The solvability of $f(p(x)) = q(f(x))$ for given strictly monotonous continuous real functions p, q

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This publication is dedicated to my wife Alenka for her lifelong support of my investigations.

Abstract. We investigate the functional equation $f(p(x)) = q(f(x))$ where p and q are given real functions. In the paper *"On solvability of $f(p(x)) = q(f(x))$ for given real functions p, q, Aequat. Math. 90 (2016), 471 - 494"*, we solved the problem of the solvability of $f(p(x)) = q(f(x))$ under the assumption that p, q are strictly increasing continuous real functions. Now, we extend the solutions of this problem for any strictly monotonous continuous real functions p, q. Thereby, we use the methods of the just mentioned paper. Further, we present computations of the so called characteristics of the given functions p, q using the results of this paper and, finally, present a quite short algorithm with input p, q and output *'solvable/not solvable'*.

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We denote the set of all real numbers by **R**. The topic of our paper is an investigation of the functional equation

$$f(p(x)) = q(f(x)) \tag{1}$$

where p and q are two given strictly monotonous continuous real functions on **R** defined totally or partially and where f is the function to be found - as a solution of (1).

These problems were investigated by J Chvalina and others, s. [1] and [2]. They were motivated by papers of F. Neuman on transformations of differential equations ([12,13]). Their works discussed certain totally defined strictly increasing continuous real functions.

Recently, J. Matkowski and P. Wójcik in paper [11] investigated the solvability of Eq. (1) for given p, q with special properties from the point of view of the so called sandwich (separation) methods. The authors came to Eq. (1) as a generalisation of the generalised property of periodicity of functions.

For strictly increasing continuous real functions p, q, a characterisation of the existence of a solution of Eq. (1) can be found in [8]. In the present paper, we 'round up' this characterisation for any strictly monotonous continuous functions p, q. The case of strictly increasing continuous functions is the basic one with more individual possibilities and more variety (e.g., increasing functions can have more fixed points, while decreasing ones have at most one). The remaining case of strictly decreasing functions can be approached by means of the results for strictly increasing functions. The basic fact in these considerations, is that the second iteration of a strictly decreasing function is a strictly increasing one.

Strictly monotonous real functions of one variable are the only ones which are invertible. For strictly monotonous functions p, q which are continuous at once, the existence of a solution of Eq. (1) can be characterised relatively simply.

Equation (1) represents many important functional equations. However, this equation plays a considerable role in its general form. If two continuous real functions p, q are given, then their topological semiconjugation means the existence of a surjective continuous function f such that Eq. (1) is fulfilled. The methods of the present paper can be a step towards solving this well-known problem.

Our paper has two parts.

The first part (a shorter one) will present a characterisation of the solvability of the equation $f(p(x)) = q(f(x))$ (i.e. (1)) for any strictly monotonous continuous functions p, q (Theorem 5). This characterisation is similar to the main assertion in paper [8] for the solvability of (1) for any strictly increasing continuous functions p, q ([8], Theorem 18).

The second part presents computations of the so called characteristics of strictly monotonous continuous functions needed for the above characterisation. To conclude, a short algorithm for the solvability of (1) will be formulated with input p, q and output '(1) is solvable' or '(1) is not solvable'.

The present paper is built on paper [8] using its methods and results. Nevertheless, it is not necessary to read this paper because all we need from this paper in the following, all the concepts and assertions, are presented here in a short form again.

1. Introduction and the theorem

First, we introduce a couple of concepts, notations and some basic facts.

Let $\mathbf{N}$ denote the set of all natural numbers (or non negative integers) and let $\mathbf{Z}$ denote the set of all integers. Further, let $\mathbf{Z}^-$ denote the set of all negative integers.

If A is a set, then $|A|$ will denote the cardinal number of A. We will use the 'smallest' ordinal numbers. They are the natural numbers, i.e. elements of $\mathbf{N}$, and the first infinite ordinal number ω_0 is the ordinal type of the ordered set $\mathbf{N}$.

Let $\alpha \in \mathbf{N} \cup \{\omega_0\}$ be an ordinal number. Then we denote $W(\alpha) = \{n \in \mathbf{N}; n < \alpha\}$. We see that $W(\omega_0) = \mathbf{N}$ holds.

The central concept of our investigations is the concept of a function. Remember, given sets X and Y the denotation $f : X \to Y$ means that the arguments of the function f are taken from X and its values belong to Y. Notice, however, that generally the *domain* (of definition) $\mathrm{dom}(f)$ of f need not be the whole X and the *codomain* $\mathrm{img}(f) = f(\mathrm{dom}(f))$ can be a proper subset of Y. If we consider a part of a function $f : X \to Y$ on a subset $X_0 \subseteq X$ only, then we denote this function $f|X_0 = \{(x, f(x)); x \in X_0\}$, i.e. with $\mathrm{dom}(f|X_0) = X_0$. (If $\mathrm{img}(f) = Y$, then f is surjective).

To solve the equation $f(p(x)) = q(f(x))$, the methods of iteration theory can be used. For basic works on iteration theory, we refer to [9, 10, 18] or [19]. But it is also possible to find algebraic methods that can be applied for this aim. These methods of both areas can be very similar in many parts and some concepts (although with different names depending on their particular purpose) can coincide.

In our paper, we will use an algebraic approach. Namely, we can use one of the simplest algebras, the so called mono-unary algebras. (A detailed consideration of mono-unary algebras can be found in [14, 15] or [16].)

A *mono-unary algebra* is a pair (A, o) where A is a set and $o : A \to A$ a (generally partial) mapping - a so called *partial unary operation* on A. If $y \in A$ is arbitrary, then we put $o^{-1}(y) = \{x \in A; o(x) = y\}$, i. e. $o^{-1}(y)$ is the set of all origins of y for the mapping o.

If $\mathrm{dom}(o) = A$, then the mono-unary algebra (A, o) is called *complete* or *totally defined*.

If $n \in \mathbf{N}$ is arbitrary, then o^n is called the n-th iteration of o. It is defined as follows. For any $x \in A$, the existence of $o^n(x)$ and $o^n(x) \in \mathrm{dom}(o)$ imply the existence of $o^{n+1}(x)$ putting $o^{n+1}(x) = o(o^n(x))$. And further, if o is injective, then o^{-1} denotes the inverse mapping of o and $o^{-n} = (o^{-1})^n$ for any $n \in \mathbf{N} \setminus \{0\}$ for which $(o^{-1})^n$ exists. In this case, (A, o^{-1}) is a mono-unary algebra as well with the injective operation o^{-1}.

We will need the following basic concepts for mono-unary algebras. A mono-unary algebra is called *connected* if, for any $x, y \in A$, there are $m, n \in \mathbf{N}$ such that $o^m(x)$ and $o^n(y)$ are defined and $o^m(x) = o^n(y)$ holds. Further, if (A, o), (A', o') are mono-unary algebras then a (totally defined) mapping

$h : A \to A'$ is called a *homomorphism* of (A, o) into (A', o') if $x \in \mathrm{dom}(o)$ implies $h(x) \in \mathrm{dom}(o')$ and $h(o(x)) = o'(h(x))$.

As usual, the 'connected blocks' of a mono-unary algebra will be called *components* of this algebra. In iteration theory, components correspond to the concepts of orbits in the sense of Kuratowski (s. [10]).

And finally now, the symbol $\simeq$ denotes the existence of an isomorphism (bijective homomorphism) between two mono-unary algebras.

By $\mathbf{C}_n$, we will denote a cycle with n elements, i.e. a mono-unary algebra $(\{a_1, a_2, ..., a_n\}, o)$ with $o(a_i) = a_{i+1}$ for $i = 1, 2, ..., n-1$ and $o(a_n) = a_1$. In iteration theory, cycles are known under the name periods, a singleton cycle as a fixed point.

Furthermore, we will use mono-unary algebras $(\mathbf{Z}, \nu)$, $(\mathbf{N}, \nu_0)$, $(\mathbf{Z}^-, \nu^-)$ and $(W(n), \nu_1)$ for some $n \in \mathbf{N}, n > 0$ which are defined in the following way. Unary operation ν on $\mathbf{Z}$ is defined by $\nu(x) = x + 1$ for any $x \in \mathbf{Z}$ and further, we denote $\nu_0 = \nu|\mathbf{N}$, $\nu^- = \nu|(\mathbf{Z}^- \setminus \{-1\})$ and $\nu_1 = \nu|(W(n) \setminus \{n-1\})$.

It is easy to see that all these algebras are simple examples of connected mono-unary algebras.

To simplify matters, we will denote all operations ν, ν_0, ν^- and ν_1 by the same symbol ν. It is always enough to consider the different domains of ν for the set $\mathbf{Z}$ and its subsets $\mathbf{N}, \mathbf{Z}^-$ and $W(n)$.

First, we formulate simple assertions which are immediate consequences of the above definitions for mono-unary algebras. In doing so, here and throughout the paper, we use the following simple elementary fact. If $p : \mathbf{R} \to \mathbf{R}$ is a given real function that is totally or partially defined on $\mathbf{R}$, then $(\mathbf{R}, p)$ is a mono-unary algebra.

Notice as well that the solvability of Eq. (1) includes implicitly the following condition. For any $x \in \mathbf{R}$, if $p(x)$ is defined, then $q(f(x))$ is defined too.

Now, we formulate the following simple assertions concerning mono-unary algebras and their homomorphisms. Assertions (a) and (b) are taken over from [8], Lemma 1 (a), (b) and assertion (c) is a direct consequence of the definition of iterations of an operation.

