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Aequationes Mathematicae

Regular solutions of a functional equation derived from the invariance problem of Matkowski means

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Abstract. The main result of the present paper is about the solutions of the functional equation

$$F\Big(\frac{x+y}{2}\Big) + f_1(x) + f_2(y) = G(g_1(x) + g_2(y)), \quad x, y \in I,$$

derived originally, in a natural way, from the invariance problem of generalized weighted quasi-arithmetic means, where $F, f_1, f_2, g_1, g_2 : I \to \mathbb{R}$ and $G : g_1(I) + g_2(I) \to \mathbb{R}$ are the unknown functions assumed to be continuously differentiable with $0 \notin g_1'(I) \cup g_2'(I)$, and the set I stands for a nonempty open subinterval of $\mathbb{R}$. In addition to these, we will also touch upon solutions not necessarily regular. More precisely, we are going to solve the above equation assuming first that F is affine on I and g_1 and g_2 are continuous functions strictly monotone in the same sense, and secondly that g_1 and g_2 are invertible affine functions with a common additive part.

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1. Introduction

Before we define the key concepts and formulate the main problem, we will introduce some notions and conventions that will be indispensable later. In our main equation and the many related results, we will have functions, constants, etc. that are distinguished only by numbering. For brevity, whenever the index

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k appears as a subscript of a function, a constant, etc., it should be understood that the statement or condition in question is fulfilled for any index k, where k is running on the two-element set $\{1, 2\}$.

Let $J \subseteq \mathbb{R}$ be a non-empty open subinterval. A function $f : J \to \mathbb{R}$ will be called *affine* on some subinterval $U \subseteq J$ if it fulfills *Jensen's Equation* on U, that is, if

$$f\Big(\frac{u+v}{2}\Big) = \frac{f(u)+f(v)}{2}, \quad u, v \in U.$$

We shall say that f is *locally affine on* J or *locally constant on* J if f is affine or constant on some subinterval $U \subseteq J$ of positive length different from J. If f is neither affine (resp. constant) nor locally affine (resp. locally constant) on J, then it will be called *nowhere affine (resp. nowhere constant) on* J.

We say that a two-place function $M : J \times J \to \mathbb{R}$ is *a two-variable mean on* J or, shortly, *a mean* if

$$\min(u, v) \leq M(u, v) \leq \max(u, v), \quad u, v \in J.$$

Elementary examples of means are the *harmonic mean* and the *geometric mean* over the set of positive reals, and the *arithmetic mean*. Rather than giving furter examples, we immediately introduce a general family of two-variable means, which covers the previous examples and which, incidentally, is also the object of our investigation.

A mean $M : J \times J \to \mathbb{R}$ will be called a *generalized weighted quasi-arithmetic mean* if one can find continuous functions $f, g : J \to \mathbb{R}$ strictly monotone in the same sense such that

$$M(u, v) = (f + g)^{-1}(f(u) + g(v)), \quad u, v \in J, \tag{1.1}$$

where the functions f and g are called the *generators* of the mean in question. If the generators differ on J in an additive constant, then we come to the notion of *quasi-arithmetic means*. To see that our previous examples are indeed of such a type, set $f{:=}g{:=}\mathrm{id}^{-1}$, $f{:=}g{:=}\ln$, and $f{:=}g{:=}\mathrm{id}$, respectively. (Obviously, the exponent -1 here stands for the multiplicative inverse.)

The class of generalized weighted quasi-arithmetic means was introduced in 2010 by Matkowski [27]. Motivated by this, means of the form (1.1) are often referred to as *Matkowski means* and, to indicate the generators too, denoted by $\mathcal{M}_{f,g}$. Note that, by the definition, any Matkowski mean is continuous and strictly monotone, furthermore is symmetric or balanced if and only if it is a quasi-arithmetic mean [19,27].

A mean $M : J \times J \to \mathbb{R}$ will be called *invariant with respect to the pair of means* $(N, K) : J \times J \to \mathbb{R}^2$ if the *invariance equation*

$$M(N(u, v), K(u, v)) = M(u, v), \quad u, v \in J$$

is fulfilled.

The most frequently cited (and an easy-to-check) example of this phenomenon in the literature is the invariance of the geometric mean with respect to the pair of arithmetic and harmonic means. A less trivial example is the arithmetic-geometric mean appearing already in Lagrange's and Gauss's works, which is invariant with respect to the pair of arithmetic and geometric means.

As more and more families of means have been introduced, over the last 20 years, the invariance problem has again become an area of active research. Without being exhaustive, we list some related papers. Interested readers can find a more detailed discussion of the invariance equation in [11, Daróczy–Páles] and in the survey [16, Jarczyk–Jarczyk].

The invariance of the arithmetic mean with respect to a pair of two quasi-arithmetic means under two times continuous differentiability of the generators was solved by Matkowski in 1999 [24]. As this regularity condition is not natural for the underlying problem, it was gradually weakened in the following years by Darôzy, Maksa, and Páles in the papers [8] and [10]. The final answer using only the necessary conditions was given in 2002 by Daróczy and Páles [11].

The invariance equation for three quasi-arithmetic means was investigated by Burai [5,6], by Jarczyk and Matkowski [17], and by Jarczyk [15], where, in the last paper, the unnecessary regularity conditions were eliminated too.

A paper closely related to the present investigation is [3], where, under four times continuous differentiability of the generators, the authors solved the problem of invariance of the arithmetic mean with respect to a pair of Matkowski means.

More studies on the invariance problem concerning different classes of means can be found in [2, Baják–Páles], [25,26,28,29, Matkowski], [7, Błasińska–Głazowska–Matkowski], [13, Głazowska–Jarczyk–Matkowski], [12, Domsta–Matkowski], and [31, Pàles–Zakaria].

2. The invariance equation and its reformulation

To make our underlying problem more manageable, relying only on continuity and strict monotonicity of the generators, we are going to give an equivalent formulation of the invariance equation of Matkowski means. This new equation will concern the composition of the initial generators and, like the original equation, will still contain six unknown functions. Then, under differentiability and a technical condition, we derive a system of two functional equations, in which the individual equations contain only 3–3 unknown functions, such that one of these will be common.

The following result states the relation between the invariance problem and our main equation mentioned in the abstract. Note that the domain of the latter is not necessarily a rectangle symmetric with respect to the diagonal.

Theorem 1. *Let* $(m_1, m_2), (n_1, n_2), (k_1, k_2) : J \to \mathbb{R}^2$ *be such that the coordinate-functions of each ordered pair are continuous and strictly monotone in the same sense. Then the invariance equation of Matkowski means*

$$\mathcal{M}_{m_1,m_2}\big(\mathcal{M}_{n_1,n_2}(u,v), \mathcal{M}_{k_1,k_2}(u,v)\big) = \mathcal{M}_{m_1,m_2}(u,v), \quad u,v \in J \tag{2.1}$$

holds if and only if for the composite functions

$$\begin{aligned} &F{:=} - m_2 \circ \Big(\frac{k_1 + k_2}{2}\Big)^{-1}, \quad && f_1{:=}m_1 \circ k_1^{-1}, \quad && f_2{:=}m_2 \circ k_2^{-1}, \\ &G{:=}m_1 \circ (n_1 + n_2)^{-1}, && g_1{:=}n_1 \circ k_1^{-1}, && g_2{:=}n_2 \circ k_2^{-1} \end{aligned} \tag{2.2}$$

we have

$$F\Big(\frac{x+y}{2}\Big) + f_1(x) + f_2(y) = G(g_1(x) + g_2(y)) \tag{2.3}$$

$forall x \in k_1(J) and y \in k_2(J).$

Proof. Assume first that Eq. (2.1) holds on $J \times J$, and let $x \in k_1(J)$ and $y \in k_2(J)$ be arbitrary. The functions k_1 and k_2 are continuous and strictly monotone, therefore we uniquely have $u, v \in J$ with $k_1(u) = x$ and $k_2(v) = y$. Then $u = k_1^{-1}(x)$ and $v = k_2^{-1}(y)$. Applying Eq. (2.1) for the pair (u, v), expanding the Matkowski means included in it by definition, and, finally, applying the function $m_1 + m_2$ on both sides, we get

$$\begin{aligned} &m_1 \circ (n_1 + n_2)^{-1}(n_1(u) + n_2(v)) + m_2 \circ (k_1 + k_2)^{-1}(k_1(u) + k_2(v)) \\ &\quad = m_1(u) + m_2(v). \end{aligned}$$

Using the definition of u and v, an obvious reformulation yields that

$$\begin{aligned} &-m_2 \circ (k_1 + k_2)^{-1}(x + y) + m_1 \circ k_1^{-1}(x) \\ &\quad + m_2 \circ k_2^{-1}(y) = m_1 \circ (n_1 + n_2)^{-1}(n_1 \circ k_1^{-1}(x) + n_2 \circ k_2^{-1}(y)). \end{aligned}$$

Replacing $x + y$ in the argument of $\gamma_2 \circ (\beta_1 + \beta_2)$ by $2 \cdot \frac{x+y}{2}$, and applying the definition of the functions in (2.2), we obtain that (2.3) is valid.

To prove that the validity of (2.3) implies the validity of (2.1), just reverse the above argumentation. □

It is important to note that Eq. (2.3), considered on an interval of the form $]0, \alpha[$ with $\alpha > 0$, has previously appeared in the paper [18] of Járai, Maksa, and Páles under the setting $-F \circ \frac{1}{2}\,\mathrm{id} = f_1 = f_2$ and $g{:=}g_1 = g_2$. In [18] the authors solved the related equation assuming that G and g are continuous and strictly monotone.

Now we simplify our problem further. Unfortunately, to do this, in the rest of the paper we have to assume that $k_1(J) = k_2(J) =: I$ holds, where due to the properties of J, k_1, and k_2, the set I is a non-empty open subinterval of $\mathbb{R}$. Without this, by the way, though not a natural technical condition, we could not derive the following system of equations. On the other hand, let us point out that, to prove the next result, we do not yet require continuity of the derivatives.

Theorem 2. *Let* $(F, f_1, f_2, G, g_1, g_2)$ *be a solution of* (2.3) *such that each coordinate-function is differentiable on its domain with* $0 \notin g_1'(I) \cup g_2'(I)$*. Then the system of functional equations*

$$\begin{cases} \varphi\left(\frac{x+y}{2}\right)(\psi_1(x) + \psi_1(y)) = \Psi_1(x) + \Psi_1(y), \\ \varphi\left(\frac{x+y}{2}\right)(\psi_2(x) - \psi_2(y)) = \Psi_2(x) - \Psi_2(y), \end{cases} \quad x, y \in I \tag{2.4}$$

holds, where

$$\varphi := \frac{1}{2}F', \quad \psi_k := \frac{1}{g_1'} + (-1)^k \frac{1}{g_2'}, \quad \text{and} \quad \Psi_k := -\frac{f_1'}{g_1'} + (-1)^{k-1}\frac{f_2'}{g_2'}. \tag{2.5}$$

Proof. Differentiating (2.3) with respect to x and y separately, and putting φ defined in (2.5), we obtain that

$$\varphi\left(\frac{x+y}{2}\right) + f_1'(x) = G'(g_1(x) + g_2(y))g_1'(x) \quad \text{and}$$
$$\varphi\left(\frac{x+y}{2}\right) + f_2'(y) = G'(g_1(x) + g_2(y))g_2'(y)$$

hold for all $x, y \in I$, respectively. Multiplying the first and the second equation by $g_2'(y)$ and $-g_1'(x)$, respectively, and then adding up side by side the equations so obtained, we get

$$\varphi\left(\frac{x+y}{2}\right)\left(g_2'(y) - g_1'(x)\right) = f_2'(y)g_1'(x) - f_1'(x)g_2'(y), \quad (x, y) \in I \times I. \tag{2.6}$$

This equation implies that

$$\varphi\left(\frac{x+y}{2}\right)\left(g_2'(x) - g_1'(y)\right) = f_2'(x)g_1'(y) - f_1'(y)g_2'(x), \quad (x, y) \in I \times I. \tag{2.7}$$

Dividing Eqs. (2.6) and (2.7) by $g_1'(x)g_2'(y) \neq 0$ and $g_1'(y)g_2'(x) \neq 0$, respectively, we obtain

$$\varphi\left(\frac{x+y}{2}\right)\left(\frac{1}{g_1'(x)} - \frac{1}{g_2'(y)}\right) = \frac{f_2'(y)}{g_2'(y)} - \frac{f_1'(x)}{g_1'(x)} \quad \text{and}$$
$$\varphi\left(\frac{x+y}{2}\right)\left(\frac{1}{g_1'(y)} - \frac{1}{g_2'(x)}\right) = \frac{f_2'(x)}{g_2'(x)} - \frac{f_1'(y)}{g_1'(y)}$$

for all $x, y \in I$. Taking the sum of the above equations and subtracting the second equation from the first, and then applying definition (2.5), we get the first and the second equation of (2.4), respectively. □

Remark 3. Note that $\psi_2 + \psi_1 = \frac{2}{g_1'} \neq 0$ and $\psi_2 - \psi_1 = \frac{2}{g_2'} \neq 0$ hold on I. Thus, by (2.5) of Theorem 2, we have

$$F' = 2\varphi, \quad g_j' = \frac{2}{\psi_2 + (-1)^{j-1}\psi_1}, \quad \text{and} \quad f_j' = -\frac{\Psi_2 + (-1)^{j-1}\Psi_1}{\psi_2 + (-1)^{j-1}\psi_1} \quad \text{on } I.$$

The individual functional equations of system (2.4) has a rich literature. In the rest of this section we recall the related results.

