpha}(t) := \alpha(-t)$, we have for all $x,y \in (-I)$ and $t>0$,

$$\begin{aligned}\bar{F}(x+t,y+t) &= F(-x+(-t),-y+(-t)) = F(-x,-y) + \alpha(-t)\\ &= \bar{F}(x,y) + \bar{\alpha}(t),\end{aligned}$$

so $\bar{F}$ is positively translative (with the function $\bar{\alpha}$). Similarly one can show the converse implication.

Due to this remark, in the sequel we consider only positively translative function. In addition, we use the term α-translative function for a positively translative function with the function α.

Remark 2. Under the conditions of Definition 2:

(i) if F is α-translative, then α is an additive function;
(ii) if F is nonnegative and α-translative, then $\alpha(t) = at$ $(t > 0)$ for some real $a \geq 0$ (called the order of translativity of F).

Proof. (i) For all $x, y \in I$ and $s, t > 0$, using (2.1) we have

$$F(x+s+t, y+s+t) = F((x+s)+t, (y+s)+t)$$
$$= F(x+s, y+s) + \alpha(t) = F(x,y) + \alpha(s) + \alpha(t),$$

and

$$F(x+s+t, y+s+t) = F(x,y) + \alpha(s+t),$$

whence $\alpha(s+t) = \alpha(s) + \alpha(t)$, so α is additive in $(0,\infty)$.
(ii) From (i), by induction, we have, for all $n \in \mathbb{N}$, $x, y \in I$, and $t > 0$,

$$F(x+nt, y+nt) = F(x,y) + n\alpha(t).$$

If $\alpha(t) < 0$ for some $t > 0$, then fixing arbitrarily $x, y \in I$ and then choosing $n \in \mathbb{N}$ such that $F(x,y) + n\alpha(t) < 0$, we would get a contradiction with the nonnegativity of $F(x+nt, y+nt)$. Thus α is nonnegative, and being additive, it must be of the form $\alpha(t) = at$ $(t > 0)$ for some $a \geq 0$ ([1, p. 33]; see also [5, p. 145]). □

3. Translativity of beta-type functions in $\mathbb{R}^2$

In this section we prove the following

Theorem 1. *Let $f : \mathbb{R} \to (0,\infty)$ and $\alpha : (0,\infty) \to \mathbb{R}$. The following conditions are pairwise equivalent:*

(i) *the beta-type function $B_f : \mathbb{R}^2 \to (0,\infty)$ is α-translative;*
(ii) *$\alpha \equiv 0$ and $\frac{f}{f(0)}$ is the exponential function, that is*

$$f(0)\, f(x+y) = f(x)\, f(y), \quad x, y \in \mathbb{R};$$

(iii) *the beta-type function B_f is constant.*

Proof. Assume (i). Remark 2 then implies that $\alpha(t) = at$ for some nonnegative a, so the translativity Eq. (2.1), by Definition 1 of B_f, takes the following form:

$$\frac{f(x+t)\, f(y+t)}{f(x+y+2t)} = \frac{f(x)\, f(y)}{f(x+y)} + at, \quad x, y \in \mathbb{R}, \quad t > 0. \tag{3.1}$$

Hence, in particular, for all $x, y \in (0,\infty)$, we get

$$\frac{f(x)\, f(y)}{f(x+y)} = \frac{f((x-y)+y)\, f(0+y)}{f(((x-y)+y)+(0+y))} = \frac{f(x-y)\, f(0)}{f(x-y)} + ay$$
$$= f(0) + ay.$$

Changing the roles of x and y in the resulting equality, we hence also get, for all $x, y \in (0, \infty)$,

$$\frac{f(x) f(y)}{f(x+y)} = f(0) + ax. \tag{3.2}$$

These two equalities imply that $a(y-x) = 0$ for all $x, y \in (0, \infty)$ and, in consequence, that $a = 0$, whence $\alpha(t) = 0$ for all $t > 0$.
Now, take arbitrary $x, y \in \mathbb{R}$. There exists a sufficiently large $t > 0$ for which $x + t$ and $y + t$ are positive. Therefore, from (3.1) and (3.2) we obtain

$$\frac{f(x) f(y)}{f(x+y)} = \frac{f(x+t) f(y+t)}{f(x+y+2t)} = f(0),$$

which proves (ii).

Since the implications $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are easy to verify, the proof is complete. □

Remark 3. The exponential function in *(ii)* need not be continuous.