Lemma 1. *The following assertions are valid.*

(a) *Let $p, q : \mathbf{R} \to \mathbf{R}$ be given real functions which are completely or partially defined on $\mathbf{R}$. Then the equation $f(p(x)) = q(f(x))$ is solvable if and only if there is a homomorphism of $(\mathbf{R}, p)$ into $(\mathbf{R}, q)$.*

(b) *Let (A, o) be a mono-unary algebra and $(B, o|B)$ be a component of (A, o) which is not isomorphic to $\mathbf{C}_1$ (a singleton cycle). Then $B \cap \mathrm{dom}(o) \neq \emptyset$ if and only if $|B| > 1$.*

(c) *Let (A, o) be a mono-unary algebra and let $x \in A$ be arbitrary. Take $m \in \mathbf{N} \setminus \{0\}$ arbitrarily. Then $o^{km}(x)$ exists for any $k \in \mathbf{N}$ if and only if $o^n(x)$ exists for any $n \in \mathbf{N}$.*

In assertion (b), we see that noncyclic components with one element play no role for homomorphisms. This is so because they are outside the domains of operations. And further (c) is a simple direct consequence of the definition of iterations of o and is useful in our considerations.

Now, we will consider a special type of mono-unary algebras. They are algebras whose unary operation is injective.

If (A, o) is a mono-unary algebra, then we denote $d_{(A,o)} = |A \setminus \text{dom}(o)|$ and $e_{(A,o)} = |A \setminus \text{img}(o)|$.

In [8], Lemma 2 (b), we find the following assertion that is basic here as well.

Lemma 2. *Let (A, o) be a connected mono-unary algebra. Then o is injective if and only if*

$$(A, o) \simeq \begin{cases} \mathbf{C}_{|A|} & \text{if } d_{(A,o)} = 0, \ e_{(A,o)} = 0 \text{ and } A \text{ is finite} \\ (\mathbf{Z}, \nu) & \text{if } d_{(A,o)} = 0, \ e_{(A,o)} = 0 \text{ and } A \text{ is infinite} \\ (\mathbf{N}, \nu) & \text{if } d_{(A,o)} = 0 \text{ and } e_{(A,o)} \neq 0 \\ (\mathbf{Z}^-, \nu) & \text{if } d_{(A,o)} \neq 0 \text{ and } e_{(A,o)} = 0 \\ (W(|A|), \nu) & \text{if } d_{(A,o)} \neq 0 \text{ and } e_{(A,o)} \neq 0 \end{cases}$$

is satisfied.

For the algebraic methods we use, an important concept is the concept of categories of algebras. Recall that by category, we mean a class of objects together with a class of morphisms between the pairs of objects with the binary operation of their composition (which has the property of associativity) and the existence of the so called identity morphisms for any object.

Here, we consider the category of all mono-unary algebras where the morphisms are their homomorphisms. The articles [4] and [5] investigate this category and its subcategory of all connected mono-unary algebras. In our considerations, we make use of the results of article [4] and implement them directly.

But we will not need the concepts of categories to speak about the classes of mono-unary algebras only. Moreover, we can confine our considerations on connected mono-unary algebras for which the operation is injective.

We will denote the class of all connected mono-unary algebras whose operations are injective by $\mathcal{L}$. Then, by Lemma 2, objects of $\mathcal{L}$ are mono-unary algebras which are isomorphic to one of the five special algebras established in this Lemma.

Any algebra $(A, o) \in \mathcal{L}$ such that $(A, o) \simeq (W(|A|), \nu)$ is called a mono-unary algebra of *finite type*.

Now, let us keep on considering a connected mono-unary algebra (A, o) whose operation o is injective, i.e. $(A, o) \in \mathcal{L}$. Then by Lemma 2, we can immediately see the following connection between the mono-unary algebras (A, o) and (A, o^{-1}).

Corollary 3. *Let* $(A, o) \in \mathcal{L}$. *Then,*

(a) *if* (A, o) *is a cycle or if it is of finite type, then* $(A, o^{-1}) \simeq (A, o)$ *holds,*

(b) *if* $(A, o) \simeq \mathbf{Z}$, *then* $(A, o^{-1}) \simeq (A, o)$ *holds as well and*

(c) $(A, o^{-1}) \simeq \mathbf{Z}^-$ *(or* $\simeq \mathbf{N}$*) if and only if* $(A, o) \simeq \mathbf{N}$ *(*$\simeq \mathbf{Z}^-$, *respectively).*

Indeed, it would be very easy to define the isomorphisms wanted. But, at the same time, the assertions are consequences of Lemma 2 because $o^{-1} \in \mathcal{L}$ too and $d_{(A,o^{-1})} = e_{(A,o)}$ and $e_{(A,o^{-1})} = d_{(A,o)}$ hold.

Now, we formulate some assertions which we will need later in particular cases for real functions.

For their presentation, the following denotations are useful. Let $(A, o) \in \mathcal{L}$ be such that it is not a cycle and let $x \in \operatorname{dom}(o)$ be arbitrary. By α_x (or β_x), we denote the greatest ordinal number such that $o^{-n}(x) \in \operatorname{dom}(o)$ for any $n \in W(\alpha_x)$ ($o^n(x) \in \operatorname{dom}(o)$ for any $n \in W(\beta_x)$, respectively). Thus, $\alpha_x \in \mathbf{N} \cup \{\omega_0\}$, $\alpha_x > 0$ ($\beta_x \in \mathbf{N} \cup \{\omega_0\}$, $\beta_x > 1$, respectively) is satisfied. For $\alpha_x = \omega_0$ (or $\beta_x = \omega_0$), we have $W(\alpha_x) = \mathbf{N}$ ($W(\beta_x) = \mathbf{N}$, respectively).

Then we see that, in the case $(A, o) \in \mathcal{L}$ where (A, o) is not a cycle, we have the following. If $x \in A$ arbitrary, then $A = \{o^{-n}(x); n \in W(\alpha_x)\} \cup \{o^n(x); n \in W(\beta_x)\}$. Thereby, if $\alpha_x \in \mathbf{N}$ (or $\beta_x \in \mathbf{N}$), then the element $o^{\alpha_x - 1}(x)$ ($o^{\beta_x - 1}(x)$) can be called the least (greatest) element of (A, o).

Lemma 4. *Let* $(A, o) \in \mathcal{L}$ *and let* $m \in \mathbf{N}, m > 0$ *be such that* o^m *exists. Let* (A', o^m) *be an arbitrary component of* (A, o^m). *Then* $(A', o^m) \in \mathcal{L}$ *and the following hold.*

(a) (A', o^m) *is a cycle if and only if* (A, o) *is a cycle.*

(b) (A', o^m) *is of finite type if and only if* (A, o) *is of finite type.*

(c) *If* A *is infinite, then* $(A', o^m) \simeq (A, o)$ *holds.*

Proof. Let $(A, o) \in \mathcal{L}$, $m \in \mathbf{N}$, $m > 0$ be arbitrary and let o^m exist. If (A', o^m) is a component of (A, o^m), then $(A', o^m) \in \mathcal{L}$ holds by Lemma 2 because (A', o^m) is connected and the injectivity of o implies the injectivity of o^m.

Now for (a), if (A', o^m) is a cycle and $x \in A'$ arbitrary, then there is $s \in \mathbf{N} \setminus \{0\}$ such that $(o^m)^s(x) = x$, i.e. $o^{sm}(x) = x$ where $x \in A$ which means that (A, o) is a cycle.

On the other hand, let (A, o) be a cycle and let $x \in A'$ be arbitrary. Then $x \in A$ and $o^n(x) = x$ for some $n \in \mathbf{N} \setminus \{0\}$. Hence $(o^m)^n(x) = o^{mn}(x) = (o^n)^m(x) = x$ and therefore, (A', o^m) is a cycle.

(b) If (A, o) is of finite type, then (A', o^m) is of finite type too because $A' \subseteq A$ holds.

On the other hand, let (A', o^m) be of finite type. Then there is $b_0 \in A' \setminus \operatorname{dom}(o^m)$. Therefore, there is $i \in \mathbf{N}$, $i \leq m$ such that the element $b = o^i(b_0) \notin \operatorname{dom}(o)$, i.e. b is the greatest element of (A, o) (in the sense of the considerations above).

Further, $(A', (o^m)^{-1}) \simeq (A', o^m)$ by Corollary 3 (a). Hence $(A', (o^m)^{-1})$ is of finite type as well. Therefore similarly, we can find the greatest element a of

(A, o^{-1}). But so we obtain $o^{-1}(a) = \emptyset$ which means that a is the least element of (A, o) (see above).

Therefore altogether, (A, o) is of finite type.

(c) If A is infinite, then (A, o) is isomorphic to $(\mathbf{Z}, \nu)$ or $(\mathbf{N}, \nu)$ or to $(\mathbf{Z}^-, \nu)$ by Lemma 2.

Further, we take $x_0 \in A'$ arbitrarily. Since $(A, o) \in \mathcal{L}$, by the denotations above, $A = \{o^{-n}(x_0); n \in W(\alpha_{x_0})\} \cup \{o^n(x_0); n \in W(\beta_{x_0})\}$ for some $\alpha_{x_0}, \beta_{x_0} \in \mathbf{N} \cup \{\omega_0\}$. Similarly, since $(A', o^m) \in \mathcal{L}$ we have $A' = \{(o^m)^{-k}(x_0); k \in W(\alpha'_{x_0})\} \cup \{(o^m)^k(x_0); k \in W(\beta'_{x_0})\}$ for some $\alpha'_{x_0}, \beta'_{x_0} \in \mathbf{N} \cup \{\omega_0\}$.