The upper equation of (2.4) Actually, this equation contains only two unknown functions, so, by substituting $x = y$, we immediately obtain that Ψ_1 must be of the form $\varphi\psi_1$. Thus it can be reformulated as

$$\varphi\Big(\frac{x+y}{2}\Big)(\psi_1(x)+\psi_1(y)) = \varphi(x)\psi_1(x)+\varphi(y)\psi_1(y), \quad x, y \in I. \tag{2.8}$$

A functional equation of a similar form was studied by Lundberg in [23]. The functions involved were *complex valued* and, in essence, *infinitely many times differentiable*. Five years later, in order to solve the equality problem of two-variable functionally weighted quasi-arithmetic means (or shortly, Bajraktarević means) and quasi-arithmetic means, Daróczy, Maksa, and Páles [9] solved the above equation. Moreover, they determined the solutions under natural conditions needed to formulate their initial problem, to be more precise, φ was supposed to be *continuous* and *strictly monotone* and ψ_1 was *positive* on its domain.

The best result known today concerning the above equation was obtained in [20]. In [20] Eq. (2.8) is totally solved under the *continuity* of φ and assuming a *regularity property of the zeros* of the function ψ_1. More specifically, the validity of the inclusion $I \setminus \mathcal{Z}_{\psi_1} \subseteq \mathrm{cl}_I \circ \mathrm{conv}\big(I \setminus \mathrm{cl}_I\, \mathcal{Z}_{\psi_1}\big)$ is required, where $\mathcal{Z}_{\psi_1}$ denotes the zeros of ψ_1 in I and the operators conv and cl_I stand for the convex hull and the closure with respect to I, respectively. At first glance, this inculsion condition may seem artificial or technical, but notice that it is trivially satisfied provided that ψ_1 is continuous or injective. Hence, instead of the above general inclusion condition, we are going to formulate the corresponding theorem of [20] under the continuity of ψ_1. For our purposes, this will be enough.

Theorem 4. *Let $\varphi, \psi_1 : I \to \mathbb{R}$ be continuous functions. Then (φ, ψ_1) solves Eq.* (2.8) *if and only if either*

(4.1) *there is an interval $L \subseteq I$ such that $\psi_1 = 0$ on $I \setminus L$ and φ is constant on $\frac{1}{2}(L + I)$, or*

(4.2) *there exist constants $a, b, c, d, \gamma \in \mathbb{R}$ with $ad \neq bc$ such that exactly one of the conditions*

(a) $\gamma < 0$ *and*

$$\varphi(x) = \frac{c\sin(\kappa x)+d\cos(\kappa x)}{a\sin(\kappa x)+b\cos(\kappa x)} \quad \text{and} \quad \psi_1(x) = a\sin(\kappa x)+b\cos(\kappa x) \neq 0, \quad \text{or}$$

(b) $\gamma = 0$ *and*

$$\varphi(x) = \frac{cx+d}{ax+b} \quad \text{and} \quad \psi_1(x) = ax+b \neq 0, \quad \text{or}$$

(c) $\gamma > 0$ *and*

$$\varphi(x) = \frac{c\sinh(\kappa x)+d\cosh(\kappa x)}{a\sinh(\kappa x)+b\cosh(\kappa x)} \quad \text{and} \quad \psi_1(x) = a\sinh(\kappa x)+b\cosh(\kappa x) \neq 0$$

holds for all $x \in I$, where $\kappa := \sqrt{|\gamma|}$.

The lower equation of (2.4) The first remarkable investigation concerning this equation is due to Aczél [1]. In 1963, he solved the equation under the assumption $\psi_2 = \mathrm{id}$. Keeping this condition, and even assuming that the right-hand side is the difference of two not necessarily equal functions, in 1979, Haruki [14] obtained the same result as Aczél.

In the early 2000s, the equation reappeared in the paper [11] of Daróczy and Páles,where, motivated by the underlying problem, the authors studied it assuming that $\Psi_2 = \varphi\psi_2$.

Later, in 2016, by Balogh, Ibrogimov, and Mityagin [4], and then, in 2018, by Łukasik [22], literally

$$\varphi\Big(\frac{x+y}{2}\Big)\big(\psi_2(x) - \psi_2(y)\big) = \Psi_2(x) - \Psi_2(y), \quad (x, y \in I) \tag{2.9}$$

was considered under the assumption that ψ_2 and Ψ_2 are three-times differentiable and continuously differentiable functions, respectively.

The result requiring the weakest regularity conditions known today can be found in [21], where the same solution was obtained as in [4,22] but assuming only that φ is continuous. To make this result easier to formulate, we introduce the following notation. For a subset $H \subseteq I$, define

$$H_-(I) := \{x \in I \mid x < \inf H\} \quad \text{and} \quad H_+(I) := \{x \in I \mid \sup H < x\}.$$

Obviously, if H is empty, then $H_-(I) = H_+(I) = I$, furthermore, we have $\inf H = \inf I$ or $\sup H = \sup I$, if and only if $H_-(I) = \emptyset$ or $H_+(I) = \emptyset$, respectively.

Theorem 5. *Let $\varphi, \psi_2, \Psi_2 : I \to \mathbb{R}$ be such that φ is continuous. Then $(\varphi, \psi_2, \Psi_2)$ solves Eq.* (2.9) *if and only if either*

(5.1) *there exist $A^* \in \varphi(I)$, a closed interval $K \subseteq I$, and $\mu^* \in \mathbb{R}$ such that $\Psi_2 = A^*\psi_2 + \mu^*$ on I, the function ψ_2 is constant on $K_-(I)$ and $K_+(I)$, and $\varphi = A^*$ on the interval $\frac{1}{2}(K+I)$, or*

(5.2) *there exist constants $a^*, b^*, c^*, d^*, \gamma^* \in \mathbb{R}$ with $a^*d^* \neq b^*c^*$ and $\mu^*, \lambda^* \in \mathbb{R}$ such that exactly one of the conditions*

(a) $\gamma^* < 0$ *and*

$$\varphi(x) = \frac{c^*\sin(\kappa^* x) + d^*\cos(\kappa^* x)}{a^*\sin(\kappa^* x) + b^*\cos(\kappa^* x)} \quad \textit{and}$$

$$\begin{aligned} \psi_2(x) &= -a^*\cos(\kappa^* x) + b^*\sin(\kappa^* x) + \lambda^*, \\ \Psi_2(x) &= -c^*\cos(\kappa^* x) + d^*\sin(\kappa^* x) + \mu^*, \end{aligned} \quad \textit{or}$$

(b) $\gamma^* = 0$ *and*

$$\varphi(x) = \frac{c^*x + d^*}{a^*x + b^*} \quad \textit{and} \quad \begin{aligned} \psi_2(x) &= \tfrac{1}{2}a^*x^2 + b^*x + \lambda^*, \\ \Psi_2(x) &= \tfrac{1}{2}c^*x^2 + d^*x + \mu^*, \end{aligned} \quad \textit{or}$$

(c) $\gamma^* > 0$ *and*

$$\varphi(x) = \frac{c^* \sinh(\kappa^* x) + d^* \cosh(\kappa^* x)}{a^* \sinh(\kappa^* x) + b^* \cosh(\kappa^* x)} \quad \text{and}$$
$$\psi_2(x) = a^* \cosh(\kappa^* x) + b^* \sinh(\kappa^* x) + \lambda^*,$$
$$\Psi_2(x) = c^* \cosh(\kappa^* x) + d^* \sinh(\kappa^* x) + \mu^*$$

holds for all $x \in I$, *where* $\kappa^* := \sqrt{|\gamma^*|}$.

The above results will be used to determine solutions that can be relevant to the invariance problem. But first, to handle the cases that arise, we need to discuss solutions that include affine functions.

3. Solutions with affine member

For the functions $g_1, g_2 : I \to \mathbb{R}$, $U \subseteq I$, and $x \in I$, define the sets

$$U_1(x) := \{y \in U \mid g_1(x) + g_2(y) \in g_1(U) + g_2(U)\} \quad \text{and}$$
$$U_2(x) := \{y \in U \mid g_1(y) + g_2(x) \in g_1(U) + g_2(U)\}.$$

Then, obviously, $U \subseteq U_1(x) = U_2(x)$ whenever $x \in U$.

Lemma 6. *If* $g_1, g_2 : I \to \mathbb{R}$ *are continuous functions strictly monotone in the same sense and* $U :=]a, b[\subseteq I$ *for some* $a < b$, *then the following statements hold.*

1. *If* $a \in I$ (*resp.* $b \in I$), *then* $U \subseteq U_k(a)$ (*resp.* $U \subseteq U_k(b)$).
2. *If* $a \in I$ (*resp.* $b \in I$), *then there exists* $x \in I$ *with* $x < a$ (*resp. with* $b < x$) *such that* $U_k(x) \neq \emptyset$.
3. *If* $x \in I$ *with* $x < a$ (*resp.* $b < x$) *and* $U_k(x) \neq \emptyset$, *then, for all* $u \in [x, a]$ (*resp. for all* $u \in [b, x]$), *we have* $U_k(x) \subseteq U_k(u)$.
4. *For all* $x \in I$ *with* $x < a$ (*resp. with* $b < x$), *we have* $a < \inf U_k(x)$ (*resp.* $\sup U_k(x) < b$). *Furthermore, if* $U_k(x) \neq \emptyset$, *then* $\sup U_k(x) = b$ (*resp.* $a = \inf U_k(x)$).

Proof. For brevity, let $H := g_1(U) + g_2(U)$. We perform the proof under the assumption $a \in I$ and for the index $k = 1$. The proof of the other sub-cases can be done analogously. For consistency, in some steps, the strictly increasing and strictly decreasing cases will be treated in parallel.

Due to the facts that U is a non-empty open subinterval of I, and g_1 and g_2 are continuous, strictly monotone, $g_1(a) + g_2(U)$ and H are non-empty open intervals as well. On the other hand, if g_1 and g_2 are strictly increasing or strictly decreasing, then

$$\begin{aligned}&\inf g_1(a)+g_2(U)=\inf H=g_1(a)+g_2(a)\in\mathbb{R}\quad\text{and}\\&\sup g_1(a)+g_2(U)\le\sup H,\quad\text{or}\\&\sup g_1(a)+g_2(U)=\sup H=g_1(a)+g_2(a)\in\mathbb{R}\quad\text{and}\\&\quad\inf g_1(a)+g_2(U)\ge\inf H,\end{aligned}$$

respectively. These show that (1) holds.

To show (2), indirectly assume that, for all $x\in I$ with $x<a$, the sum $g_1(x)+g_2(y)$ is not contained in H for all $y\in U$. The two-place function $(x,y)\mapsto g_1(x)+g_2(y)$ is continuous on $]\inf I,a[\times U$, hence the image of this product by the function in question is a subinterval of $\mathbb{R}$. Moreover, by the strict monotonicity of g_1 and g_2, the image is of positive length. Therefore, in view of our indirect assumption, it follows that we have either $g_1(x)+g_2(y)\le\inf H$ or $\sup H\le g_1(x)+g_2(y)$ for all $x\in]\inf I,a[$ and for all $y\in U$.