4. Translativity of beta-type functions in $[0, \infty)^2$

Lemma 1. *Assume that* $f : [0, \infty) \to (0, \infty)$ *and* $\alpha : (0, \infty) \to \mathbb{R}$. *If* $B_f : [0, \infty)^2 \to (0, \infty)$ *is* α*-translative, i.e.*

$$B_f(x+t, y+t) = B_f(x, y) + \alpha(t), \quad x, y \geq 0, \quad t > 0, \tag{4.1}$$

then $\alpha \equiv 0$*, that is*

$$B_f(x+t, y+t) = B_f(x, y), \quad x, y \geq 0, \quad t > 0.$$

Proof. From (4.1), the definition of B_f and Remark 2, we easily conclude that $\alpha(t) = at$ for some real a and all $t > 0$, whence

$$\frac{f(x+t) f(y+t)}{f(x+y+2t)} = \frac{f(x) f(y)}{f(x+y)} + at, \quad x, y \geq 0, \quad t > 0.$$

Setting here $y = 0$ gives

$$\frac{f(x+t) f(t)}{f(x+2t)} = f(0) + at, \quad x \geq 0, \quad t > 0,$$

whence, putting $b := f(0)$ and $s := x + t$, we get

$$\frac{f(s) f(t)}{f(s+t)} = b + at, \quad 0 < t \leq s. \tag{4.2}$$

Replacing here s by $s + t$ with $t \leq s$, we get

$$f(s+2t) = f(s+t) \frac{f(t)}{b+at} = f(s) \left(\frac{f(t)}{b+at} \right)^2, \quad 0 < t \leq s. \tag{4.3}$$

Taking t such that $t \le s \le 2t$, in view of (4.2) we also have

$$f(2t+s) = f(2t)\,\frac{f(s)}{b+as}.$$

Since, taking $s = t$ in (4.2) gives

$$f(2t) = \frac{(f(t))^2}{b+at},$$

we hence get

$$f(2t+s) = \frac{(f(t))^2}{b+at}\,\frac{f(s)}{b+as}, \quad t \le s \le 2t. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$f(s)\left(\frac{f(t)}{b+at}\right)^2 = \frac{(f(t))^2}{b+at}\,\frac{f(s)}{b+as}, \quad 0 < t \le s \le 2t,$$

which reduces to $a(s-t) = 0$. This proves that $a = 0$. □

The main result of this section reads as follows:

Theorem 2. *Let $f : [0,\infty) \to (0,\infty)$ and $\alpha : (0,\infty) \to \mathbb{R}$. The following conditions are pairwise equivalent:*

(i) *the beta-type function $B_f : [0,\infty)^2 \to (0,\infty)$ is α-translative;*
(ii) *$\alpha \equiv 0$ and $\frac{f}{f(0)}$ is an exponential function, that is*

$$f(0)\,f(x+y) = f(x)\,f(y), \quad x, y \ge 0;$$

(iii) *the beta-type function B_f is constant.*

Proof. Assume (i). By Lemma 1 we have $\alpha \equiv 0$ and

$$\frac{f(x+t)\,f(y+t)}{f(x+y+2t)} = \frac{f(x)\,f(y)}{f(x+y)}, \quad x, y \ge 0, \quad t > 0.$$

Take arbitrary $x, y \in [0,\infty)$. Assuming that $y \le x$, which can be done without any loss of generality, and applying the above equality with $t = y$, we get

$$\frac{f(x)\,f(y)}{f(x+y)} = \frac{f((x-y)+y)\,f(0+y)}{f(((x-y)+y)+(0+y))} = \frac{f(x-y)\,f(0)}{f(x-y)} = f(0),$$

which proves (ii).

As the remaining two implications $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are simple to check, the proof is complete. □

5. Translativity of beta-type functions in $(0,\infty)^2$

We begin with an analogue of Lemma 1.

Lemma 2. *Assume that* $f:(0,\infty)\to(0,\infty)$ *and* $\alpha:(0,\infty)\to\mathbb{R}$. *If* $B_f:(0,\infty)^2\to(0,\infty)$ *is* α*-translative, i.e.*

$$B_f(x+t,y+t)=B_f(x,y)+\alpha(t),\quad x,y>0,\quad t>0, \tag{5.1}$$

then $\alpha\equiv 0$, *that is*

$$B_f(x+t,y+t)=B_f(x,y),\quad x,y>0,\quad t>0.$$

Proof. In view of Remark 2, $\alpha(t)=at$ for some nonnegative a and for all $t>0$. Assume, on the contrary, that $a>0$ and that there exists a function $f:(0,\infty)\to(0,\infty)$ satisfying (5.1), i.e. that