Now, by Lemma 1 (c), $o^{km}(x_0) = (o^m)^k(x_0)$ exists for any $k \in \mathbf{N}$ if and only if $o^n(x_0)$ exists for any $n \in \mathbf{N}$ and similarly, $o^{-km}(x_0) = (o^m)^{-k}(x_0)$ exists for any $k \in \mathbf{N}$ if and only if $o^{-n}(x_0)$ exists for any $n \in \mathbf{N}$. Therefore, $\beta_{x_0} = \omega_0$, i.e. $d_{(A,o)} = 0$ if and only if $\beta'_{x_0} = \omega_0$, i.e. $d_{(A',o^m)} = 0$ is satisfied. And similarly, $\alpha_{x_0} = \omega_0$, i.e. $e_{(A,o)} = 0$ if and only if $\alpha'_{x_0} = \omega_0$, i.e. $e_{(A',o^m)} = 0$ hold.

It means by Lemma 2 that both (A', o^m) and (A, o) are isomorphic to the same algebra $(\mathbf{Z}, \nu)$ or $(\mathbf{N}, \nu)$ or $(\mathbf{Z}^-, \nu)$. Thus, $(A', o^m) \simeq (A, o)$ holds. $\square$

These last assertions will be used in the second part of the article. The second section deals with a practical use of Theorem 5 (the last one of the first part) which is an answer to our problem of the solvability of Eq. (1). The second part has the name 'Computations of characteristics and an algorithm'.

From now on, we want to focus our considerations on special mono-unary algebras, namely algebras with the carrier $\mathbf{R}$ and strictly monotonous real functions as operations. Thereby, we will confine our considerations to strictly monotonous functions which are continuous.

Let $p : \mathbf{R} \to \mathbf{R}$ be a (strictly) monotonous function defined on an interval (a, b).

Then we define

$$p(a) = \lim_{x \to a_+} p(x) \quad \text{and} \quad p(b) = \lim_{x \to b_-} p(x).$$

The values $p(a)$ and $p(b)$ exist because of the monotony of p on (a, b). So the domain of p can be extended onto the interval $[a, b]$ (where, for $\text{dom}(p) = \mathbf{R}$, $[-\infty, \infty] = \mathbf{R} \cup \{-\infty, \infty\}$ with $-\infty < x < \infty$ for any $x \in \mathbf{R}$). Further, if p is continuous on (a, b), then, in this way, it is right continuous at a and left continuous at b.

Let $p : \mathbf{R} \to \mathbf{R}$ be a strictly monotonous function. Let $\mathbf{A} = (A, p|A)$ be a component of the mono-unary algebra $(\mathbf{R}, p)$. Remember the assertion of Lemma 2 for a mono-unary algebra with an injective operation (and recall that $|A|$ denotes the cardinal number of the set A). Then we put

$$\chi(\mathbf{A}) = \begin{cases} \infty_2 & \text{if } \mathbf{A} \text{ } isacycle \\ \infty_1 & \text{if } \mathbf{A} \simeq \mathbf{Z} \\ \omega & \text{if } \mathbf{A} \simeq \mathbf{N} \\ \omega^- & \text{if } \mathbf{A} \simeq \mathbf{Z}^- \\ |A| & \text{if } \mathbf{A} \simeq W(|A|). \end{cases}$$

The symbol ∞_2 is used in the theory of papers [3–5] for certain characteristics of cyclic elements. And the symbol ∞_1 is used in those papers for certain characteristics of elements with infinite sequences of predecessors.

Further, ω is an abbreviation for the ordinal type ω_0 (of $\mathbf{N}$) and ω^- is the symbol of the type of the ordered set $\mathbf{Z}^-$.

χ is a mapping $\chi : \mathcal{L} \to (\mathbf{N} \setminus \{0\}) \cup \{\omega, \omega^-, \infty_1, \infty_2\}$. However, for a mono-unary algebra $(\mathbf{R}, p)$, we need a part of mapping χ only. If $\mathcal{K}_p$ is the system of all components of $(\mathbf{R}, p)$, then it is the part $\chi|\mathcal{K}_p$. But we will not mention this fact because the association of the considered connected mono-unary algebras to $(\mathbf{R}, p)$ as its components is clear.

Now, let $\mathcal{K}_p$ be the system of all components of $(\mathbf{R}, p)$. Then we define

$$\text{char}(\mathbf{R}, p) = \{\chi(\mathbf{A}); \mathbf{A} \in \mathcal{K}_p, \chi(\mathbf{A}) \neq 1\}$$

and call $\text{char}(\mathbf{R}, p)$ the *characteristic* of p. We will use a shorter notation $\text{char}(p)$ for $\text{char}(\mathbf{R}, p)$.

To be brief in our next denotations, mainly in some formulas later, we put $\mathbf{N}_{1+} = \mathbf{N} \setminus \{0\}$ and $\mathbf{N}_{2+} = \mathbf{N} \setminus \{0, 1\}$ now.

We denote the set $\Gamma = \mathbf{N}_{2+} \cup \{\omega, \omega^-, \infty_1, \infty_2\}$. Then $\text{char}(p) \subseteq \Gamma$.

For constructions of the solutions of Eq. (1) if there are any, it is - concerning cycles - necessary to have a 'finer' characterisation for them as it is the value ∞_2 only. But for the question of solvability of (1), the value ∞_2 alone is enough. This is because of the simplest consequence of the famous Theorem of A. N. Sharkovskyi ([17]) (actually Ukrainian O. M. Sharkovskyi). Namely, if a continuous function has cycles, then it has singleton cycles - called fixed points as well. (We can say that the singleton cycles 'represent' the existence of cycles of a function.)

Now, we consider the set Γ of the characteristic values of components of continuous strictly monotonous functions, i.e. the set $\Gamma = \mathbf{N}_{2+} \cup \{\omega, \omega^-, \infty_1, \infty_2\}$. In connection with our questions, there is a hierarchy of these values.

We define a relation $\leq$ on the set Γ in the following way.

The relation $<$ is defined so that

$n < n + 1$ for any $n \in \mathbf{N}_{2+}$, further,

$n < \omega$ and $n < \omega^-$ for any $n \in \mathbf{N}_{2+}$ and finally,

$\omega < \infty_1$, $\omega^- < \infty_1$ and $\infty_1 < \infty_2$.

Now, let the relation $\leq$ be the reflexive and transitive closure of relation $<$. Then $\leq$ is a partial ordering on Γ, i.e. $(\Gamma, \leq)$ is a (partially) ordered set.

The relation $<$ is described by Fig. 1 where the symbols ∞_1, ∞_2 are abbreviated by **1** and **2**, respectively.
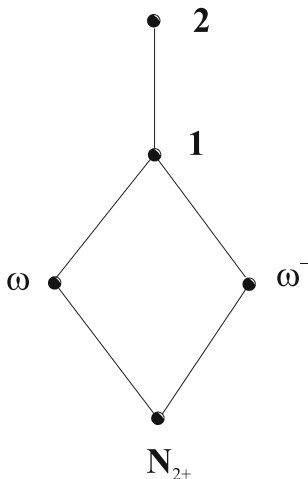
FIGURE 1. The structure of relation $<$ on $(\Gamma, \leq)$

The Theorem of this section (Theorem 5) will be obtained as a consequence of the results of article [4]. Actually, Theorem 5 is the reduction of Theorem 1.28 in [4] for our particular problem for strictly monotonous continuous functions. We do it in the same way as for the problem of solvability of Eq. (1) for strictly increasing continuous functions in paper [8].

Article [4] investigates a general form of our problem for connected mono-unary algebras. There, we could begin with definition 1.19. It was written in concepts of categories (classes) but, in our reduced situation for mono-unary algebras of real functions and in our denotation, it can be rewritten as follows. For this purpose, let us recall the denotations $d_{(A,o)} = |A \setminus \mathrm{dom}(o)|$ and $e_{(A,o)} = |A \setminus \mathrm{img}(o)|$ which we introduced before Lemma 2.

Let (A, o) be a connected mono-unary algebra. Then, we define

$$\chi(A, o) = \begin{cases} |A| & \text{if } (A, o) \text{ is a cycle} \\ \bar{d} & \text{if } A \text{ is infinite with } d_{(A,o)} = 0 \\ d & \text{if } A \text{ is infinite with } d_{(A,o)} \neq 0 \\ \omega_0 & \text{if } e_{(A,o)} \neq 0 \text{ with } d_{(A,o)} = 0 \\ |W(A, o)| & \text{if } e_{(A,o)} \neq 0 \text{ with } d_{(A,o)} \neq 0 \end{cases}$$

where $d, \bar{d}, d \neq \bar{d}$ are two symbols that are neother cardinal nor ordinal number.

Additionally, we see that A is finite for (A, o) as a cycle or for $e_{(A,o)} \neq 0, d_{(A,o)} \neq 0$ and, on the other hand, A is infinite for $e_{(A,o)} \neq 0, d_{(A,o)} = 0$.

Now, if we have a strictly monotonous continuous function p, then its char(p) is defined by means of the function χ of the components of the mono-unary algebra $(\mathbf{R}, p)$ (s. above). Here, we use the same symbol χ as in paper

[4] for components of mono-unary algebras generally. And we see that both have the same structure.