If g_1 and g_2 are strictly increasing and *the first inequality holds*, then, taking $x\to a^-$ and using the continuity of g_1, we get that $g_2(y)\le g_2(a)$ holds for all $y\in U$. This contradicts the fact that g_2 is strictly increasing. *If the second inequality is valid*, then necessarily $\sup H\in\mathbb{R}$, and, particularly, for all $y\in U$ and for all $x\in]\inf I,a[$, we have $g_1(y)+g_2(y)\le\sup H\le g_1(x)+g_2(y)$. This reduces to $g_1(y)\le g_1(x)$, which is a contradiction again.

The proof of (2) for the strictly decreasing case can be easily constructed by modifying the preceding reasoning.

To prove (3), let $x\in I$ be such that $x<a$ and $U_1(x)\ne\emptyset$. For any $y\in U_1(x)$, the function $\xi\mapsto g_1(\xi)+g_2(y)$ is continuous on $[x,a]$. Then, by the choice of y, we have $g_1(x)+g_2(y)\in H$ and, by statement (1), $g_1(a)+g_2(y)\in H$. Hence, due to the Darboux Property of $\xi\mapsto g_1(\xi)+g_2(y)$, (3) follows.

To prove the first statement of (4), take $x\in I$ with $x<a$. If $U_1(x)=\emptyset$, then there is nothing to prove, hence we may assume that $U_1(x)\ne\emptyset$. Furthermore, assume indirectly that $\inf U_1(x)=a$ holds. Then, for some $r_0>0$, we have $a+r\in U_1(x)$ for all $0<r<r_0$. Hence $g_1(x)+g_2(a+r)\in H$. Consequently, if g_1 and g_2 are strictly increasing or strictly decreasing, by continuity, it follows that $g_1(a)\le g_1(x)$ or $g_1(x)\le g_1(a)$, respectively, contradicting that $x<a$.

The second statement of (4) is obvious. □

Now, keeping the previous notation, let

$$U^+:=\{x\in I\mid U_1(x)\cap U_2(x)\ne\emptyset\}.$$

By $U\subseteq U^+$, the set U^+ is non-empty. Furthermore, by (2) and (3) of Lemma 6, this inclusion is strict whenever $U\ne I$, and U^+ is a subinterval of I. Thus, roughly speaking, U^+ is a proper continuation of the interval U in I provided that $U\ne I$.

Proposition 7. *Let $F : I \to \mathbb{R}$ be a function affine on some non-empty open subinterval $U \subseteq I$, and $g_1, g_2 : I \to \mathbb{R}$ be continuous functions strictly monotone in the same sense. If $(F, f_1, f_2, G, g_1, g_2)$ solves functional Eq. (2.3) then there exist an additive function $B : \mathbb{R} \to \mathbb{R}$ and constants $\lambda_1, \lambda_2 \in \mathbb{R}$, such that*

$$f_k = -\frac{1}{2}F + B \circ g_k + \lambda_k \quad \textit{and} \quad G = B + \Lambda \tag{3.1}$$

hold on $U^+ \cap (2U - U) \cap I$ and on $g_1(U) + g_2(U)$, respectively, where $\Lambda{:=}\lambda_1 + \lambda_2$.

Proof. First, we are going to show that the formulas concerning the functions f_k and G in (3.1) hold on U and on $g_1(U) + g_2(U)$, respectively.

Define $\ell_k{:=}F + 2f_k$ on U. Expressing f_k from here, substituting it back into (2.3), and using the fact that F is affine on U, we obtain that

$$G\big(g_1(x) + g_2(y)\big) = \frac{\ell_1(x) + \ell_2(y)}{2}, \quad x, y \in U.$$

Putting $u{:=}g_1(x)$ and $v{:=}g_2(y)$, and using that g_1 and g_2 are continuous and strictly monotone, it follows that

$$2G(u + v) = \ell_1 \circ g^{-1}(u) + \ell_2 \circ g_2^{-1}(v), \quad (u, v) \in g_1(U) \times g_2(U).$$

Then, in view of the celebrated Theorem 1 of [30, Radó–Baker], there exist $B^* : \mathbb{R} \to \mathbb{R}$ additive and $\lambda_1^*, \lambda_2^* \in \mathbb{R}$ such that

$$\ell_k = B^* \circ g_k + \lambda_k^* \quad \text{and} \quad G = \frac{1}{2}B^* + \frac{1}{2}(\lambda_1^* + \lambda_2^*)$$

hold on U and on $g_1(U) + g_2(U)$, respectively. Define $B{:=}\frac{1}{2}B^*$, $\lambda_k{:=}\frac{1}{2}\lambda_k^*$, and let $\Lambda{:=}\lambda_1 + \lambda_2$. Then, expressing f_k in terms of ℓ_k and F, we obtain the desired decompositions listed in (3.1).

Now we are going to extend the form of f_k to the subinterval $V{:=}U^+ \cap (2U - U) \cap I$. If $U = I$, then $U^+ = I$, hence, in this case, there is nothing to prove. Therefore assume that this is not the case and let $x \in V$ be any point. We may assume that $x \notin U$, moreover, without loss of generality, it can be assumed that $x < \inf U \in I$.

Then $U_1(x) \cap U_2(x) \neq \emptyset$ and there exists $v \in U$ such that $\frac{1}{2}(x + v) \in U$. Let $u \in U_1(x) \cap U_2(x)$ be any point and $y{:=}\max(u, v) \in U$. Then

$$\frac{x + y}{2} \in U \quad \text{and} \quad y \in U_1(x) \cap U_2(x).$$

Indeed, if $u \le v$, then the first inclusion is trivially fulfilled and the second inclusion is implied by the fact $\sup U_k(x) = \sup U$. If $v \le u$, then the second inclusion holds automatically, furthermore, since U is an interval, the validity of the first inclusion is trivial.

Applying (2.3) for (x, y) and using that F is affine on U and the definition of y, the left hand side of (2.3) can be written as

$$\begin{aligned} F\Big(\frac{x+y}{2}\Big) + f_1(x) + f_2(y) &= \frac{1}{2}\big(F(x) + F(y)\big) + f_1(x) \\ &\quad -\frac{1}{2}F(y) + B \circ g_2(y) + \lambda_2 \\ &= \frac{1}{2}F(x) + f_1(x) + B \circ g_2(y) + \lambda_2. \end{aligned}$$

Thus

$$\frac{1}{2}F(x) + f_1(x) + B \circ g_2(y) + \lambda_2 = G(g_1(x) + g_2(y)) = B(g_1(x) + g_2(y)) + \Lambda$$

follows, which, after subtracting $B \circ g_2(y)$ from both sides and using that $\Lambda - \lambda_2 = \lambda_1$, implies that

$$f_1(x) = -\frac{1}{2}F(x) + B \circ g_1(x) + \lambda_1.$$

To get a similar conclusion for f_2, apply Eq. (2.3) for the pair (y, x) and perform the same reasoning. □

Corollary 8. *Let $F : I \to \mathbb{R}$ be affine and $g_1, g_2 : I \to \mathbb{R}$ be continuous functions strictly monotone in the same sense. Then $(F, f_1, f_2, G, g_1, g_2)$ solves functional Eq. (2.3) if and only if there exist $B : \mathbb{R} \to \mathbb{R}$ additive and $\lambda_1, \lambda_2, \Lambda \in \mathbb{R}$ with $\Lambda = \lambda_1 + \lambda_2$ such that (3.1) of Proposition 7 holds with $U{:=}I$.*

Proof. The statement is a direct consequence of Proposition7. *(At this point, it is worth noting that the proof of sufficiency does not require that g_1 and g_2 be continuous or strictly monotone.)* □

The following proposition is stated only for a subinterval of I, because we want to use it later in this form. The statement concerning the whole interval I is formulated as a corollary after the proposition.

Proposition 9. *Let $U \subseteq I$ be a nonempty open subinterval, $\mu_1, \mu_2 \in \mathbb{R}$, $D : \mathbb{R} \to \mathbb{R}$ be an invertible additive function, and $g_1, g_2 : I \to \mathbb{R}$ such that $g_k{:=}D + \mu_k$ on U. If $(F, f_1, f_2, G, g_1, g_2)$ solves (2.3) then there exist an additive function $C : \mathbb{R} \to \mathbb{R}$ and constants $\lambda_1, \lambda_2 \in \mathbb{R}$, such that*

$$\begin{aligned} &f_k(x) = Cx + \lambda_k \quad \textit{and} \\ &G(u) = F \circ \frac{1}{2}D^{-1}(u - \mu) + C \circ D^{-1}(u - \mu) + \Lambda \end{aligned} \tag{3.2}$$

hold for all $x \in U$ and for all $u \in g_1(U) + g_2(U)$, where $\mu{:=}\mu_1 + \mu_2$ and $\Lambda{:=}\lambda_1 + \lambda_2$.

Proof. If $(F, f_1, f_2, G, g_1, g_2)$ solves (2.3), then

$$h(x + y) = f_1(x) + f_2(y), \quad x, y \in U$$

with $h(v){:=}G(Dv+\mu)-F(\frac{1}{2}v)$, where $v\in 2U$. Hence, by Theorem 1 of [30, Radó–Baker], it follows that there exist an additive function $C:\mathbb{R}\to\mathbb{R}$ and constants $\lambda_1,\lambda_2\in\mathbb{R}$ such that

$$f_k(x)=Cx+\lambda_k \quad \text{and} \quad h(v)=Cv+\Lambda$$

hold for all $x\in U$ and for all $v\in 2U$. Using the definition of h and putting $u{:=}Dv+\mu\in D(2U)+\mu=g_1(U)+g_2(U)$, we obtain the desired formula for G. □

Corollary 10. *Let $\mu_1,\mu_2\in\mathbb{R}$, $D:\mathbb{R}\to\mathbb{R}$ be an invertible additive function, and define $g_k{:=}D+\mu_k$ on I. Then (F,f_1,f_2,G,g_1,g_2) solves (2.3) if and only if there exist $C:\mathbb{R}\to\mathbb{R}$ additive and $\lambda_1,\lambda_2\in\mathbb{R}$ such that (3.2) of Proposition 9 holds with $U{:=}I$.*

Before we formulate and prove the main result of this section, let us recall a simple lemma, whose precise proof can be found in [20].

Lemma 11. *For any subset $H\subseteq I$, the sum $H+I$ is an open subinterval of I such that*

$$H+I=\begin{cases}\emptyset & \text{if } H=\emptyset,\\ ]\inf H+\inf I,\sup H+\sup I[& \text{otherwise.}\end{cases}$$

In the sequel, a 6-tuple of functions (F,f_1,f_2,G,g_1,g_2) will be called regular *if $F,f_k,g_k:I\to\mathbb{R}$ and $G:g_1(I)+g_2(I)\to\mathbb{R}$ are continuously differentiable with $0\notin g_1'(I)\cup g_2'(I)$.*

In the next proof, $\operatorname{diam} H$ *stands for the diameter of a set H, more precisely,* $\operatorname{diam} H{:=}0$ *if $H=\emptyset$ and* $\operatorname{diam} H{:=}\sup\{x-y\mid x,y\in H\}$ *otherwise.*

Theorem 12. *If (F,f_1,f_2,G,g_1,g_2) is a regular solution of (2.3), then F is either affine or nowhere affine on I.*

Proof. Let (F,f_1,f_2,G,g_1,g_2) be a regular solution of Eq. (2.3) and, indirectly, assume that F is locally affine on I. Then, by Theorem 2, (φ,ψ_1,Ψ_1) and (φ,ψ_2,Ψ_2) solve the first and the second functional equations of system (2.4), where the coordinate-functions in question are defined in (2.5) of Theorem 2.

By our indirect assumption, $\varphi=\frac{1}{2}F'$ is locally constant on I. Thus, by Theorem 4 and Theorem 5, it follows that there exist nonempty subintervals $L,K\subseteq I$ different from I and constants $A^*,\mu^*\in\mathbb{R}$ such that ψ_1 is identically zero on $L_-(I)\cup L_+(I)$, $\Psi_1=\varphi\psi_1$ holds on I, and φ is constant on $L'{:=}\frac{1}{2}(L+I)$, furthermore that ψ_2 is constant on $K_-(I)$ and $K_+(I)$, $\Psi_2=A^*\psi_2+\mu^*$ on I, and $\varphi=A^*$ on $K'{:=}\frac{1}{2}(K+I)$.