$$\frac{f(x+t)\,f(y+t)}{f(x+y+2t)}=\frac{f(x)\,f(y)}{f(x+y)}+at,\quad x,y,t>0.$$

Writing this equality in the form

$$\frac{\frac{f(x+t)}{a}\,\frac{f(y+t)}{a}}{\frac{f(x+y+2t)}{a}}=\frac{\frac{f(x)}{a}\,\frac{f(y)}{a}}{\frac{f(x+y)}{a}}+t,\quad x,y,t>0,$$

we see that, without any loss of generality, we can assume that $a=1$, i.e. that

$$\frac{f(x+t)\,f(y+t)}{f(x+y+2t)}=\frac{f(x)\,f(y)}{f(x+y)}+t,\quad x,y,t>0. \tag{5.2}$$

Replacing here x, y, t, respectively by $2x$, $2y$, $2t$, we obtain

$$\frac{f(2(x+t))f(2(y+t))}{f(2(x+y+2t))}=\frac{f(2x)f(2y)}{f(2(x+y))}+2t,\quad x,y,t>0. \tag{5.3}$$

Setting $y=x$ in (5.2) we get

$$\frac{(f(x+t))^2}{f(2(x+t))}=\frac{(f(x))^2}{f(2x)}+t,\quad x,t>0,$$

which shows that the function $g(x):=\frac{(f(x))^2}{f(2x)}$ satisfies the functional equation

$$g(x+t)=g(x)+t,\quad x,t>0. \tag{5.4}$$

Changing the roles of x and t in (5.4) gives us

$$g(t+x)=g(t)+x,\quad x,t>0, \tag{5.5}$$

whence, by (5.4) and (5.5), we obtain that $g(x)-x$ is constant. It follows that, for some $b\geq 0$,

$$\frac{(f(x))^2}{f(2x)}=x+b,\quad x>0.$$

Hence, using (5.3), we have

$$\frac{\frac{(f(x+t))^2}{x+t+b}\frac{(f(y+t))^2}{y+t+b}}{\frac{(f(x+y+2t))^2}{x+y+2t+b}} = \frac{\frac{(f(x))^2}{x+b}\frac{(f(y))^2}{y+b}}{\frac{(f(x+y))^2}{x+y+b}} + 2t, \quad x,y,t>0,$$

which is equivalent to

$$\left(\frac{f(x+t)f(y+t)}{f(x+y+2t)}\right)^2 = \frac{(x+t+b)(y+t+b)(x+y+b)}{(x+y+2t+b)(x+b)(y+b)}\left(\frac{f(x)f(y)}{f(x+y)}\right)^2 + \frac{2t\,(x+t+b)\,(y+t+b)}{x+y+2t+b}, \quad x,y,t>0. \tag{5.6}$$

On the other hand, squaring both sides of (5.2), gives us

$$\left(\frac{f(x+t)f(y+t)}{f(x+y+2t)}\right)^2 = \left(\frac{f(x)f(y)}{f(x+y)}\right)^2 + 2t\frac{f(x)f(y)}{f(x+y)} + t^2. \tag{5.7}$$

From (5.6) and (5.7) we obtain, for all $x,y,t>0$,

$$\left(\frac{(x+t+b)\,(y+t+b)\,(x+y+b)}{(x+y+2t+b)\,(x+b)\,(y+b)} - 1\right)\left(\frac{f(x)f(y)}{f(x+y)}\right)^2 - 2t\frac{f(x)f(y)}{f(x+y)} + \frac{2t\,(x+t+b)\,(y+t+b)}{x+y+2t+b} - t^2 = 0. \tag{5.8}$$

To finish the proof, we consider two cases separately: $b>0$ and $b=0$. In the first case equality (5.8), which must hold for all $x,y,t>0$, in particular must also be valid for $x=b$, $y=3b$, $t=b$. Putting these values to (5.8) we get

$$\frac{19}{56}\left(\frac{f\,(b)\,f\,(3b)}{f\,(4b)}\right)^2 - 2b\left(\frac{f\,(b)\,f\,(3b)}{f\,(4b)}\right) + \frac{23}{7}b^2 = 0.$$

However, because the quadratic polynomial $\frac{19}{56}p^2 - 2bp + \frac{23}{7}b^2$ of the variable p does not have real roots, we get a desired contradiction. Now, if $b=0$, equality (5.8) simplifies to

$$\left(\frac{(x+t)\,(y+t)\,(x+y)}{(x+y+2t)\,xy} - 1\right)\left(\frac{f(x)f(y)}{f(x+y)}\right)^2 - 2t\frac{f(x)f(y)}{f(x+y)} + \frac{2t\,(x+t)\,(y+t)}{x+y+2t} - t^2 = 0,$$

but once again, there does not exist a real-valued function f defined on $(0,\infty)$, satisfying this equality for all $x,y>0$ (to see that, set, for instance, $x=t=1$, $y=2$ and repeat the reasoning from the former case). □

Now we are ready to give a complete solution to the translativity problem of B_f in $(0,\infty)^2$.