The new function χ is a reduction of the old one. So we could easily prove that the ordered set $(\Gamma, \leq)$ of characteristics of components of strictly monotonous continuous functions (s. eg. Fig. 1) can be isomorphically embedded into the ordered set of characteristics of all connected mono-unary algebras with injective operations. Actually, the isomorphism ϕ is such that $\phi|\mathbf{N}_{2+} = \mathrm{id}_{\mathbf{N}_{2+}}$, $\phi(\omega) = \omega_0$, $\phi(\omega^-) = d$, $\phi(\infty_1) = \bar{d}$ and $\phi(\infty_2) = 1$ where 1 denotes the cardinal number of singleton cycles (fixed points).

On the other hand, consider Fig. 1 above for a representation of the ordered set $(\Gamma, \leq)$. By means of ordering $\leq$ on set Γ, we now define a relation ρ on set 2^Γ. If $\theta, \theta' \in 2^\Gamma$ are arbitrary, then $\theta \, \rho \, \theta'$ holds if and only if, for any $\alpha \in \theta$, there is $\alpha' \in \theta'$ with $\alpha \leq \alpha'$.

So we obtain the set $(2^\Gamma, \rho)$ with a quasi-ordering ρ on 2^Γ.

And now, using the existence of an isomorphic embedding for any component of the mono-unary algebra $(\mathbf{R}, p)$, we can obtain almost immediately the following assertion as a consequence of Theorem 1.28 in [4].

Theorem 5. *Let* $p, q : \mathbf{R} \to \mathbf{R}$ *be strictly monotonous continuous functions. Then the equation* $f(p(x)) = q(f(x))$ *has a solution if and only if* $\mathrm{char}(p) \, \rho$ $\mathrm{char}(q)$ *holds.*

This Theorem is an extension of Theorem 1.18 in [8] which is formulated for strictly increasing continuous functions. It presents an answer to the problem formulated in the title of our paper.

2. Computations of characteristics and an algorithm

Theorem 5 would not have much impact if we didn't have the possibility to find out the values of the characteristics of the given strictly monotonous continuous functions in a not very complicated way. Therefore, we have to dedicate ourselves to these computations now.

For our computation, we need the following central definition. Let p be a strictly monotonous continuous function defined on (a, b). Putting

$$|a, b| = \begin{cases} [a, b] & \text{if } p \text{ is increasing} \\ (a, b) & \text{if } p \text{ is decreasing} \end{cases}$$

we define $I_p = \{x \in |a, b|; p(x) = x\}$.

The reason for making a distinction, in the definition of I_p, between an increasing and a decreasing function is that, for a decreasing function p, the neighborhoods of a and b play no direct role in the components of the mono-unary algebra $(\mathbf{R}, p)$. For an increasing function p, it can be the case.

To begin with, the set of all possible characteristics for strictly monotonous continuous functions as a subset of 2^Γ can be described in the following way.

In [8], we find the assertions for strictly increasing continuous functions (Corollaries 12 and 13) as follows. (Recall the denotation $\mathbf{N}_{2+} = \mathbf{N} \setminus \{0, 1\}$.)

Lemma 6. *Let p be a strictly increasing continuous function. Then the following assertions hold.*

(a)
$$\text{char}(p) \subseteq \begin{cases} \mathbf{N}_{2+} & if \quad I_p = \emptyset \\ \{\omega, \omega^-, \infty_1, \infty_2\} & if \quad I_p \neq \emptyset \end{cases}$$

(b) *Let $I_p \neq \emptyset$. If $|\text{char}(p)| \geq 2$ is satisfied, then $\infty_2 \in \text{char}(p)$ holds.*

In this connection, we will have a closer look at strictly decreasing continuous functions. We can formulate the following assertion.

Lemma 7. *Let $p : \mathbf{R} \to \mathbf{R}$ be a strictly decreasing continuous function defined on (a, b). Then the following assertions are valid.*

(a) $|I_p| \leq 1$ *holds.*

(b) *It holds that*
$$|I_p| = \begin{cases} 0 \ if \ p(a) \leq a \ or \ p(b) \geq b \\ 1 \ otherwise. \end{cases}$$

Proof. (a) If $x_1, x_2 \in (a, b), x_1 \neq x_2$ are arbitrary, then $x_1 < x_2$ implies $p(x_1) > p(x_2)$ because p is strictly decreasing and thus, both $p(x_1) = x_1$ and $p(x_2) = x_2$ can not be fulfilled. Hence $|I_p| \leq 1$ holds.

(b) Let $p(a) \leq a$. Then, for any $x \in (a, b)$, $p(x) < p(a) \leq a < x$ which implies $I_p = \emptyset$, i.e. $|I_p| = 0$. Dually, the same holds in the case $p(b) \geq b$.

On the other hand, let $p(a) > a$ and $p(b) < b$ be satisfied. Then there is $x_0 \in (a, b)$ such that $p(x_0) = x_0$ by the Intermediate value theorem because p is continuous on $[a, b]$. So we have $I_p \neq \emptyset$, i.e. $|I_p| = 1$ by (a). □

Lemma 6 shows us that values of characteristics of strictly increasing continuous functions have a special structure. For values of strictly decreasing continuous functions, this can be shown by means of Lemma 4, where we have considered some properties of mono-unary algebras.

For this, we recall some well-known properties of strictly monotonous functions.

Lemma 8. *Let $f, g : \mathbf{R} \to \mathbf{R}$ be real functions. Then the following assertions hold.*

(a) *If $f(x)$ and $g(x)$ are both strictly increasing or if they are both strictly decreasing, then $f(g(x))$ is strictly increasing.*

(b) *If $f(x)$ is strictly monotonous, then $f^2(x)$ is strictly increasing.*

Now, we can use a consequence of Lemma 4 for mono-unary algebras whose operations are injective. It could be formulated more generally but for our considerations, we need it only for the second iteration of the given functions.

If we consider a strictly monotonous continuous function $p : \mathbf{R} \to \mathbf{R}$ (defined on an interval (a, b)), and consider the comparison of values $\chi(\ )$ of the components of $(\mathbf{R}, p)$ and values $\chi(\ )$ of the components of $(\mathbf{R}, p^2)$, then we obtain the following assertions.

Lemma 9. *Let $p : \mathbf{R} \to \mathbf{R}$ be a strictly monotonous continuous function. Let (A, p) be a component of $(\mathbf{R}, p)$. Further, let p^2 on A exist and let (A', p^2) be a component of $(\mathbf{R}, p^2)$ such that $A' \subseteq A$ holds. Then the following is satisfied.*
(a) *If A is finite, then either $\chi((A, p)) = \chi((A', p^2)) = \infty_2$ or $\chi((A, p))$, $\chi((A', p^2)) \in \mathbf{N}_{2+}$ holds.*
(b) *If A is infinite, then $\chi((A, p)) = \chi((A', p^2))$ holds.*

Proof. By Lemma 2, if A is finite, then (A, p) is a cycle or it is of finite type. Thus, (a) is satisfied by Lemma 4 (a) and (b). Further for (b), if A is infinite, then $\chi((A', p^2)) = \chi((A, p))$ holds by Lemma 4 (c). □

Hence, by the definition of char(p) of a strictly monotonous continuous function p, we obtain the following.

Corollary 10. *Let $p : \mathbf{R} \to \mathbf{R}$ be a strictly monotonous continuous function. If p^2 exists, then char$(p) =$ char(p^2) holds.*

So we can formulate a similar assertion to Lemma 6 for strictly decreasing continuous functions as well.

Corollary 11. *Let $p : \mathbf{R} \to \mathbf{R}$ be a strictly decreasing continuous function. Then the following assertions hold.*
(a) char$(p) \subseteq \{2, \omega, \omega^-, \infty_1, \infty_2\}$ *(for $2 \in \mathbf{N}_{2+}$).*
(b) *If $I_p = \emptyset$, then char$(p) = \{2\}$ holds.*
(c) *If $I_p \neq \emptyset$, then $\infty_2 \in$ char(p) is satisfied.*

Indeed, let p be defined on (a, b).

Then for (a), if (A, p) is a component of $(\mathbf{R}, p)$ such that $|A| > 2$, then there is p^2 on A and $\chi((A, p)) = \chi((A, p^2))$ holds by Lemma 9. Thus, $\chi((A, p)) \in \{\omega, \omega^-, \infty_1, \infty_2\}$ by Lemma 6 because p^2 is increasing. On the other hand, if $|A| \leq 2$, then $|A| = 2$ because $A \cap (a, b) \neq \emptyset$ and so, for $x \in A \cap (a, b)$, $A = \{a, p(x)\}$ holds. Hence, char$(p) \subseteq \{2, \omega, \omega^-, \infty_1, \infty_2\}$.

Further, (b) and (c) are consequences of Lemma 7 (b) because, by the definition of I_p, $I_p \neq \emptyset$ and $(a, b) \cap (p(b), p(a)) \neq \emptyset$ are equivalent.

Thus by Lemma 6 and Corollary 11, we obtain the following.

Corollary 12. *Let p be a strictly monotonous continuous function. Then*

$$\text{char}(p) \in 2^{\mathbf{N}_{2+}} \cup \{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \{X \cup \{\infty_2\}; X \in 2^{\{\omega, \omega^-, \infty_1\}}\}$$

holds.

Indeed, by Lemma 6, the characteristic of a strictly increasing continuous function can be, on the one hand, a subset of $\mathbf{N}_{2+}$ and on the other hand, a subset of $\{\omega, \omega^-, \infty_1, \infty_2\}$ whereby the singletons $\{\omega\}, \{\omega^-\}, \{\infty_1\}, \{\infty_2\}$ are possible. Further, by Corollary 11, the characteristic of a strictly decreasing continuous function can be either a subset $\{2\}$ (for $2 \in \mathbf{N}_{2+}$) or a subset of $\{\omega, \omega^-, \infty_1, \infty_2\}$ whereby the singletons $\{\omega\}, \{\omega^-\}, \{\infty_1\}$ are not possible (by (c) of this Corollary).