Then I cannot be $\mathbb{R}$, otherwise, by Lemma 11, $L'=K'=I$ follows, which contradicts our indirect assumption. If I is unbounded, then, regardless of the exact position of L relative to K, we have $L'\subseteq K'$ or $K'\subseteq L'$. Finally, if I is bounded, then $L'\cap K'$ is an interval of positive length. To see this, it is enough to observe that, by Lemma 11, $\min\{\operatorname{diam} L', \operatorname{diam} K'\} > \frac{1}{2}\operatorname{diam} I$ follows. Consequently, there exists a maximal closed subinterval $U \neq I$ of I such that $L' \cup K' \subseteq U$ and φ is constant on U. We may and do assume that $\alpha := \inf I < \min U$ holds. (The proof for the complementary case is analogous.) Then, obviously, we have the chain of inequalities

$$-\infty < \alpha < \inf U < \min\{\inf L, \inf K\} =: \beta.$$

Define $W :=]\alpha, \beta[$. Then, on the one hand, we have $W \subseteq L_-(I) \cap K_-(I)$, consequently, for all $x \in W$, we get

$$\psi_1(x) = \frac{1}{g_1'(x)} - \frac{1}{g_2'(x)} = 0 \quad \text{and}$$
$$\psi_2(x) = \frac{1}{g_1'(x)} + \frac{1}{g_2'(x)} = d \tag{3.3}$$

for some $d \in \mathbb{R}$. Solving this system of differential equations, by $0 \notin g_1'(I) + g_2'(I)$, we obtain that $d \neq 0$, and g_1 and g_2 are continuous affine functions on W with the common slope $\frac{2}{d}$, where condition $0 \notin g_1'(I) + g_2'(I)$ provides that $d \neq 0$. Thus the conditions of Proposition 9 are fulfilled over the interval W, hence f_1 and f_2 are affine on W as well.

On the other hand, in view of Proposition 7, one can find $\varepsilon > 0$ with $\inf U - \varepsilon > \alpha$ such that, for all points $x \in]\inf U - \varepsilon, \beta[$, we have

$$F(x) = 2A \circ g_k(x) - 2f_k(x) + 2\lambda_k$$

for some additive function $A : \mathbb{R} \to \mathbb{R}$ and for some constant $\lambda_k \in \mathbb{R}$. The function G is continuously differentiable, hence, in view of (3.1), A must be continuous. Thus the above formula yields that F is affine, that is, φ is constant on $]\inf U - \varepsilon, \beta[$. This contradicts the maximality of U. □

4. Regular solutions of (2.3)

In this section, we are going to determine the regular solutions of 2.3). In each cases, the proof of the sufficiency of the listed functions will be obvious as a matter of substitution. In contrast, for the necessity part, that is, to get some information about the shape of the functions in question, we are going to actively use our previous results. Let us detail the exact schedule below.

Having a regular solution $(F, f_1, f_2, G, g_1, g_2)$ of (2.3), by definition (2.5), consider the triplets of continuous functions $(\varphi, \psi_1, \Psi_1)$ and $(\varphi, \psi_2, \Psi_2)$, which, in view of Theorem 2, solve the first and the second equations of system (2.4), respectively. In view of Theorem 12, F is either affine or nowhere affine on I, which exactly means that φ is either constant or nowhere constant on I. In the case when φ is nowhere constant on I, the functions ψ_1 and ψ_2 can still be zero and constant on the interval I, respectively. Finally, the complementary case will be when φ is nowhere constant and ψ_1 or ψ_2 is nowhere zero or constant on I, respectively.

Motivated by this, having a solution $(F, f_1, f_2, G, g_1, g_2)$, for the functions defined in (2.5), we are going to distinguish the following main cases: either

(A) φ is constant on I, or

(B) φ is nowhere constant on I and either

- (B.1) (4.1) of Theorem 4 with $L = \emptyset$ and (5.1) of Theorem 5 with $K = \emptyset$ hold simultaneously, or
- (B.2) at least one of the cases (4.2) of Theorem 4 or (5.2) of Theorem 5 is valid.

As we shall see, solutions satisfying conditions (A) or (B.1) will contain arbitrary members too. Motivated by this behavior, such solutions will be formulated within a theorem and case (B.2) will be discussed separately.

Theorem 13. *If $(F, f_1, f_2, G, g_1, g_2)$ is a regular solution of (2.3) such that either (A) or (B.1) is met then either*

(1) *g_1 and g_2 are arbitrary functions and there exist constants $A, B \in \mathbb{R}$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$ such that*

$$F(x) = Ax + \lambda, \quad f_k(x) = -\tfrac{1}{2}F(x) + Bg_k(x) + \lambda_k, \quad \textit{and} \quad G(u) = Bu + \Lambda, \ \textit{or}$$

(2) *F is an arbitrary function and there exist $C, D \in \mathbb{R}$ with $D \neq 0$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that*

$$f_k(x) = Cx + \lambda_k, \quad g_k(x) = Dx + \mu_k, \quad \textit{and} \quad G(u) = F\left(\tfrac{u-\mu}{2D}\right) + C\tfrac{u-\mu}{D} + \Lambda$$

holds for all $x \in I$ and for all $u \in g_1(I)+g_2(I)$ with $\mu:=\mu_1+\mu_2$ and $\Lambda:=\lambda_1+\lambda_2$.

Conversely, $(F, f_1, f_2, G, g_1, g_2)$ with members defined as either in (1) or (2) solves (2.3).

Proof. If condition (A) holds, then, by (2.5), F is affine on I. Thus, applying Corollary 8, we obtain solution (1).

If condition (B.1) is valid, then the system (3.3) holds on I. Consequently, g_1 and g_2 are continuous affine functions with some common slope $D \in \mathbb{R}$ with $D \neq 0$. Thus, by Corollary 10, we obtain solution (2).

As we mentioned before, sufficiency of (1) and (2) is a matter of substitution. □

To treat the sub-case of (B.2) when (4.2) of Theorem 4 and (5.2) of Theorem 5 are valid simultaneously, we need to formulate and prove two lemmas. First we recall the notion of Schwarzian derivative. Let $U \subseteq \mathbb{R}$ be a non-empty open subinterval and $f : U \to \mathbb{R}$ be an at least three-times differentiable function with non-vanishing first derivative. Then the *Schwarzian derivative of f* is defined by the formula

$$\mathcal{S}f := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

As it is well known, the ,,Schwarzian lines" are exactly the *Möbius transformations* or *linear fractions*, that is, functions $g : U \to \mathbb{R}$ of the form

$$g(x) = \frac{cx + d}{ax + b}, \quad x \in U,$$

with $a, b, c, d \in \mathbb{R}$ such that $ad \neq bc$. More generally, it can be shown that if g is a linear fraction, then

$$\mathcal{S}(g \circ f) = \mathcal{S}f \tag{4.1}$$

holds on U for all functions f on U with appropriate properties. Roughly speaking, the Schwarzian derivative is invariant under linear fractions or, equivalently, linear fractions preserve Schwarzian derivatives.

We shall say that a function $f : U \to \mathbb{R}$ is a *trigonometric fraction* or a *hyperbolic fraction* if there exist constants $a, b, c, d, \kappa \in \mathbb{R}$ with $ad \neq bc$ and $\kappa > 0$ such that

$$f(x) = \frac{c\sin(\kappa x) + d\cos(\kappa x)}{a\sin(\kappa x) + b\cos(\kappa x)} \quad \text{or}$$
$$f(x) = \frac{c\sinh(\kappa x) + d\cosh(\kappa x)}{a\sinh(\kappa x) + b\cosh(\kappa x)}, \quad x \in U,$$

respectively. Obviously, on some subinterval of U, these functions can be rewritten as $g \circ \tan \circ (\kappa\,\mathrm{id})$ and $g \circ \tanh \circ (\kappa\,\mathrm{id})$, respectively, with some Möbius transformation g. Thus, in view of the property (4.1) the next lemma is suitable to give the Schwarzian derivative of trigonometric and hyperbolic fractions.

Lemma 14. *For a given constant $\kappa \in \mathbb{R}$ with $\kappa \neq 0$, we have*

$$\mathcal{S}\tan(\kappa x) = 2\kappa^2 \quad \textit{and} \quad \mathcal{S}\tanh(\kappa y) = -2\kappa^2$$

for all $x \in \mathbb{R} \setminus \{\frac{(2\ell-1)\pi}{2\kappa} \mid \ell \in \mathbb{Z}\}$ and for all $y \in \mathbb{R}$. Furthermore, if $\kappa = 0$, then the above formulas hold on $\mathbb{R}$.

Proof. Simple calculation. □

Within case (B.2), at most one of the functions ψ_1 and ψ_2 can be constant on I, which, by Theorem 4, Theorem 5, and Lemma 14, yields that $\varphi = \frac{1}{2}F'$ is either (B.2.1) a trigonometric fraction, or (B.2.2) a linear fraction, or (B.2.3) a hyperbolic fraction on I. In addition, following from Lemma 14, we must have $\gamma = \gamma^*$ provided that (4.2) of Theorem 4 and (5.2) of Theorem 5 are true simultaneously. Motivated by this, for simplicity and tractability, depending on the exact form of $F' = 2\varphi$, case (B.2) will be discussed in three parts.

Before we turn to these results, we formulate and prove the following lemma which will help us to handle the different representations of the function φ.

Lemma 15. *Let $U \subseteq \mathbb{R}$ be a nonempty open interval, $a, b, c, d \in \mathbb{R}$ and $a^*, b^*, c^*, d^* \in \mathbb{R}$ such that $ad \neq bc$ and $a^*d^* \neq b^*c^*$, and let $t \in \{\tan, \mathrm{id}, \tanh\}$ be defined on U. Then we have*

$$\frac{ct(x)+d}{at(x)+b} = \frac{c^*t(x)+d^*}{a^*t(x)+b^*}, \quad x \in U \tag{4.2}$$

if and only if there exists $p \neq 0$ such that $(a, b, c, d) = p(a^*, b^*, c^*, d^*)$.

Proof. The sufficiency is trivial. For the necessity, observe that (4.2) holds if and only if

$$\begin{pmatrix} -c^* & 0 & a^* & 0 \\ -d^* & -c^* & b^* & a^* \\ 0 & -d^* & 0 & b^* \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is valid. Solving this equation, we get that (a, b, c, d) must be of the form $p(a^*, b^*, c^*, d^*)$, where

$$p := \begin{cases} \frac{b}{b^*} & \text{if } a^* = 0, \\ \frac{a}{a^*} & \text{if } b^* = 0, \\ \frac{a}{a^*} = \frac{b}{b^*} & \text{if } a^*b^* \neq 0. \end{cases}$$

By condition $a^*d^* \neq b^*c^*$, the constants a^* and b^* cannot be zero simultaneously. Hence p is well-defined. On the other hand, by condition $ad \neq bc$, it follows that $a^* = 0$ or $b^* = 0$ if and only if $a = 0$ or $b = 0$, respectively. Consequently $p \neq 0$, which finishes the proof. □

First we are dealing with the so-called *Trigonometric Solutions*, that is, with the case when $\gamma = \gamma^* < 0$.