Theorem 3. *Let $f:(0,\infty)\to(0,\infty)$ and $\alpha:(0,\infty)\to\mathbb{R}$. The following conditions are equivalent:*

(i) *the beta-type function* $B_f : (0,\infty)^2 \to (0,\infty)$ *is* α*-translative;*

(ii) $\alpha \equiv 0$ *and there exists* $C > 0$ *such that the function* $\frac{f}{C}$ *is an exponential function, that is*

$$Cf(x+y) = f(x)f(y), \quad x, y > 0;$$

(iii) *the function* B_f *is constant.*

Proof. Assume (i). Then, based on Lemma 2, we have

$$\frac{f(x+t)f(y+t)}{f(x+y+2t)} = \frac{f(x)f(y)}{f(x+y)}, \quad x, y, t > 0. \tag{5.9}$$

It is easy to observe that (5.9) implies that the related beta-type function B_f for arbitrary $b \in \mathbb{R}$ must be constant on the set $\mathcal{D}_b = \{(x,y) \in (0,\infty)^2 : y = x + b\}$. For a given real number b we denote this constant value by C_b. Hence (5.9) can be equivalently written as the following infinite system of equalities, indexed by $b \in \mathbb{R}$,

$$\frac{f(x)f(x+b)}{f(2x+b)} = C_b, \quad x > \max\{0, -b\}. \tag{5.10}$$

Now we prove that in fact $C_b = C_0$ for each $b \in \mathbb{R}$. First we consider the case $b > 0$. By (5.10) the function f satisfies the following two equalities:

$$f(x)f(x) = C_0 f(2x), \quad x > 0, \tag{5.11}$$

and

$$f(x)f(x+b) = C_b f(2x+b), \quad x > 0, \tag{5.12}$$

where $b > 0$ is arbitrarily fixed. Replacing x in (5.11) by b and $2b$, respectively, we immediately get

$$f(b) = \frac{\left(f\left(\frac{1}{2}b\right)\right)^2}{C_0}$$

and

$$f(2b) = \frac{(f(b))^2}{C_0} = \frac{\left(f\left(\frac{1}{2}b\right)\right)^4}{C_0^3}. \tag{5.13}$$

These two formulas, together with the following one, obtained by replacing x by b in (5.12):

$$f(b)f(2b) = C_b f(3b)$$

allow us to write

$$f(3b) = \frac{f(b)f(2b)}{C_b} = \frac{\left(f\left(\frac{1}{2}b\right)\right)^6}{C_b C_0^4}. \tag{5.14}$$

On the other hand, after putting $x = \frac{3}{2}b$ in (5.11) and $x = \frac{1}{2}b$ in (5.12), we get

$$f(3b) = \frac{\left(f\left(\frac{3}{2}b\right)\right)^2}{C_0}$$

and

$$f\left(\frac{3}{2}b\right) = \frac{C_b f(2b)}{f\left(\frac{1}{2}b\right)},$$

which, using (5.13), we can write as

$$f(3b) = \frac{C_b^2}{C_0^7}\left(f\left(\frac{1}{2}b\right)\right)^6. \tag{5.15}$$

Now, comparing the right-hand sides of (5.14) and (5.15), we obtain

$$\frac{1}{C_b C_0^4} = \frac{C_b^2}{C_0^7},$$

whence, by the positivity of C_0 and C_b, we get $C_b = C_0$, that is the desired equality.

The proof in the case $b < 0$ runs quite similarly. The main difference is that we are now looking for two different formulas for $f(-3b)$ expressed in terms of only $f\left(-\frac{1}{2}b\right)$. The first formula of this kind has the form

$$f(-3b) = \frac{\left(f\left(-\frac{1}{2}b\right)\right)^6}{C_b C_0^4}, \tag{5.16}$$

which can be obtained by putting $x = -\frac{3}{2}b$ and $x = -b$ in (5.11) and $x = -\frac{3}{2}b$ in (5.12). In turn, replacing $x = -\frac{1}{2}b$ in (5.11) and $x = -2b$ in (5.12) leads to the second formula

$$f(-3b) = \frac{C_b^2}{C_0^7}\left(f\left(-\frac{1}{2}b\right)\right)^6. \tag{5.17}$$

Finally, equalities (5.16) and (5.17) imply that also in the case of arbitrary $b < 0$ we have $C_b = C_0$. This shows that, for all $b \in \mathbb{R}$,

$$\frac{f(x)f(y)}{f(x+y)} = C_0, \quad (x, y) \in \mathcal{D}_b.$$

Because $\bigcup_b \mathcal{D}_b = (0, \infty)^2$, the proof of (ii) is completed.