Now, we denote the set from Corollary 12 by Θ. Thus, if we denote $\Theta^* = \{X \cup \{\infty_2\}; X \in 2^{\{\omega, \omega^-, \infty_1\}}\}$ (which is a set with eight elements), then we obtain the set Θ from this Corollary in the form

$$\Theta = 2^{\mathbf{N}_{2+}} \cup \{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \Theta^*.$$

Further, we come back to the quasi-ordered set $(2^\Gamma, \rho)$ where $\Gamma = \mathbf{N}_{2+} \cup \{\omega, \omega^-, \infty_1, \infty_2\}$. Then we see, that $\Theta \subset 2^\Gamma$ holds and so we can reduce quasi-ordering ρ on set Θ, i.e. put $\rho_\Theta = \rho \cap (\Theta \times \Theta)$.

Thereby, coming from the ordered set $(\Gamma, \leq)$ we see as well that the relation ρ_Θ on Θ is defined so that, for any $\theta, \theta' \in \Theta$, $\theta \rho_\Theta \theta'$ holds if and only if, for any $\alpha \in \theta$, there is $\alpha' \in \theta'$ with $\alpha \leq \alpha'$ (cf. Fig. 1 for $(\Gamma, \leq)$).

For the purpose of simplicity, we use the same denotation ρ for its reduction ρ_Θ and come to the quasi-ordered set (Θ, ρ).

Further especially, $\infty_2 \in \theta$ for any $\theta \in \Theta^*$. Thus by the definition of ρ, $\theta \ \rho \ \theta'$ is satisfied for any $\theta, \theta' \in \Theta^*$ and so the relation ρ is complete on Θ^*. Moreover, we recognize a congruence on the quasi-ordered set (Θ, ρ) with congruence classes $2^{\mathbf{N}_{2+}}, \{\omega\}, \{\omega^-\}, \{\infty_1\}$ and Θ^*.

The relation ρ is represented in Fig. 2. There we use the possibility of a graphical representation by means of the above congruence classes. Also, the reflexive arrows are omitted. (The symbol ∞_1 is abbreviated by $\mathbf{1}$ again).
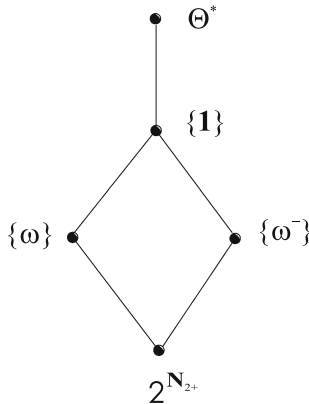

FIGURE 2. The structure of relation ρ of (Θ, ρ)

Comparing this Figure and Fig. 1 we can see a similarity between the representations of the relational sets $(\Gamma, \leq)$ and (Θ, ρ).

And now, we can come to computations of characteristics of our considered strictly monotonous continuous functions. The more difficult part of this question was treated in paper [8]. Moreover, we can use some results from [8] now. There, we find Corollary 16 where this problem is solved for strictly increasing continuous function.

In this connection, we consider the set $\mathbf{I}$ of all intervals on $\mathbf{R}$ with the partial order $\subseteq$ on $\mathbf{I}$. We use $\subset$ or $\supset$ for $\subseteq$ or $\supseteq$ respectively and $\neq$. Further, $\parallel$ denotes $\not\subseteq$ and $\not\supseteq$, i.e. it means the incomparability of two intervals with respect to the order $\subseteq$. Then the assertion ([8], 16) is as follows.

Theorem 13. *Let p be a strictly increasing continuous function with $\mathrm{dom}(p) = (a,b)$. Let $I_p \neq \emptyset$. Denote $I_p^{()} = I_p \cap (a,b)$ and let $\mathbf{I}$ be the set of all intervals on set $\mathbf{R}$. Then*

$$
\mathrm{char}(p) = \begin{cases}
\{\infty_1\} & \text{if } a, b \in I_p \text{ and } I_p^{()} = \emptyset \\
\{\infty_2\} & \text{if } a, b \in I_p \text{ and } I_p \in \mathbf{I} \\
\{\infty_1, \infty_2\} & \text{if } a, b \in I_p, \ I_p^{()} \neq \emptyset \text{ and } I_p \notin \mathbf{I} \\
\{\omega\} & \text{if } (p(a), p(b)) \subset (a,b) \text{ and } I_p^{()} = \emptyset \\
\{\omega, \infty_2\} & \text{if } (p(a), p(b)) \subset (a,b), \ I_p^{()} \neq \emptyset \text{ and } I_p \in \mathbf{I} \\
\{\omega, \infty_1, \infty_2\} & \text{if } (p(a), p(b)) \subset (a,b), \ I_p^{()} \neq \emptyset \text{ and } I_p \notin \mathbf{I} \\
\{\omega^-\} & \text{if } (a,b) \subset (p(a), p(b)) \text{ and } I_p^{()} = \emptyset \\
\{\omega^-, \infty_2\} & \text{if } (a,b) \subset (p(a), p(b)), \ I_p^{()} \neq \emptyset \text{ and } I_p \in \mathbf{I} \\
\{\omega^-, \infty_1, \infty_2\} & \text{if } (a,b) \subset (p(a), p(b)), \ I_p^{()} \neq \emptyset \text{ and } I_p \notin \mathbf{I} \\
\{\omega^-, \omega, \infty_2\} & \text{otherwise with } I_p \in \mathbf{I} \\
\{\omega, \omega^-, \infty_1, \infty_2\} & \text{otherwise with } I_p \notin \mathbf{I}
\end{cases}
$$

is satisfied.

For a more practical use of this assertion, we will add a new natural property of I_p for a strictly monotonous continuous function p. If p is such a function, then we assume that $I_p \neq (a,b)$ implies $|I_p| \leq \aleph_0$ and say that I_p *is countable*. Then we can formulate the following consequence of Theorem 13.

Theorem 14. *Let p be a strictly increasing continuous function with $\mathrm{dom}(p) = (a,b)$. Let $I_p \neq \emptyset$ and be countable. Denote $I_p^{()} = I_p \cap (a,b)$. Then*

$$\text{char}(p) = \begin{cases} \{\infty_2\} & \textit{if } \ I_p^{()} = (a,b) \\ \{\infty_1\} & \textit{if } \ (p(a),p(b)) = (a,b) \ \textit{and } |I_p^{()}| = 0 \\ \{\infty_1,\infty_2\} & \textit{if } \ (p(a),p(b)) = (a,b) \ \textit{and } |I_p^{()}| > 0 \\ \{\omega\} & \textit{if } \ (p(a),p(b)) \subset (a,b) \ \textit{and } |I_p^{()}| = 0 \\ \{\omega,\infty_2\} & \textit{if } \ (p(a),p(b)) \subset (a,b) \ \textit{and } |I_p^{()}| = 1 \\ \{\omega,\infty_1,\infty_2\} & \textit{if } \ (p(a),p(b)) \subset (a,b) \ \textit{and } |I_p| > 1 \\ \{\omega^-\} & \textit{if } \ (p(a),p(b)) \supset (a,b) \ \textit{and } |I_p^{()}| = 0 \\ \{\omega^-,\infty_2\} & \textit{if } \ (p(a),p(b)) \supset (a,b) \ \textit{and } |I_p^{()}| = 1 \\ \{\omega^-,\infty_1,\infty_2\} & \textit{if } \ (p(a),p(b)) \supset (a,b) \ \textit{and } |I_p| > 1 \\ \{\omega^-,\omega,\infty_2\} & \textit{if } \ (p(a),p(b)) \parallel (a,b) \ \textit{and } |I_p^{()}| = 1 \\ \{\omega,\omega^-,\infty_1,\infty_2\} & \textit{if } \ (p(a),p(b)) \parallel (a,b) \ \textit{and } |I_p^{()}| > 1 \end{cases}$$

is satisfied.

Proof. The assertion is a direct consequence of Theorem 13.

The conditions for the computations of the value of $\text{char}(p)$ in Theorem 13 can be separated into two parts. The first ones give the relationship between the intervals (a,b) and $(p(a),p(b))$. The others are combinations of conditions for $I_p^{()}$ and the conditions for I_p to belong to the set **I** of all intervals in **R**.

The conditions $a,b \in I_p$ in the first three cases of the computation of $\text{char}(p)$ mean $p(a) = a$ and $p(b) = b$ or also, $(p(a),p(b)) = (a,b)$. Further, it is $|I_p \cap \{a,b\}| = 2$.

On the other hand, the conditions 'otherwise' (for the relationship among $(p(a),p(b))$ and (a,b)) in the two last cases of this computation mean $(p(a), p(b)) \nsubseteq (a,b)$ and $(p(a),p(b)) \nsupseteq (a,b)$, i.e. $(p(a),p(b))\|(a,b)$. Now, $|I_p \cap \{a,b\}| = 0$ holds because for $p(a) = a$, we would have $(p(a),p(b)) \subseteq (a,b)$ or $(a,b) \subseteq (p(a),p(b)$ and similarly it is for $p(b) = b$.

The other conditions, for $I_p^{()}$ and I_p, are simple in our case. Namely, the condition that I_p is countable implies that an interval in **R** can be a singleton only. Therefore, the condition $I_p^{()} \neq \emptyset$, $I_p \in$ **I** is equivalent to $|I_p^{()}| = 1$. And further, the condition $I_p^{()} \neq \emptyset$, $I_p \notin$ **I** is equivalent to $|I_p^{()}| > 1$ or to $|I_p^{()}| = 1$ (if $|I_p \cap \{a,b\}| \geq 1$) and thus, it is equivalent to $|I_p| > 1$.