Theorem 16. *If $(F, f_1, f_2, G, g_1, g_2)$ is a regular solution of (2.3) such that, for the functions defined in (2.5), condition* (B.2.1) *holds, then there exist $A, B, C, D, T \in \mathbb{R}$ with $AD \neq 0$, $\alpha, \beta, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha \neq 0$ and $\beta_1 + \beta_2 \in \mathbb{Z}\pi + \beta$, and $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that F is of the form*

$$F(x) = 2A \ln|\sin(2\alpha x + \beta)| + 2Bx + \lambda, \quad x \in I$$

and exactly one of the following holds true:

(T.1) *$T^2 < 1$ and, for all $x \in I$ and $u \in g_1(I) + g_2(I)$,*

$$f_k(x) = -A \ln|\cos(2\alpha x + 2\beta_k) + T| - Bx + C \ln\left|\tfrac{\tau\tan(\alpha x+\beta_k)-1}{\tau\tan(\alpha x+\beta_k)+1}\right| + \lambda_k,$$

$$g_k(x) = D \ln\left|\tfrac{\tau\tan(\alpha x+\beta_k)-1}{\tau\tan(\alpha x+\beta_k)+1}\right| + \mu_k, G(u) = -2A \ln\left|T^* \sinh(\tfrac{u-\mu}{2D})\right|$$

$$+C\tfrac{u-\mu}{D} + \Lambda$$

(T.2) $T^2 = 1$ *and, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$,

$$f_k(x) = -A \ln|\cos(2\alpha x + 2\beta_k) + T| - Bx + C \tan^T(\alpha x + \beta_k) + \lambda_k,$$
$$g_k(x) = D \tan^T(\alpha x + \beta_k) + \mu_k, G(u)$$
$$= 2A \ln|\tfrac{u-\mu}{2D}| + C\tfrac{u-\mu}{D} + \Lambda$$

(T.3) $T^2 > 1$ *and, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$,

$$f_k(x) = -A \ln|\cos(2\alpha x + 2\beta_k) + T| - Bx + C \arctan(\tau \tan(\alpha x + \beta_k)) + \lambda_k,$$
$$g_k(x) = D \arctan(\tau \tan(\alpha x + \beta_k)) + \mu_k,$$
$$G(u) = 2A \ln\left|T^* \sin(\tfrac{u-\mu}{D})\right| + C\tfrac{u-\mu}{D} + \Lambda,$$

where $\tau := \left|\frac{T+1}{T-1}\right|^{1/2}$, $T^* := |T^2 - 1|^{-1/2}$, $\mu := \mu_1 + \mu_2$, *and* $\Lambda := \lambda + \lambda_1 + \lambda_2$.

Conversely, in each of the above possibilities we obtain a regular solution of Eq. (2.3).

Proof. The proof of sufficiency of (T.1), or (T.2), or (T.3) is a simple calculation. Therefore we are going to focus on the necessity part.

Assume that $(F, f_1, f_2, G, g_1, g_2)$ is a regular solution of (2.3) such that, for the functions defined in (2.5), condition (B.2.1) holds. To measure the behavior of the functions ψ_1 and ψ_2 defined in (2.5), we introduce the following parameters. Let $p := 1$ and $q_k := 0$ if ψ_1 is zero on I and let $p \in \mathbb{R}$ and $q_k := (-1)^{k-1}$ otherwise. In this terminology, $p = 0$ stands for the case when ψ_2 is constant on I. Furthermore if $q_k \neq 0$ holds, then $\psi_1 \neq 0$ and ψ_2 is not constant on I. Then, by our assumption, there exist constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$, $\kappa > 0$, and $\lambda^*, \mu^* \in \mathbb{R}$ such that

$$F'(x) = \frac{c \sin(\kappa x) + d \cos(\kappa x)}{a \sin(\kappa x) + b \cos(\kappa x)},$$
$$g'_k(x) = \frac{2}{\omega_{2,k} \sin(\kappa x) + \omega_{1,k} \cos(\kappa x) + \omega_0}, \quad \text{and}$$
$$f'_k(x) = -\frac{1}{2} \cdot \frac{\theta_{2,k} \sin(\kappa x) + \theta_{1,k} \cos(\kappa x) + \theta_0}{\omega_{2,k} \sin(\kappa x) + \omega_{1,k} \cos(\kappa x) + \omega_0}$$

hold for all $x \in I$, where

$$(\omega_{2,k}, \omega_{1,k}, \omega_0) := (pb + q_k a, -pa + q_k b, \lambda^*)$$
$$\text{and} \quad (\theta_{2,k}, \theta_{1,k}, \theta_0) := (pd + q_k c, -pc + q_k d, 2\mu^*).$$

To make it easier to handle the different sub-cases, first we are going to reformulate the right hand side of the above differential equations. Condition $ad \neq bc$ provides that

$$t := \omega_{1,k}^2 + \omega_{2,k}^2 = (p^2 + 1)(a^2 + b^2) > 0 \quad \text{and}$$
$$s := \theta_{1,k}^2 + \theta_{2,k}^2 = (p^2 + 1)(c^2 + d^2) > 0,$$

therefore there exist $\rho_k, \sigma_k \in [0, 2\pi[$ such that

$$(\cos\rho_k, \sin\rho_k) = \tfrac{1}{\sqrt{t}}(\omega_{1,k}, \omega_{2,k}) \quad \text{and} \quad (\cos\sigma_k, \sin\sigma_k) = \tfrac{1}{\sqrt{s}}(\theta_{1,k}, \theta_{2,k}).$$

We note that, in general, t and s are independent from k, and $\sigma_1 = \sigma_2 =: \sigma$ and $\rho_1 = \rho_2 =: \rho$ provided that ψ_1 is zero on I. In light of this, we obtain that

$$\begin{aligned} g_k'(x) &= \frac{2}{\sqrt{t}} \cdot \frac{1}{\cos(\kappa x - \rho_k) + T} \quad \text{and} \\ f_k'(x) &= -\frac{1}{2}\sqrt{\frac{s}{t}} \cdot \frac{\cos(\kappa x - \sigma_k) + S}{\cos(\kappa x - \rho_k) + T}, \quad x \in I, \end{aligned} \tag{4.3}$$

where $T := \frac{1}{\sqrt{t}}\omega_0$ and $S := \frac{1}{\sqrt{s}}\theta_0$. Similarly, the function F' can be written as

$$\begin{aligned} F'(x) = &\sqrt{\frac{s}{t}} \cos(\rho_0 - \sigma_0) \\ &- \sqrt{\frac{s}{t}} \sin(\rho_0 - \sigma_0) \tan(\kappa x - \rho_0), \quad x \in I \end{aligned} \tag{4.4}$$

with $\rho_0, \sigma_0 \in [0, 2\pi[$ satisfying the identities

$$(\sin\rho_0, \cos\rho_0) = \tfrac{1}{\sqrt{a^2+b^2}}(b, a) \quad \text{and} \quad (\sin\sigma_0, \cos\sigma_0) = \tfrac{1}{\sqrt{c^2+d^2}}(d, c).$$

Equation (4.4) yields that there exists a constant $\lambda \in \mathbb{R}$ such that

$$F(x) = 2A \ln|\sin(2\alpha x + \beta)| + 2Bx + \lambda, \quad x \in I, \tag{4.5}$$

where $\alpha := \frac{\kappa}{2} > 0$, $\beta = \rho_0$,

$$A := \frac{ad - bc}{2\kappa(a^2 + b^2)} \neq 0, \quad \text{and} \quad B := \frac{ac + bd}{2(a^2 + b^2)}.$$

It turns out that the exact form of the solutions of the differential equations in (4.3) strongly depends on the value of T. Therefore, in the rest of the proof, we are going to distinguish three cases: $T^2 < 1$ or $T^2 = 1$ or $T^2 > 1$.

Case 1. Assume that $T^2 < 1$ holds. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$\begin{aligned} g_k(x) &= D \ln\left|\tfrac{\tau\tan(\alpha x + \beta_k) - 1}{\tau\tan(\alpha x + \beta_k) + 1}\right| + \mu_k \quad \text{and} \\ f_k(x) &= -A\ln|\cos(2\alpha x + 2\beta_k) + T| - Bx + C\ln\left|\tfrac{\tau\tan(\alpha x + \beta_k) - 1}{\tau\tan(\alpha x + \beta_k) + 1}\right| + \lambda_k \end{aligned}$$

hold for all $x \in I$, where $\beta_k := -\frac{\rho_k}{2}$, $\tau := \sqrt{\frac{1-T}{1+T}}$,

$$\begin{aligned} C &:= -\frac{\operatorname{sgn}(T-1)}{\kappa(a^2+b^2)\sqrt{1-T^2}}\left(S\sqrt{(a^2+b^2)(c^2+d^2)} - \tfrac{1}{2}T(ac+bd)\right), \quad \text{and} \\ D &:= \frac{4\operatorname{sgn}(T-1)}{\kappa\sqrt{t(1-T^2)}} \neq 0. \end{aligned}$$

Substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned}
&G\big(D\tfrac{\tau\tan\xi-1}{\tau\tan\xi+1}\tfrac{\tau\tan\eta-1}{\tau\tan\eta+1}+\mu\big)=2A\ln|\sin(\alpha(x+y)+\beta)|+B(x+y)+\lambda\\
&\qquad -A\ln|\cos(2\alpha x+2\beta_1)+T|-Bx+C\ln\big|\tfrac{\tau\tan(\alpha x+\beta_1)-1}{\tau\tan(\alpha x+\beta_1)+1}\big|+\lambda_1\\
&\qquad -A\ln|\cos(2\alpha y+2\beta_2)+T|-By+C\ln\big|\tfrac{\tau\tan(\alpha y+\beta_2)-1}{\tau\tan(\alpha y+\beta_2)+1}\big|+\lambda_2\\
&\quad =A\ln\big|\tfrac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}\big|+C\ln\Big|\tfrac{\tau\tan\xi-1}{\tau\tan\xi+1}\tfrac{\tau\tan\eta-1}{\tau\tan\eta+1}\Big|+\Lambda,
\end{aligned}$$

where in the last step, we used that $\beta_1+\beta_2-\beta$ is of the form $\ell\pi$ for some $\ell\in\mathbb{Z}$. *(To see this, check that* $\sin^2(\beta_1+\beta_2-\beta)=0$ *is valid.)* In view of the identities

$$\frac{\tau\tan\xi-1}{\tau\tan\xi+1}\frac{\tau\tan\eta-1}{\tau\tan\eta+1}=\Big(1-\frac{\tau\tan\xi+\tau\tan\eta}{1+\tau^2\tan\xi\tan\eta}\Big)\Big(1+\frac{\tau\tan\xi+\tau\tan\eta}{1+\tau^2\tan\xi\tan\eta}\Big)^{-1}$$

and

$$\begin{aligned}
\frac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}&=\frac{1}{1-T^2}\Big(\frac{\tau\tan\xi+\tau\tan\eta}{1+\tau^2\tan\xi\tan\eta}\Big)^2\\
&\quad\times\Big(1-\Big(\frac{\tau\tan\xi+\tau\tan\eta}{1+\tau^2\tan\xi\tan\eta}\Big)^2\Big)^{-1},
\end{aligned}$$

for $u{:=}D\ln\big|\frac{\tau\tan\xi-1}{\tau\tan\xi+1}\frac{\tau\tan\eta-1}{\tau\tan\eta+1}\big|+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=-2A\ln\big|\tfrac{1}{\sqrt{1-T^2}}\sinh(\tfrac{u-\mu}{2D})\big|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (T.1).

Case 2. Assume that $T^2=1$ *holds.* Then there exist $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$\begin{aligned}
&g_k(x)=D\tan^T(\alpha x+\beta_k)+\mu_k\ \text{ and}\\
&f_k(x)=-A\ln|\cos(2\alpha x+2\beta_k)+T|-Bx+C\tan^T(\alpha x+\beta_k)+\lambda_k
\end{aligned}$$

hold for $x\in I$, where $\beta_k{:=}-\frac{\rho_k}{2}$,

$$C=-\frac{1}{2\kappa(a^2+b^2)}\big(S\sqrt{(a^2+b^2)(c^2+d^2)}+T(ac+bd)\big),\quad\text{and}\quad D{:=}\frac{2}{\kappa\sqrt{t}}\neq 0.$$

Particularly, $\beta_1+\beta_2=\mathbb{Z}\pi+\beta$ holds. Substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned}
&G(D(\tan^T\xi+\tan^T\eta)+\mu)\\
&\quad=2A\ln|\sin(\alpha(x+y)+\beta)|+B(x+y)+\lambda\\
&\qquad -A\ln|\cos(2\alpha x+2\beta_1)+T|-Bx+C\tan^T(\alpha x+\beta_1)+\lambda_1\\
&\qquad -A\ln|\cos(2\alpha y+2\beta_2)+T|-By+C\tan^T(\alpha y+\beta_2)+\lambda_2\\
&\quad=A\ln\big|\tfrac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}\big|\\
&\qquad +C(\tan^T\xi+\tan^T\eta)+\Lambda.
\end{aligned}$$

Using

$$\frac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}=\tfrac{1}{4}(\tan^T\xi+\tan^T\eta)^2,$$

for $u{:=}D(\tan^T\xi+\tan^T\eta)+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln\left|\tfrac{u-\mu}{2D}\right|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (T.2).