Implications $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are obvious. □

6. Extension theorem for translative functions

To look at the problem of translativity of beta-type functions from a broader perspective, we begin with the following

Theorem 4. *Assume that the functions* $F : (0,\infty)^2 \to \mathbb{R}$ *and* $\alpha : (0,\infty) \to \mathbb{R}$ *satisfy the equation*

$$F(x+t, y+t) = F(x,y) + \alpha(t), \quad x, y, t > 0. \tag{6.1}$$

Then

(i) α *is additive;*

(ii) *the function* $G : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$G(x,y) := F(x+n, y+n) - \alpha(n) \quad \text{if} \quad (x,y) \in (-n, +\infty)^2, \quad n \in \mathbb{N}, \tag{6.2}$$

is well defined, and it is a unique function such that

$$G(x+t, y+t) = G(x,y) + \alpha(t), \quad x, y \in \mathbb{R}, \quad t > 0, \tag{6.3}$$

and

$$G(x,y) = F(x,y), \quad x, y > 0; \tag{6.4}$$

(iii) *if* $\alpha \equiv 0$ *and* F *is nonnegative (positive), then so is* G.

Proof. To prove (i) we can argue as in Remark 2.

(ii) For every $n \in \mathbb{N}$ define $G_n : (-n,\infty)^2 \to \mathbb{R}$ by

$$G_n(x,y) := F(x+n, y+n) - \alpha(n), \quad x, y > -n.$$

Note that for every $n \in \mathbb{N}$

$$G_{n+1}(x,y) = G_n(x,y), \quad x, y > -n.$$

Indeed, if $x, y > -n$ then, from the definition of G_n, (6.1), and the additivity of α (see Remark 2), we have

$$\begin{aligned} G_{n+1}(x,y) &= F(x+n+1, y+n+1) - \alpha(n+1) \\ &= F((x+n)+1, (y+n)+1) - \alpha(n+1) \\ &= (F(x+n, y+n) + \alpha(1)) - (\alpha(n) + \alpha(1)) \\ &= F(x+n, y+n) - \alpha(n) = G_n(x,y). \end{aligned}$$

Moreover from (6.1), for every n and $x, y > 0$,

$$G_n(x,y) := F(x+n, y+n) - \alpha(n) = (F(x,y) + \alpha(n)) - \alpha(n) = F(x,y).$$

It follows that the function $G : \mathbb{R}^2 \to \mathbb{R}$ given by (6.2), that is

$$G(x,y) := G_n(x,y) \quad \text{if} \quad (x,y) \in (-n,\infty)^2,$$

is well defined and satisfies condition (6.4).

To see that G satisfies (6.3), take arbitrary $x, y \in \mathbb{R}$ and $t > 0$. Choosing $n \in \mathbb{N}$ such that $x, y \in (-n, \infty)$, making use of the definition of G and applying property (6.1) of F, we have

$$\begin{aligned} G(x+t, y+t) &= G_n(x+t, y+t) = F(x+t+n, y+t+n) - \alpha(n) \\ &= (F(x+n, y+n) + \alpha(t)) - \alpha(n) \\ &= (F(x+n, y+n) - \alpha(n)) + \alpha(t) = G_n(x, y) + \alpha(t) \\ &= G(x, y) + \alpha(t), \end{aligned}$$

which proves (24). Since the uniqueness of G is obvious, result (ii) is proved.

To prove (iii), assume $\alpha \equiv 0$ and the nonnegativity (positivity) of F. From (6.3) and (6.4) we have

$$G(x, y) = F(x+t, y+t), \quad t+x > 0, \quad t+y > 0,$$

so G is nonnegative (positive). □

Remark 4. The counterpart of this result holds true on replacing the interval $(0, \infty)$ by (c, ∞), where $c \geq 0$ is arbitrarily fixed.

As a simple consequence of Theorems 4 and 1 we obtain that for a α-translative beta-type function B_f defined on $(0, \infty)^2$ its unique α-translative extension G is a beta-type function on $\mathbb{R}^2$, provided that $\alpha \equiv 0$.

Declarations

Conflict of interest The authors have no conflicts of interest to declare.

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