Finally, in the first two cases of the computation here with $(p(a),p(b)) = (a,b)$, we have either $|I_p^{()}| = 0$ or $|I_p^{()}| > 0$ (because $|I_p \cap \{a,b\}| = 2$ holds) and in the last two cases with $(p(a),p(b))\|(a,b)$, we have $|I_p^{()}| = 1$ or $|I_p^{()}| > 1$ (because $|I_p \cap \{a,b\}| = 0$ holds now). □

The following rules could serve as a mnemonic for the numerous possible values of $\text{char}(p)$ of a given p in the Theorem.

If the function p, its $\text{dom}(p)$

contracts, then $\omega \in \text{char}(p)$ (in any $x \in \text{dom}(p) \setminus \text{img}(p)$ begins a component),

expands, then $\omega^- \in \text{char}(p)$ (in any $x \in \text{img}(p) \backslash \text{dom}(p)$ ends a component),
moves sideways (*neither* $\subset$ *nor* $\supset$), then both $\omega, \omega^- \in \text{char}(p)$.
And, $|I_p| > 1$ *adds* $\infty_1 \in \text{char}(p)$.

Now, let us look at the case of strictly decreasing continuous functions.

For strictly decreasing continuous functions, it is possible to use Corollary 10. Therefore, the second iteration of a strictly decreasing continuous function is important for our computations.

Let p be a strictly decreasing continuous function defined on (a, b). Then the following can be proven.

Since $\text{dom}(p^2) = p^{-1}((a,b) \cap (p(b), p(a)))$ holds and p^2 is increasing, Theorem 14 implies that the possible values of $\text{char}(p^2)$ are $\{\infty_2\}$, $\{\infty_1, \infty_2\}$, $\{\omega, \infty_2\}$, $\{\omega, \infty_1, \infty_2\}$, $\{\omega^-, \infty_2\}$ and $\{\omega^-, \infty_1, \infty_2\}$. Hence, these six values are the only possible ones for strictly decreasing continuous functions - and not all of the eleven in Theorem 14. It is different for strictly increasing functions. In [8], we showed that strictly increasing functions can take all the values listed in Theorem 14.

Example 15. In [8], we can find several examples of computations of $\text{char}(p)$ if p is a strictly increasing continuous function (Example 17). There, we have an example for any possibility of the characteristic of a strictly increasing continuous function. The number of such possibilities is 11 (Theorem 14), actually ten, because the first one is trivial.

Now, we can continue this 'tradition' for strictly decreasing continuous function. If we look at the remark just mentioned above, then we see that we have less work this time. The number of such possibilities is 6, actually five, because the value $\{\infty_2\}$ as the characteristic of a strictly decreasing continuous function, is trivial again.

(a) Consider the (partially defined) strictly decreasing continuous function $p(x) = -\sqrt{x} + 1$ and let us compute $\text{char}(p)$.

Firstly, by Corollary 11 (b), (c), we look at whether $I_p \neq \emptyset$ or not. So we set $p(x) = x$, i.e. $-\sqrt{x} + 1 = x$ and find that the equation is solvable. Hence, $I_p \neq \emptyset$ and so $\infty_2 \in \text{char}(p)$ by 11 (c).

Further, by Corollary 10, we can compute $\text{char}(p^2)$. Then, since p^2 is a strictly increasing continuous function we will use Theorem 14. Denote $(a, b) = \text{dom}(p)$.

Using Theorem 14 we must determine the relation between $\text{dom}(p^2)$ and $\text{img}(p^2)$. Denote $(a_1, b_1) = \text{dom}(p^2)$ and so $\text{img}(p^2) = (p^2(a_1), p^2(b_1))$.

We see that $(a_1, b_1) = p^{-1}((a,b) \cap (p(b), p(a))) = p^{-1}(\max\{a, p(b)\}, \min\{b, p(a)\}) = (p^{-1}(\min\{b, p(a)\}), p^{-1}(\max\{a, p(b)\}))$.

Therefore,
$a_1 = p^{-1}(\min\{b, p(a)\})$ and
$b_1 = p^{-1}(\max\{a, p(b)\})$.

So, we have carried out this computation for the next examples as well.

Here, we have $a = 0, b = \infty$ and $p(a) = p(0) = 1, p(b) = p(\infty) = -\infty$.

Thus, $\max\{a, p(b)\} = \max\{0, -\infty\} = 0$ and $\min\{b, p(a)\} = \min\{\infty, 1\} = 1$.

Now, we need $p^{-1}(x) = (1 - x)^2 |\, (-\infty, 1)$ and determine $a_1 = p^{-1}(1) = 0$ and $b_1 = p^{-1}(0) = 1$. Hence, we obtain $(a_1, b_1) = (0, 1)$.

Further, we compute $p^2(a_1) = p^2(0) = p(p(0)) = p(1) = 0$ and $p^2(b_1) = p^2(1) = p(p(1)) = p(0) = 1$ which implies $(p^2(a_1), p^2(b_1)) = (a_1, b_1)$.

This relation corresponds to the condition for the value $\{\infty_1\}$ or $\{\infty_1, \infty_2\}$ of a characteristic in Theorem 14. But we already know from before that $\infty_2 \in \mathrm{char}(p)$ and so we obtain $\mathrm{char}(p) = \mathrm{char}(p^2) = \{\infty_1, \infty_2\}$ by Corollary 10.

Altogether, we get the result $\mathrm{char}(-\sqrt{x} + 1) = \{\infty_1, \infty_2\}$.

Let us still notice that, in the case of the characteristic $\{\infty_1, \infty_2\}$, we did not need to know the help-function p^2 exactly, but only the values $p^2(a_1)$ and $p^2(b_1)$.

(b) Consider the (totally defined) strictly decreasing continuous function $p(x) = -\arctan(x)$. We will compute $\mathrm{char}(p)$ and can do so in a shorter way now (by means of some computations from (a)).

Since the equation $-\arctan(x) = x$ has a solution we have $I_p \neq \emptyset$ and so $\infty_2 \in \mathrm{char}(p)$.

Further, $\mathrm{dom}(p) = (-\infty, \infty)$ and $p(-\infty) = \frac{\pi}{2}$, $p(\infty) = -\frac{\pi}{2}$. We need $p^{-1}(x) = \tan(-x)$.

If we denote $(a_1, b_1) = \mathrm{dom}(p^2)$ like in example (a) , then we have, by the computations in (a), $a_1 = p^{-1}(\min\{\infty, \frac{\pi}{2}\}) = p^{-1}(\frac{\pi}{2}) = -\infty$ and $b_1 = p^{-1}(\max\{-\infty, -\frac{\pi}{2}\}) = p^{-1}(-\frac{\pi}{2}) = \infty$.

Thus, $(a_1, b_1) = (-\infty, \infty)$. Thereby, $p^2(a_1) = p^2(-\infty) = p(\frac{\pi}{2}) = -\arctan(\frac{\pi}{2})$ and $p^2(b_1) = p^2(\infty) = p(-\frac{\pi}{2}) = \arctan(\frac{\pi}{2})$ hold. Hence, $(p^2(a_1), p^2(b_1)) = (-\arctan(\frac{\pi}{2}), \arctan(\frac{\pi}{2}))$.

It implies the relation $(p^2(a_1), p^2(b_1)) \subset (a_1, b_1)$. If we check in Theorem 14 the conditions with this relation, then we see that they can have the characteristics with values $\{\omega\}$, $\{\omega, \infty_2\}$ or $\{\omega, \infty_1, \infty_2\}$. But we already know that $\infty_2 \in \mathrm{char}(p)$ and so $\{\omega\}$ is omitted.

On the other hand, the equation $p^2(x) = -\arctan(-\arctan(x)) = x$ for I_{p^2} has the only solution 0 and thus, $|I_{p^2}^{()}| = 1$. Therefore, $\mathrm{char}(p^2) = \{\omega, \infty_2\}$.

Hence, $\mathrm{char}(-\arctan(x)) = \{\omega, \infty_2\}$ by Corollary 10.

(c) Now, let as look at the (partially defined) strictly decreasing continuous function $p(x) = -\sqrt[3]{x} \,|(-8, 8)$.

Here again, $p(x) = x$ has one solution (actually, 0) which means for the computation of $\mathrm{char}(p)$ that $\infty_2 \in \mathrm{char}(p)$. However, this result also appears later.

By (a) and similarly to (a) and (b), we come to the relation $(p^2(a_1), p^2(b_1)) \subset (a_1, b_1)$ where $(a_1, b_1) = \mathrm{dom}(p^2)$. Looking at the conditions in Theorem 14

for char(p^2) we see (knowing that ∞_2 is also included) that the possible values would be $\{\omega, \infty_2\}$ and $\{\omega, \infty_1, \infty_2\}$.

But further for $|I_{p^2}|$, the equation $\sqrt[9]{x} = x$ has the solutions $-1, 0$ and 1 and thus, $|I_{p^2}| > 1$.

That implies the result char$(-\sqrt[3]{x} \,|(-8, 8)) = \{\omega, \infty_1, \infty_2\}$ (Corollary 10).

(d) Let us compute char(p) for $p(x) = -\ln(x)$ in the same way but as an example of a strictly decreasing continuous function with another characteristic.

Here again, by (a) and similarly to (a) and (b), we come to the relation $(p^2(a_1), p^2(b_1)) \supset (a_1, b_1)$ where $(a_1, b_1) = \mathrm{dom}(p^2)$.