Case 3. Finally, assume that $T^2>1$ holds. Then there exist $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$g_k(x)=D\arctan(\tau\tan(\alpha x+\beta_k))+\mu_k \quad \text{and}$$
$$f_k(x)=-A\ln|\cos(2\alpha x+2\beta_k)+T|-Bx+C\arctan(\tau\tan(\alpha x+\beta_k))+\lambda_k$$

hold for all $x\in I$, where $\beta_k{:=}-\frac{\rho_k}{2}$, $\tau{:=}\sqrt{\frac{T-1}{T+1}}$,

$$C{:=}-\frac{\operatorname{sgn}(T+1)}{\kappa(a^2+b^2)\sqrt{T^2-1}}\big(S\sqrt{(a^2+b^2)(c^2+d^2)}-\tfrac{1}{2}T(ac+bd)\big),\quad\text{and}$$
$$D{:=}\frac{4\operatorname{sgn}(T+1)}{\kappa\sqrt{t(T^2-1)}}\neq 0.$$

Particularly, $\beta_1+\beta_2=\mathbb{Z}\pi+\beta$ holds. Substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned}G\big(D\arctan\big(\tfrac{\tau\tan\xi+\tau\tan\eta}{1-\tau^2\tan\xi\tan\eta}\big)+\mu\big)&=2A\ln|\sin(\alpha(x+y)+\beta)|+B(x+y)+\lambda\\&\quad-A\ln|\cos(2\alpha x+2\beta_1)+T|-Bx+C\arctan(\tau\tan(\alpha x+\beta_1))+\lambda_1\\&\quad-A\ln|\cos(2\alpha y+2\beta_2)+T|-By+C\arctan(\tau\tan(\alpha y+\beta_2))+\lambda_2\\&=A\ln\left|\tfrac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}\right|+C\arctan\big(\tfrac{\tau\tan\xi+\tau\tan\eta}{1-\tau^2\tan\xi\tan\eta}\big)+\Lambda.\end{aligned}$$

In view of the identity

$$\frac{\sin^2(\xi+\eta)}{(\cos 2\xi+T)(\cos 2\eta+T)}=\frac{1}{T^2-1}\Big(\frac{\tau\tan\xi+\tau\tan\eta}{1-\tau^2\tan\xi\tan\eta}\Big)^2\Big(1+\Big(\frac{\tau\tan\xi+\tau\tan\eta}{1-\tau^2\tan\xi\tan\eta}\Big)^2\Big)^{-1},$$

for $u=D\arctan\big(\frac{\tau\tan\xi+\tau\tan\eta}{1-\tau^2\tan\xi\tan\eta}\big)+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln\Big|\tfrac{1}{\sqrt{T^2-1}}\sin(\tfrac{u-\mu}{D})\Big|+C(\tfrac{u-\mu}{D})+\Lambda.$$

Thus we obtained the solutions listed in (T.3). □

Now we turn to the case of *Polynomial Solutions* of Eq. (2.3), that is, when $\gamma=\gamma^*=0$. As we will see, in terms of F, solutions can be classified into two groups. In one case, F is a linear combination of an affine function and the logarithm of an affine function. This group will include four solutions. In the other case, F is a second order polynomial, which gives two further solutions.

Since the behavior of the latter two solutions is fundamentally different, they will be formulated separately within the theorem.

Theorem 17. *If* $(F, f_1, f_2, G, g_1, g_2)$ *is a regular solution of* (2.3) *such that, for the functions defined in* (2.5)*, condition* (B.2.2) *holds, then there exist* $A, B, C, D \in \mathbb{R}$ *with* $AD \neq 0$*,* $\alpha, \beta, \beta_1, \beta_2 \in \mathbb{R}$ *with* $\alpha \neq 0$ *and* $\beta_1 + \beta_2 = \beta$*, and* $\lambda, \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ *such that either* F *is of the form*

$$F(x) = 2A \ln|2\alpha x + \beta| + 2Bx + \lambda, \quad x \in I$$

and either

$$\begin{aligned} f_k(x) &= -A \ln|(\alpha x + \beta_k)^2 + 1| - Bx + C \arctan(\alpha x + \beta_k) + \lambda_k, \\ g_k(x) &= D \arctan(\alpha x + \beta_k) + \mu_k, \quad G(u):=2A \ln|\sin(\tfrac{u-\mu}{D})| + C\tfrac{u-\mu}{D} + \Lambda, \end{aligned} \tag{P1.1}$$

or

$$\begin{aligned} f_k(x) &= -2A \ln|\alpha x + \beta_k| - Bx + C(\alpha x + \beta_k)^{-1} + \lambda_k, \\ g_k(x) &= D(\alpha x + \beta_k)^{-1} + \mu_k, \quad G(u) = 2A \ln|\tfrac{u-\mu}{D}| + C\tfrac{u-\mu}{D} + \Lambda, \end{aligned} \tag{P1.2}$$

or

$$\begin{aligned} f_k(x) &= -A \ln|(\alpha x + \beta_k)^2 - 1| - Bx + C \ln|\tfrac{\alpha x+\beta_k-1}{\alpha x+\beta_k+1}| + \lambda_k, \\ g_k(x) &= D \ln|\tfrac{\alpha x+\beta_k-1}{\alpha x+\beta_k+1}| + \mu_k, \quad G(u) = 2A \ln|\sinh(\tfrac{u-\mu}{2D})| + C\tfrac{u-\mu}{D} + \Lambda, \end{aligned} \tag{P1.3}$$

or there exist $A_1, A_2 \in \mathbb{R}$ *with* $\frac{1}{2}(A_1 + A_2) = A$ *such that*

$$\begin{aligned} f_k(x) &= -A_k \ln|\alpha x + \beta_k| - Bx + \lambda_k, \\ g_k(x) &= (-1)^{k-1} D \ln|\alpha x + \beta_k| + \mu_k, \\ G(u) &= 2A \ln|\exp(\tfrac{u-\mu}{D}) + 1| - A_1\tfrac{u-\mu}{D} + \Lambda, \end{aligned} \tag{P1.4}$$

hold for all $x \in I$ *and for all* $u \in g_1(I) + g_2(I)$ *or there exist* $A_k, B_k, D_k \in \mathbb{R}$ *with* $D_1 D_2(A + 4A_k) = D_k^2 A$ *and* $B + 2B_k = 2D_k C$ *such that*

$$\begin{aligned} F(x) &= Ax^2 + 2Bx + \lambda, \quad & f_k(x) &= A_k x^2 + B_k x + \lambda_k, \\ g_k(x) &= D_k x + \mu_k, & G(u) &= \tfrac{1}{4D_1 D_2} A(u-\mu)^2 + C(u-\mu) + \Lambda, \end{aligned} \tag{P2.1}$$

or

$$\begin{aligned} F(x) &= A(2x + \beta)^2 + 2B + \lambda, \quad & f_k(x) &= -A(x + \beta_k)^2 - Bx + C \ln|x + \beta_k| + \lambda_k, \\ g_k(x) &= D \ln|x + \beta_k| + \mu_k, & G(u) &= 2A \exp(\tfrac{u-\mu}{D}) + C\tfrac{u-\mu}{D} + \Lambda \end{aligned} \tag{P2.2}$$

hold for all $x \in I$ *and for all* $u \in g_1(I) + g_2(I)$*, where* $\mu:=\mu_1 + \mu_2$ *and* $\Lambda:=\lambda + \lambda_1 + \lambda_2$*.*

Conversely, in each of the above possibilities we obtain a regular solution of Eq. (2.3)

Proof. The proof of sufficiency is a simple calculation, therefore we will focus only on the necessity.

Similarly to the previous proof, to indicate the behavior of ψ_1 and ψ_2, let $p{:=}1$ and $q_k{:=}0$ if ψ_1 is zero on I and let $p \in \mathbb{R}$ and $q_k{:=}(-1)^{k-1}$ otherwise. Again, $p = 0$ corresponds to the case when ψ_2 is constant on I. Furthermore, if $q_k \neq 0$, then we get the case when ψ_1 and ψ_2 are not constant on I. Then, in view of our assumption concerning the derivative of F, there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that

$$F'(x) = \frac{cx+d}{ax+b}, \quad g_k'(x) = \frac{4}{\omega_2 x^2 + \omega_{1,k}x + \omega_{0,k}}, \quad \text{and}$$
$$f_k'(x) = -\frac{1}{2} \cdot \frac{\theta_2 x^2 + \theta_{1,k}x + \theta_{0,k}}{\omega_2 x^2 + \omega_{1,k}x + \omega_{0,k}}$$

hold for all $x \in I$, where

$$(\omega_2, \omega_{1,k}, \omega_{0,k}){:=}(pa, 2pb + 2q_k a, 2q_k b + 2\lambda^*) \quad \text{and}$$
$$(\theta_2, \theta_{1,k}, \theta_{0,k}){:=}(pc, 2pd + 2q_k c, 2q_k d + 4\mu^*).$$

Then, an elementary calculation yields that there exists a constant $\lambda_0 \in \mathbb{R}$, such that

$$F(x) = \begin{cases} A_0 x^2 + B_0 x + \lambda_0 & \text{if } a = 0, \\ 2A_0 \ln|ax+b| + 2B_0 + \lambda_0 & \text{if } a \neq 0, \end{cases} \quad x \in I, \tag{4.6}$$

where

$$0 \neq A_0{:=}\begin{cases} \frac{c}{2b} & \text{if } a = 0, \\ \frac{ad-bc}{2a^2} & \text{if } a \neq 0, \end{cases} \quad \text{and} \quad B_0{:=}\begin{cases} \frac{d}{b} & \text{if } a = 0, \\ \frac{c}{2a} & \text{if } a \neq 0. \end{cases} \tag{4.7}$$

The exact form of the functions g_k and f_k strongly depends on the degree of the polynomial in their denominators. Therefore we are going to distinguish two main cases: either $\omega_2 = 0$ or $\omega_2 \neq 0$.

Case 1. Assume that $\omega_2 = 0$ holds. Then $a = 0$ or $p = 0$ holds such that whenever $p = 0$ is valid then $g_1 - g_2$ cannot be constant on I.

(1.1) *Suppose that $a = 0$ and $p \neq 0$.* Then condition $ad \neq bc$ implies that $bc \neq 0$ and there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = D \ln|x + \beta_k| + \mu_k \quad \text{and}$$
$$f_k(x) = -A(x+\beta_k)^2 - Bx + C \ln|x+\beta_k| + \lambda_k$$

hold for all $x \in I$, where $A{:=}\frac{1}{4}A_0$, $\beta_k{:=}\frac{\omega_{0,k}}{2pb}$, $B{:=}\frac{d}{2b} - \lambda^* \frac{c}{2pb^2}$,

$$C{:=} -\frac{1}{4b^3} \cdot \begin{cases} 4\mu^* b^2 - \lambda^*(2bd - c\lambda^*) & \text{if } g_1 - g_2 \text{ is constant on } I, \\ p^{-2}(4p\mu^* b^2 - \lambda^*(2pbd - c\lambda^*) - cb^2) & \text{otherwise,} \end{cases}$$

$$\text{and} \quad D{:=}\frac{2}{pb} \neq 0.$$

Observe that F in (4.6) can be reformulated as

$$F(x) = A(2x+\beta)^2 + 2Bx + \lambda, \quad x \in I,$$

with $\beta{:=}\beta_1+\beta_2$ and $\lambda{:=}\lambda_0-A\beta^2$. Substituting $\xi{:=}x+\beta_1$, $\eta{:=}y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(D\ln|\xi\eta|+\mu) &= A\big(x+y+\beta\big)^2+B(x+y)+\lambda \\ &\quad -A(x+\beta_1)^2-Bx+C\ln|x+\beta_1|+\lambda_1 \\ &\quad -A(y+\beta_2)^2-By+C\ln|y+\beta_2|+\lambda_2 \\ &= A(\xi+\eta)^2-A(\xi^2+\eta^2)+C\ln|\xi\eta|+\Lambda = 2A\xi\eta+C\ln|\xi\eta|+\Lambda. \end{aligned}$$

Consequently, for $u{:=}D\ln|\xi\eta|+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\exp(\tfrac{u-\mu}{D})+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (P2.2).

(1.2) Suppose that $a\neq 0$ and $p=0$. Then there exist constants $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$\begin{aligned} g_k(x) &= (-1)^{k-1}D\ln|\alpha x+\beta_k|+\mu_k \quad \text{and} \\ f_k(x) &= -A_k\ln|\alpha x+\beta_k|-Bx+\lambda_k \end{aligned}$$

for all $x\in I$, where $\alpha{:=}a\neq 0$, $B{:=}B_0$, $D{:=}\frac{2}{a}\neq 0$, $\beta_k{:=}b+(-1)^{k-1}\lambda^*$, and

$$A_k{:=}\tfrac{1}{2a^2}(ad-bc+(-1)^{k-1}(2a\mu^*-c\lambda^*)).$$

Observe that $\frac{1}{2}(A_1+A_2)=A_0=:A$ and that the function F in (4.6) can be written as

$$F(x)=2A\ln|2\alpha x+\beta|+2Bx+\lambda,$$

with $\beta{:=}\beta_1+\beta_2$, and $\lambda{:=}\lambda_0-\ln 4A$. Hence, substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(D\ln|\tfrac{\xi}{\eta}|+\mu) &= 2A\ln|\alpha(x+y)+\beta|+B(x+y)+\lambda \\ &\quad -A_1\ln|\alpha x+\beta_1|-Bx+\lambda_1 \\ &\quad -A_2\ln|\alpha y+\beta_2|-By+\lambda_2 \\ &= 2A\ln|\tfrac{\xi}{\eta}+1|-A_1\ln|\tfrac{\xi}{\eta}|+\Lambda. \end{aligned}$$

Consequently, for $u{:=}D\ln|\frac{\xi}{\eta}|+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln|\exp(\tfrac{u-\mu}{D})+1|-A_1\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (P1.4).

(1.3) *Finally, suppose that $a=p=0$.* Then, on the one hand, by $ad\neq bc$, we must have $c\neq 0$. On the other hand, in view of

Remark 18. $\omega_{0,k}\neq 0$ follows. Thus there exist constants $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$g_k(x)=D_kx+\mu_k \quad \text{and} \quad f_k(x)=A_kx^2+B_kx+\lambda_k$$

hold for all $x \in I$, where

$$A_k := \tfrac{(-1)^k}{2}\omega_{0,k}^{-1} c \neq 0, \quad B_k := (-1)^k \omega_{0,k}^{-1}(d + (-1)^{k-1} 2\mu^*),$$
$$\text{and} \quad D_k := 4\omega_{0,k}^{-1} \neq 0.$$

Observe that the identity $D_1 D_2 (A + 4A_k) = D_k^2 A$ holds. Note further that the constants $\frac{1}{2D_1}(B + 2B_1)$ and $\frac{1}{2D_2}(B + 2B_2)$ are equal to each other. Denote their common value by C. Substituting $\xi := 2\alpha_1 x$, $\eta := 2\alpha_2 y$, $\mu := \mu_1 + \mu_2$, and $\Lambda := \lambda + \lambda_1 + \lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(\xi + \eta + \mu) &= \tfrac{1}{4}A(x+y)^2 + \tfrac{1}{2}B(x+y) + \lambda + A_1 x^2 + B_1 x + \lambda_1 \\ &\quad + A_2 y^2 + B_2 y + \lambda_2 \\ &= \tfrac{1}{4D_1D_2} A(\xi + \eta)^2 + C(\xi + \eta) + \Lambda. \end{aligned}$$

Consequently, for $u := \xi + \eta + \mu \in g_1(I) + g_2(I)$, we get

$$G(u) = \tfrac{1}{4D_1D_2} A(u - \mu)^2 + C(u - \mu) + \Lambda.$$

Thus we obtained the solutions listed in (P2.1).

Case 2. Assume now that $\omega_2 \neq 0$. Then the polynomials appearing in the denominator in the formulas concerning g_k and f_k are of degree two and have a common discriminant

$$\Delta := \begin{cases} 4b^2 - 8a\lambda^* & \text{if } g_1 - g_2 \text{ is constant on } I, \\ 4p^2b^2 - 8pa\lambda^* + 4a^2 & \text{otherwise.} \end{cases}$$

Accordingly, depending on the sign of Δ, within Case 2., we are going to distinguish three further sub-cases.

(2.1) *Suppose that* $\Delta < 0$. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$\begin{aligned} g_k(x) &= D \arctan(\alpha x + \beta_k) + \mu_k \quad \text{and} \\ f_k(x) &= -A \ln|(\alpha x + \beta_k)^2 + 1| - Bx + C \arctan(\alpha x + \beta_k) + \lambda_k \end{aligned}$$

hold for all $x \in I$, where $A := A_0$, $B := B_0$, $\alpha := \frac{2}{\sqrt{-\Delta}}\omega_2 \neq 0$, $\beta_k := \frac{1}{\sqrt{-\Delta}}\omega_{1,k}$, $D := \frac{4}{\sqrt{-\Delta}} \neq 0$, and

$$C := \frac{1}{a\sqrt{-\Delta}}(c\lambda^* - 2a\mu^*) + \frac{pb}{a^2\sqrt{-\Delta}}(ad - bc).$$

Note that F can be reformulated as

$$F(x) = 2A \ln|2\alpha x + \beta| + 2Bx + \lambda, \quad x \in I$$

with $\beta{:=}\beta_1+\beta_2$ and $\lambda{:=}\lambda_0+2A\ln|\frac{\sqrt{-\Delta}}{4p}|$. Hence, substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(D\arctan(\tfrac{\xi+\eta}{1-\xi\eta})+\mu) &= 2A\ln|\alpha(x+y)+\beta|+B(x+y)+\lambda \\ &\quad -A\ln|(\alpha x+\beta_1)^2+1|-Bx+C\arctan(\alpha x+\beta_1)+\lambda_1 \\ &\quad -A\ln|(\alpha y+\beta_2)^2+1|-By+C\arctan(\alpha y+\beta_2)+\lambda_2 \\ &= A\ln\Big|\tfrac{(\xi+\eta)^2}{(\xi^2+1)(\eta^2+1)}\Big|+C\arctan(\tfrac{\xi+\eta}{1-\xi\eta})+\Lambda. \end{aligned}$$

Consequently, for $u{:=}D\arctan(\frac{\xi+\eta}{1-\xi\eta})+\mu\in g_1(I)+g_2(I)$, we get

$$\frac{(\xi+\eta)^2}{(\xi^2+1)(\eta^2+1)}=\frac{\left(\frac{\xi+\eta}{1-\xi\eta}\right)^2}{1+\left(\frac{\xi+\eta}{1-\xi\eta}\right)^2}=\frac{\tan^2\left(\frac{u-\mu}{D}\right)}{1+\tan^2\left(\frac{u-\mu}{D}\right)}=\sin^2\Big(\frac{u-\mu}{D}\Big),$$

and hence

$$G(u){:=}2A\ln|\sin(\tfrac{u-\mu}{D})|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (P1.1).

(2.2) *Suppose that* $\Delta=0$. Then there exist constants $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$\begin{aligned} g_k(x) &= D(\alpha x+\beta_k)^{-1}+\mu_k \quad \text{and} \\ f_k(x) &= -2A\ln|\alpha x+\beta_k|-Bx+C(\alpha x+\beta_k)^{-1}+\lambda_k \end{aligned}$$

hold for all $x\in I$, where $A{:=}A_0$, $B{:=}B_0$, $\alpha{:=}\omega_2\neq 0$, $\beta_k{:=}\frac{1}{2}\omega_{1,k}$, $D{:=}-4\neq 0$, and

$$C{:=}-\frac{1}{a}(c\lambda^*-2a\mu^*)-\frac{pb}{a^2}(ad-bc).$$

Observe that F can be written as

$$F(x)=2A\ln|2\alpha x+\beta|+2Bx+\lambda,\quad x\in I$$

with $\beta{:=}\beta_1+\beta_2$ and $\lambda{:=}\lambda_0-\ln(4p^2)A$. Hence, substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G\left(D\tfrac{\xi+\eta}{\xi\eta}+\mu\right) &= 2A\ln|\alpha(x+y)+\beta|+B(x+y)+\lambda \\ &\quad -2A\ln|\alpha x+\beta_1|-Bx+C(\alpha x+\beta_1)^{-1}+\lambda_1 \\ &\quad -2A\ln|\alpha y+\beta_2|-By+C(\alpha y+\beta_2)^{-1}+\lambda_2=2A_0\ln|\tfrac{\xi+\eta}{\xi\eta}| \\ &\quad +C\tfrac{\xi+\eta}{\xi\eta}+\Lambda. \end{aligned}$$

Consequently, for $u{:=}D\frac{\xi+\eta}{\xi\eta}+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln|\tfrac{u-\mu}{D}|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (P1.2).

(2.3) *Finally, suppose that* $\Delta > 0$. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = D \ln\left|\tfrac{\alpha x+\beta_k-1}{\alpha x+\beta_k+1}\right| + \mu_k \quad \text{and}$$
$$f_k(x) = -A\ln\left|(\alpha x+\beta_k)^2-1\right| - Bx + C\ln\left|\tfrac{\alpha x+\beta_k-1}{\alpha x+\beta_k+1}\right| + \lambda_k$$

for all $x \in I$, where $A{:=}A_0$, $B{:=}B_0$, $\alpha{:=}\frac{2}{\sqrt{\Delta}}\omega_2 \neq 0$, $\beta_k{:=}\frac{1}{\sqrt{\Delta}}\omega_{1,k}$, $D{:=}\frac{4}{\sqrt{\Delta}} \neq 0$, and

$$C{:=}\frac{1}{a\sqrt{\Delta}}(c\lambda^* - 2a\mu^*) + \frac{pb}{a^2\sqrt{\Delta}}(ad-bc)$$

Observe that F can be written as

$$F(x) = 2A\ln|2\alpha x+\beta| + 2Bx + \lambda, \quad x \in I,$$

with $\beta{:=}\beta_1+\beta_2$ and $\lambda{:=}\lambda_0 + 2A_0\ln\left|\frac{\sqrt{\Delta}}{4p}\right|$. Hence, substituting $\xi{:=}\alpha x+\beta_1$, $\eta{:=}\alpha y+\beta_2$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G\big(D\ln\left|\tfrac{(\xi-1)(\eta-1)}{(\xi+1)(\eta+1)}\right| + \mu\big) &= 2A\ln|\alpha(x+y)+\beta| + B(x+y) + \lambda \\ &\quad -A\ln|(\alpha x+\beta_1)^2-1| - Bx + C\ln\left|\tfrac{\alpha x+\beta_1-1}{\alpha x+\beta_1+1}\right| + \lambda_1 \\ &\quad -A\ln|(\alpha y+\beta_2)^2-1| - By + C\ln\left|\tfrac{\alpha y+\beta_2-1}{\alpha y+\beta_2+1}\right| + \lambda_2 \\ &= A\ln\left|\tfrac{(\xi+\eta)^2}{(\xi^2-1)(\eta^2-1)}\right| + C\ln\left|\tfrac{(\xi-1)(\eta-1)}{(\xi+1)(\eta+1)}\right| + \Lambda. \end{aligned}$$

Putting $u{:=}D\ln\left|\frac{(\xi-1)(\eta-1)}{(\xi+1)(\eta+1)}\right| + \mu \in g_1(I)+g_2(I)$, we obtain that

$$\frac{(\xi+\eta)^2}{(\xi^2-1)(\eta^2-1)} = \frac{\left(1-\frac{(\xi-1)(\eta-1)}{(\xi+1)(\eta+1)}\right)^2}{4\cdot\frac{(\xi-1)(\eta-1)}{(\xi+1)(\eta+1)}} = \frac{(1-\exp(\frac{u-\mu}{D}))^2}{4\exp(\frac{u-\mu}{D})} = \sinh^2\Big(\frac{u-\mu}{2D}\Big),$$

consequently

$$G(u) = 2A\ln\left|\sinh(\tfrac{u-\mu}{2D})\right| + C\tfrac{u-\mu}{D} + \Lambda$$

holds.

Thus we obtained the solutions listed in (P1.3). □

Finally, consider the so-called *Hyperbolic Solutions*, that is, the case when $\gamma = \gamma^* > 0$ holds.