Now, we look at the possible characteristics in Theorem 14. To the condition for the relation $\supset$ we add the fact that the equation $p^2(x) = x$ has one solution which implies $|I_{p^2}| = 1$.

So we obtain the result char$(-\ln x) = \{\omega^-, \infty_2\}$ (by Corollary 10).

(e) We can show the following. If p is a strictly monotonous continuous function, then char(p) and char(p^{-1}) are conjugated to each other in the sense that - as the only difference - the value ω will be exchanged by ω^- and vice versa.

Therefore, char$(-x^3 \,|(-2, 2)) = \{\omega^-, \infty_1, \infty_2\}$ holds by example (c) because $-x^3 \,|(-2, 2)$ and $-\sqrt[3]{x} \,|(-8, 8)$ are mutually inverse functions.

$\{\omega^-, \infty_1, \infty_2\}$ is the last possible value of a characteristic of a strictly decreasing continuous function out of the five.

Now, let us come back to the problem of our paper. Remember that the equation $f(p(x)) = q(f(x))$ is denoted by (1). An answer to the question of the solvability of (1) for given strictly monotonous continuous functions p, q was given in Theorem 5. In the second section of the paper, we considered the structure of the relation set (Θ, ρ) which plays the main role in Theorem 5. Further, we have investigated computations of the characteristics of strictly monotonous continuous functions.

By Theorem 14 and by means of Corollary 10, char(p) can be computed for any strictly monotonous continuous function p under the assumption that $I_p \neq \emptyset$ holds.

But, on the other hand, for a strictly monotonous continuous function p with $I_p = \emptyset$, the situation is easier because of the simplicity of the structure of the relation set (Θ, ρ) - see Fig. 2. Namely, the area of values of char(p) is $\mathbf{N}_{2+}$ by Lemma 6 and Corollary 11. (See Corollary 17 later.)

Not just any solution of (1) can be as important or interesting as others. For instance, this may be the set of trivial solutions of Eq. (1) if there is any.

A solution f of (1) is called *trivial* if $f(\mathbf{R}) \subseteq I_q$ holds. If any solution of Eq. (1) is trivial, then we say that (1) is *solvable trivially*.

Here, the naming 'solvable trivially' concerns only the existence of solutions of (1) and does not imply a way to find or constructions of solutions of (1) if $f(\mathbf{R}) \subseteq I_q$ holds.

In this connection, it can be seen that the number of all trivial solutions of Eq. (1) can be countable. It is in contrast with the fact that, on the other hand, it can be shown that the number of all non trivial solutions (if there is one) is uncountable (cf. [6]).

We find the subrelation of relation ρ on the set Θ (of all characteristics of strictly monotonous continuous functions), which represents only trivially solvable equations and which we will denote by ρ_t, as follows. In fact, if $\theta, \theta' \in \Theta$, then we write $\theta \rho_t \theta'$ in the case that $\theta \rho \theta'$ holds and if, for any $\alpha \in \theta$, $\alpha' \in \theta'$, $\alpha \leq \alpha'$ implies $\alpha' = \infty_2$.

We will demonstrate this in the next table. The elements of the subset $\{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \Theta^*$ of Θ are written without curly brackets and commas and symbols ∞_1, ∞_2 are abbreviated by **1**, **2**, respectively. If $\theta, \theta' \in \{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \Theta^*$, then we denote the case that $\theta \rho \theta'$ is not fulfilled by '$-$', further by '$+$' if $\theta \rho \theta'$ holds and finally, by '$(+)$' if $\theta \rho_t \theta'$ is satisfied. (Actually, '$-$' stands for 'not solvable', '$+$' for 'solvable' and '$(+)$' for 'solvable trivially'.)

So we obtain the table

$\theta \mid \theta'$	ω	ω^-	**1**	**2**	ω**2**	ω^-**2**	**12**	$\omega\omega^-$**2**	ω**12**	ω^-**12**	$\omega\omega^-$**12**
ω	+	−	+	(+)	+	(+)	+	+	+	+	+
ω^-	−	+	+	(+)	(+)	+	+	+	+	+	+
1	−	−	+	(+)	(+)	(+)	+	(+)	+	+	+
2	−	−	−	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
ω**2**	−	−	−	(+)	+	(+)	+	+	+	+	+
ω^-**2**	−	−	−	(+)	(+)	+	+	+	+	+	+
12	−	−	−	(+)	(+)	(+)	+	(+)	+	+	+
$\omega\omega^-$**2**	−	−	−	(+)	+	+	+	+	+	+	+
ω**12**	−	−	−	(+)	+	(+)	+	+	+	+	+
ω^-**12**	−	−	−	(+)	(+)	+	+	+	+	+	+
$\omega\omega^-$**12**	−	−	−	(+)	+	+	+	+	+	+	+

This table presents a part of relation ρ on set Θ, namely this one on $\Theta \setminus 2^{\mathbf{N}_{2+}} = \{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \Theta^*$ and its subrelation ρ_t. We denote this table by T_{ρ, ρ_t} because we will use it. The part of T_{ρ, ρ_t} under the middle line is the part of ρ and ρ_t on set Θ which is the complete relation on Θ^*.

Example 16. In the paper [8], we find several examples of whether an Eq. (1) is solvable for given strictly increasing continuous functions (Example 19). Now, we can add examples where the given functions can be strictly decreasing continuous function as well.

(a) We take the equation

$$f(-\sqrt{x} + 1) = -\sqrt{f(x) + 1}$$

where $p(x) = -\sqrt{x} + 1$ and $q(x) = -\sqrt{x+1}$ are strictly decreasing continuous functions and ask about the solvability of the equation.

In Example 15 (a), we have determined $\mathrm{char}(-\sqrt{x}+1) = \{\infty_1, \infty_2\}$.

Now, we still need to know $\mathrm{char}(-\sqrt{x+1})$.

Like in Example 15 (a), we come to the relation $(q^2(a_1), q^2(b_1)) = (a_1, b_1))$ where $(a_1, b_1) = \mathrm{dom}(q^2)$.

In Theorem 14, this condition together with the solvability of the equation $q(x) = x$ for $I_q^{()}$ correspond to the characteristic $\mathrm{char}(q) = \{\infty_1, \infty_2\}$.

Since the table T_{ρ, ρ_t} represents the relation ρ (and its subrelation ρ_t) it is simple to turn to this table now. In our case, in the 7-th row for $\mathrm{char}(p) = \{\infty_1, \infty_2\}$ (denoted by **12**) and (coincidentally) in the 7-th column for $\mathrm{char}(q) = \{\infty_1, \infty_2\}$ (**12**), we find the value $'+'$ of ρ (of course, in the diagonal of T_{ρ, ρ_t}) and thus, the given equation is solvable.

When considering an equation $f(p(x))) = q(f(x))$ from the point of view of its solvability using $\mathrm{char}(p)$ and $\mathrm{char}(q)$, the solvability of the equation $f(q(x)) = p(f(x))$ (with the given functions interchanged) can be decided simultaneously.

In our case, the situation is trivial because $\mathrm{char}(q) = \mathrm{char}(p)$, i.e. $(\mathrm{char}(q), \mathrm{char}(p)) \in \rho$ too and so the equation $f(-\sqrt{x+1}) = -\sqrt{f(x)} + 1$ is solvable.

(b) Now, we investigate the solvability of the equation

$$f\left(\sqrt{\frac{x}{1-x}}\right) = -\sqrt{f(x) + 1}.$$

The function $p(x) = \sqrt{\frac{x}{1-x}}$ is strictly increasing and its characteristic $\mathrm{char}(p) = \{\omega^-, \infty_2\}$ can be found in [8], Example 17 (d). The function $q(x) = -\sqrt{x+1}$ was considered just now in (a). Its characteristic is $\mathrm{char}(q) = \{\infty_1, \infty_2\}$

Therefore for $\mathrm{char}(p)$, we look at the 6-th row and, for $\mathrm{char}(q)$, at the 7-th column of the table T_{ρ, ρ_t} and find the value $'+'$. It means that $(\mathrm{char}(p), \mathrm{char}(q)) \in \rho$ and so the considered equation is solvable.

The equation $f(-\sqrt{x+1}) = \sqrt{\frac{f(x)}{1-f(x)}}$ with the given functions interchanged has the ordered pair of functions $(q(x), p(x))$ now. Hence, we look for $q(x)$ in the 7-th row and for $p(x)$ in the 6-th column of the table T_{ρ, ρ_t} and find the value $'(+)'$. Thus, $(\mathrm{char}(q), \mathrm{char}(p)) \in \rho_t$ holds which means that this equation is solvable, but only trivially.

(c) None of the following equations is non trivially solvable.

$$f(\pm \ln x) = \pm \arctan f(x), f(\pm \arctan x) = \pm \ln f(x)$$
$$f(\ln x) = e^{f(x)}, f(e^x) = \ln f(x)$$

Indeed, this is the case because $\mathrm{char}(\ln x) = \{\omega^-\}$ (by [8], Example 17 (c)), $\mathrm{char}(-\ln x) = \{\omega^-, \infty_2\}$ (by Example 15 here), $\mathrm{char}(\arctan x) = \mathrm{char}$

$(-\arctan x) = \{\omega, \infty_2\}$ (by [7], Example 19 and Example 15 here) and char $(e^x) = \{\omega\}$ (by [7], Example 19).

And further, the values of relation ρ for the corresponding ordered pairs of these characteristics can be found in the table T_{ρ, ρ_t}.

Moreover, we see that all equations in the first row are solvable trivially except $f(\pm \arctan x) = \ln f(x)$.

Let p, q be given strictly monotonous continuous functions, such that $\operatorname{char}(p), \operatorname{char}(q) \in \{\{\omega\}, \{\omega^-\}, \{\infty_1\}\} \cup \Theta^*$ holds. In this case $I_p \neq \emptyset$ and $I_q \neq \emptyset$ hold. Then, using the table T_{ρ, ρ_t}, we see that we can decide whether $(\operatorname{char}(p), \operatorname{char}(q)) \in \rho$ or $(\operatorname{char}(p), \operatorname{char}(q)) \notin \rho$ is satisfied.

But the relation for $\operatorname{char}(p)$ and $\operatorname{char}(q)$ can be decided easily in two other cases without the table as well. Namely, by means of Lemma 6 and Corollary 11. The two cases are the following.

Corollary 17. *Let p, q be strictly monotonous continuous functions. Then the following hold.*
  (a) *If $I_p = \emptyset$ and $I_q \neq \emptyset$, then $(\operatorname{char}(p), \operatorname{char}(q)) \in \rho$ is satisfied.*
  (b) *If $I_p \neq \emptyset$ and $I_q = \emptyset$, then $(\operatorname{char}(p), \operatorname{char}(q)) \notin \rho$ is satisfied.*

Indeed, both assertions are consequences of the definition of relation ρ (cf. Fig. 2) and the assertions in Lemma 6 (a) and Corollary 11 (b) and (c).

Recall the denotation $I_p^{()} = I_p \cap (a, b)$ for any function p defined on (a, b). By the definition of I_p, if p is strictly decreasing, then $I_p^{()} = I_p$ holds. For a strictly monotonous continuous function p, the condition $I_p^{()} \neq \emptyset$ is equivalent to the existence of a fixed point of p. The solvability of Eq. (1) where both given functions p, q are without any fixed point, (i.e. if they have components of finite type only) is not investigated further.

Now, we can summarize our results in a quite short algorithm.

It would be easy to formulate an algorithm for the computations of this section in a pseudo programming language. But we refrain from writing it in detail and describe it in the form of some steps. This will have two parts.

In the first part, given two strictly monotonous continuous functions p, q, we realize the first considerations using Corollary 17. We consider the four different cases for computations depending on whether $I_p = \emptyset$ or not and $I_q = \emptyset$ or not. And in the second part, we process the computations in the (most common) case $I_p \neq \emptyset$ and $I_q \neq \emptyset$.

Algorithm 18. *Solvability of $f(p(x)) = q(f(x))$.*
  Input: strictly monotonous continuous functions p, q.
  Output: $f(p(x)) = q(f(x))$ is 'not solvable'/'solvable trivially'/'solvable'.
  The first part.
  1. (To decide whether $I_p = \emptyset$ or not:)
     denote $(a, b) = \operatorname{dom}(p)$,
     decide the solvability of $p(x) = x$ and

if p is increasing and $I_p^{()} = \emptyset$,
then determine whether $p(a) = a$ or $p(b) = b$ or not.
2. (To decide whether $I_q = \emptyset$ or not:)
   denote $(c, d) = \mathrm{dom}(q)$ and
   do the same for q as for p in step 1.
   (In programming, use the same sub-procedure.)
3 . Write the *answer*

$$answer = \begin{cases} 'solvable/solvable\ trivially' & if\ \ I_p = \emptyset\ and\ I_q \neq \emptyset \\ 'not\ solvable' & if\ \ I_p \neq \emptyset\ and\ I_q = \emptyset \\ 'use\ the\ second\ part' & if\ \ I_p \neq \emptyset\ and\ I_q \neq \emptyset \\ 'use\ a\ paperandpencil\ method' & if\ \ I_p = \emptyset\ and\ I_q = \emptyset. \end{cases}$$

The second part.
1. (For the computation of char(p), do:)
1.1. If p is strictly decreasing, then
   compute p^2 with the computation of $\mathrm{dom}(p^2)$;
   further, put

$$r = \begin{cases} p & if\ p\ is\ increasing \\ p^2 & if\ p\ is\ decreasing \end{cases}$$

and denote

$$(a, b) = \begin{cases} \mathrm{dom}(p) & if\ p\ is\ increasing \\ \mathrm{dom}(p^2) & if\ p\ is\ decreasing. \end{cases}$$

1.2. (The function r defined on (a, b) is strictly increasing.)
   Compute $r(a), r(b)$ and
   compare $(r(a), r(b))$ and (a, b) for their relationship $=, \subset, \supset$ or $||$.
   Further, find solutions of $r(x) = x$ until two, if they exist and
   check the cardinalities $|I_r^{()}|$ and $|I_r| = |I_r^{()}| + |I_r \cap \{a, b\}|$ for 0, 1 or > 1.

1.3. Find the corresponding characteristic char(r) in the list in Theorem 14.
   Put char(p) = char(r).

2. (For the computation of char(q), do:)
   In steps 2.1 to 2.3 do the same for q as in steps 1.1 to 1.3 for p.
   (By programming, use the same sub-procedure.)
3. Find the value v for the pair (char(p), char(q)) in the table T_{ρ, ρ_t}.
4. Write the *answer*

$$answer = \begin{cases} 'not\ solvable' & if\ \ v = - \\ 'solvable\ trivially' & if\ \ v = (+) \\ 'solvable' & if\ \ v = + \ . \end{cases}$$

Example 19. Consider Algorithm 18 in more detail.

(a) Let us leave the equation $f(-\sqrt{x+1}) = \ln(f(x)+1)$ to a computer which contains a program with Algorithm 18.

As *input*, it gets the functions $p(x) = -\sqrt{x+1}$ and $q(x) = \ln(x+1)$.

Then in the *first part* of the algorithm,

in 1., it decides that $p(x) = x$ is solvable, and since p is not increasing nothing more. The result is $I_p \neq \emptyset$. Then

in 2., it does the same for q and the result is $I_q \neq \emptyset$;

in 3. the *answer* is 'use the second part' and the run goes to the second part of the algorithm.

In the *second part* of the algorithm,

in 1., i.e. in the *1. block* with the following steps,

in 1.1., it recognizes that p is strictly decreasing and thus, it applies its computation on p^2; here it finds a, b such that $(a, b) = \mathrm{dom}(p^2)$,

in 1.2., it computes $p^2(a), p^2(b)$ and compares $(p^2(a), p^2(b))$ and (a, b) with respect to their relation $=, \subset, \supset$ or $||$; the result is '$=$' (cf. Example 16 (a)); further here, it determines that $|I_{p^2}| > 1$, and

in 1.3., it finds that the corresponding characteristic value for p in the list of Theorem 14 is (∞_1, ∞_2), and

in 2., i.e. in the *2. block* with the following steps,

in 2.1., it recognizes that q is strictly increasing and thus, it applies its computation on q directly; here it finds a, b such that $(a, b) = \mathrm{dom}(q)$,

in 2.2., it computes $q(a), q(b)$ and compares $(q(a), q(b))$ and (a, b) with respect to their relation $=, \subset, \supset$ or $||$; the result is '$\supset$'; further here, it determines that $|I_{q^2}| > 1$ (cf. [8], Example 17 (e)),

in 2.3., it finds that the corresponding characteristic value for q in the list of Theorem 14 is $(\omega^-, \infty_1, \infty_2)$, and, in the last steps,

in 3., 4,, it finds in T_{ρ, ρ_t} (r. 7, c. 10) the value '$+$' and writes 'solvable'.

(b) In (a), we have seen an example of strictly monotonous continuous functions p, q for which $I_p \neq \emptyset$ and $I_q \neq \emptyset$ is satisfied. The two possibilities $I_p \neq \emptyset$ and $I_q = \emptyset$ or $I_p \neq \emptyset$ and $I_q = \emptyset$ are simple because the output-answer already appears in the first part of the algorithm.

What remains is the case $I_p = \emptyset$ and $I_q = \emptyset$ which refers to 'a paper and pencil method'. This is because an investigation of computations of $\mathrm{char}(p)$ for a function p with $I_p = \emptyset$ was still left out. By Corollaries 6 and 11, we have that $\mathrm{char}(p) \subseteq \mathbf{N}_{2+}$ holds. Since the set of strictly monotonous continuous functions with this property is pretty limited, let us help ourselves in the way shown in the following example.

As can be seen soon, it can only concern strictly increasing functions. So let us take for example $p(x) = -e^{-\sqrt{x}}$ and $q(x) = \arctan(\ln(x))$.

It can be seen as well, all iterations (components) of this kind of functions form strictly decreasing sequences. So we can consider their lengths by starting

with ever larger numbers. Here, we determine $\text{char}(\arctan(\ln(x))) = \{2, 3, 4\}$. It is even easier for $\text{char}(-e^{-\sqrt{x}}) = \{2\}$.

Thus, by relation ρ, the equation

$$f(-e^{-\sqrt{x}}) = \arctan(\ln(f(x)))$$

is solvable and the other one, $f(\arctan(\ln(x))) = -e^{-\sqrt{f(x)}}$, is not.

In future considerations, it is obvious to focus on solutions of the equation $f(p(x) = q(f(x))$ for given p, q if they exist - and their constructions, which is pretty tedious (s. [7]). Because usually there are uncountably many of them, it is advantageous to limit these to given p, q with special properties or boundary value conditions. An other natural topic, would be to continue the investigations of the solvability of $f(p(x) = q(f(x))$ for given continuous p, q which are piecewise monotonous or then mainly, if they are polynomials or rational functions.

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