Theorem 19. *If* $(F, f_1, f_2, G, g_1, g_2)$ *is a regular solution of* (2.3) *such that, for the functions defined in* (2.5)*, condition* (B.2.3) *holds, then there exist* $A, B, C, D \in \mathbb{R}$ *with* $AD \neq 0$*,* $\kappa > 0$*, and* $\lambda, \lambda_k, \mu_k \in \mathbb{R}$ *such that either* F *is of the form*

$$F(x) = Ae^{-2q\kappa x} + 2Bx + \lambda, \quad x \in I$$

for some $|q| = 1$ *and exactly one of the following holds true:*

(H1.1) *there exist* $\alpha, \beta_k, A_k \in \mathbb{R}$ *with* $\alpha\beta_k A_k \neq 0$, $\beta_2 A = \alpha q A_1$, *and* $\beta_1 A = \alpha q A_2$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= qA_k e^{-q\kappa x} - Bx + C\ln|\alpha e^{-q\kappa x} + \beta_k| + \lambda_k, \\ g_k(x) &= D\ln|\alpha e^{-q\kappa x} + \beta_k| + \mu_k, \\ G(u) &= \alpha^{-2}A(\exp(\tfrac{u-\mu}{D}) - \beta_1\beta_2) + C\tfrac{u-\mu}{D} + \Lambda \end{aligned}$$

(H1.2) *there exist* $|p| \notin \{0,1\}$, $A_k, C_k, D_k \in \mathbb{R}$ *with* $-\frac{1}{2}p^*A = q_k^2 A_k$, $p^*C = q_k C_k$, *and* $p^*D = q_k D_k$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= -A_k e^{-2q\kappa x} - Bx + qC_k e^{-q\kappa x} + \lambda_k, \\ g_k(x) &= qD_k e^{-q\kappa x} + \mu_k, \quad G(u) = \tfrac{1}{2p^*}A(\tfrac{u-\mu}{D})^2 + C\tfrac{u-\mu}{D} + \Lambda, \end{aligned}$$

where $(p^*, q_k){:=}(1,1)$ *if* $\psi_1 = 0$ *on* I *and* $(p^*, q_k){:=}(1-p^2, q+(-1)^{k-1}p)$ *otherwise,*

or F *is of the form*

$$F(x) = 2A\ln|\alpha e^{2\kappa x} - \beta| + 2Bx + \lambda, \quad x \in I$$

for some $\alpha, \beta \in \mathbb{R}$ *with* $\alpha\beta \neq 0$ *and exactly one of the following holds true:*

(H2.1) *there exist* $\alpha_k, \gamma \in \mathbb{R}$ *with* $\alpha_1\alpha_2 = \alpha$ *and* $\gamma^2 = \beta$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= -2A\ln|\alpha_k e^{\kappa x} + \gamma| - Bx + C(\alpha_k e^{\kappa x} + \gamma)^{-1} + \lambda_k, \\ g_k(x) &= D(\alpha_k e^{\kappa x} + \gamma)^{-1} + \mu_k, \quad G(u) = 2A\ln|1 - \gamma\tfrac{u-\mu}{D}| + C\tfrac{u-\mu}{D} + \Lambda \end{aligned}$$

(H2.2) *there exist* $\alpha_k, \gamma \in \mathbb{R}$ *with* $\alpha_1\alpha_2 = \alpha$, $\gamma \neq 0$, *and* $\gamma^2 + 1 = \beta$, *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= -A\ln|(\alpha_k e^{\kappa x} + \gamma)^2 + 1| - Bx + C\arctan(\alpha_k e^{\kappa x} + \gamma) + \lambda_k, \\ g_k(x) &= D\arctan(\alpha_k e^{\kappa x} + \gamma) + \mu_k, \\ G(u) &= 2A\ln|\gamma\sin(\tfrac{u-\mu}{D}) + \cos(\tfrac{u-\mu}{D})| + C\tfrac{u-\mu}{D} + \Lambda \end{aligned}$$

(H2.3) *there exist* $\alpha_k, \gamma_k, A_k \in \mathbb{R}$ *with* $\alpha_1\alpha_2 = \alpha$, $\gamma_1\gamma_2 = \beta$, *and* $\frac{1}{2}(A_1 + A_2) = A$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= -A_1\ln|\alpha_k e^{\kappa x} + \gamma_1| - A_2\ln|\alpha_k e^{\kappa x} + \gamma_2| - Bx + \lambda_k, \\ g_k(x) &= D\ln\left|\tfrac{\alpha_k e^{\kappa x} + \gamma_1}{\alpha_k e^{\kappa x} + \gamma_2}\right| + \mu_k, \\ G(u) &= 2A\ln|\tfrac{\gamma_2}{\gamma_2 - \gamma_1}\exp(\tfrac{u-\mu}{D}) - \tfrac{\gamma_1}{\gamma_2 - \gamma_1}| - A_1\tfrac{u-\mu}{D} + \Lambda \end{aligned}$$

(H2.4) *there exist* $|p| = 1$ *and* $\gamma, \beta_k, A_k, B_k \in \mathbb{R}$ *with* $\frac{1}{2}(A_1 + A_2) = A$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$, *we have*

$$\begin{aligned} f_k(x) &= -A_k\ln|\gamma e^{(-1)^k p\kappa x} + \beta_k| - B_k x + \lambda_k, \\ g_k(x) &= (-1)^k D\ln|\gamma e^{(-1)^k p\kappa x} + \beta_k| + \mu_k, \\ G(u) &= 2A\ln|\beta_1\exp(\tfrac{u-\mu}{D}) - \beta_2\Big| - A_2\tfrac{u-\mu}{D} + \Lambda \end{aligned}$$

with $(B_1, B_2) = \frac{1+p}{2}(2\kappa A + B, B) + \frac{1-p}{2}(B, 2\kappa A + B)$ *and* $\gamma(\beta_1, \beta_2) = \frac{1+p}{2}(\alpha, -\beta) + \frac{1-p}{2}(-\beta, \alpha)$

(H2.5) *there exist* $|p| = 1$ *and* $B_k, C_k, D_k \in \mathbb{R}$ *with* $-\frac{p}{2\kappa}(B_2 - B_1) = A$ *such that, for all* $x \in I$ *and* $u \in g_1(I) + g_2(I)$*, we have*

$$f_k(x) = C_k e^{(-1)^k p\kappa x} - B_k x + \lambda_k,$$
$$g_k(x) = D_k e^{(-1)^k p\kappa x} + \mu_k, \quad G(u) = 2A\ln\left|\tfrac{u-\mu}{D}\right| + C\tfrac{u-\mu}{D} + \Lambda,$$

where $B = \frac{1+p}{2}B_2 + \frac{1-p}{2}B_1$ *and*

$$\alpha\beta(C, D) = -\tfrac{1+p}{2}\alpha(C_1, D_1) - \tfrac{1-p}{2}\alpha(C_2, D_2) = \tfrac{1+p}{2}\beta(C_2, D_2) + \tfrac{1-p}{2}\beta(C_1, D_1).$$

Proof. For brevity, in this proof, whenever we write $\psi_1 \equiv 0$, we mean that $\psi_1 = 0$ on the whole interval I. The sufficiency of the listed functions is a matter of substitution.

To show the necessity, let $(F, f_1, f_2, G, g_1, g_2)$ be a regular solution of (2.3). Then there exist constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$, $\kappa > 0$, and $\lambda^*, \mu^* \in \mathbb{R}$ such that

$$F'(x) = \frac{(d+c)e^{2\kappa x} + d - c}{(b+a)e^{2\kappa x} + b - a},$$
$$g_k'(x) = \frac{4e^{\kappa x}}{(\omega_{1,k} + \omega_{2,k})e^{2\kappa x} + 2\omega_{0,k}e^{\kappa x} + \omega_{1,k} - \omega_{2,k}},$$

and

$$f_k'(x) = -\frac{1}{2}\frac{(\theta_{1,k} + \theta_{2,k})e^{2\kappa x} + 2\theta_{0,k}e^{\kappa x} + \theta_{1,k} - \theta_{2,k}}{(\omega_{1,k} + \omega_{2,k})e^{2\kappa x} + 2\omega_{0,k}e^{\kappa x} + \omega_{1,k} - \omega_{2,k}}$$

hold for all $x \in I$, where

$$(\omega_{2,k}, \omega_{1,k}, \omega_0) := \begin{cases} (b, a, \lambda^*) & \text{if } \psi_1 \equiv 0, \\ (pb + (-1)^{k-1}a, pa + (-1)^{k-1}b, \lambda^*) & \text{otherwise} \end{cases}$$

and

$$(\theta_{2,k}, \theta_{1,k}, \theta_0) := \begin{cases} (d, c, 2\mu^*) & \text{if } \psi_1 \equiv 0, \\ (pd + (-1)^{k-1}c, pc + (-1)^{k-1}d, 2\mu^*) & \text{otherwise.} \end{cases}$$

Note that $p = 0$ corresponds to the case when the function ψ_2 is constant on I. On the other hand, if ψ_1 is different from zero and ψ_2 is not constant on I, by Lemma 15, parameter p is different from zero.

From the differential equation concerning F we directly get that there exist constants $\lambda, \lambda_0 \in \mathbb{R}$ such that

$$F(x) = \begin{cases} A\exp(-2q\kappa x) + 2Bx + \lambda & \text{if } |a| = |b|, \\ 2A\ln|\alpha_0\exp(2\kappa x) + \beta_0| + 2Bx + \lambda_0 & \text{if } |a| \neq |b|, \end{cases} \quad x \in I, \quad (4.8)$$

where $q:=\operatorname{sgn}(ab)\neq 0$, $\alpha_0:=a+b\neq 0$, $\beta_0:=b-a\neq 0$,

$$0\neq A:=\begin{cases}\frac{c-qd}{2\kappa(b+qa)} & \text{if } |a|=|b|,\\ \frac{bc-ad}{2\kappa(b^2-a^2)} & \text{if } |a|\neq|b|,\end{cases} \quad\text{and}\quad B:=\begin{cases}\frac{d+qc}{2(b+qa)} & \text{if } |a|=|b|,\\ \frac{d-c}{2(b-a)} & \text{if } |a|\neq|b|.\end{cases} \tag{4.9}$$

Correspondingly, we are going to distinguish the following main cases: either $|a|=|b|$ or $|a|\neq|b|$.

Case 1. Assume first that $|a|=|b|$ *is valid.* In this case, the form of g_k and f_k strongly depends on the value of the coefficient ω_0, therefore, within Case 1., we will distinguish two sub-cases: either $\omega_0=0$ or $\omega_0\neq 0$.

(1.1) Assume that $\omega_0=0$ *holds.* Then we must have $|p|\neq 1$whenever ψ_1 is not zero on I. Indeed, our assumption $|a|=|b|$ and condition $ad\neq bc$ imply that either $a=b$ or $a=-b$ holds. Consequently, for a given $k=1,2$, at least one of $\omega_{1,k}+\omega_{2,k}=(p+(-1)^{k-1})(a+b)$ and $\omega_{1,k}-\omega_{2,k}=(p-(-1)^{k-1})(a-b)$ must be zero. If $|p|=1$ were true, then, it is easy to see, that there would exist $k\in\{1,2\}$, for which both of the previous coefficients are zero. It would then follow that exactly one of the functions $\psi_2-\psi_1$ or $\psi_2+\psi_1$ is identically zero on I, which, in view of Remark 18, is impossible.

Thus there exist $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$g_k(x):=qD_ke^{-q\kappa x}+\mu_k \quad\text{and}\quad f_k(x):=-A_ke^{-2q\kappa x}-Bx-qC_ke^{-q\kappa x}+\lambda_k,$$

hold for all $x\in I$, where

$$(C,D):=\begin{cases}\frac{1}{\kappa(a+qb)}\big(2\mu^*,-4\big) & \text{if } \psi_1\equiv 0,\\ \frac{1}{\kappa(a+qb)p(p^2-1)}\big(2\mu^*,4p\big) & \text{otherwise,}\end{cases}$$

and $-\frac{1}{2}p^*A=q_k^2A_k$, $p^*C=q_kC_k$, and $p^*D=q_kD_k$, with

$$(p^*,q_k):=\begin{cases}(1,1) & \text{if } \psi_1\equiv 0\\ (1-p^2,q+(-1)^{k-1}p) & \text{otherwise.}\end{cases} \tag{4.10}$$

Observe that we have $0\notin\{q-p,q+p\}$. Now, substituting $(\xi,\eta):=(e^{-q\kappa x},e^{-q\kappa y})$ if $\psi_1\equiv 0$ holds or $(\xi,\eta):=((q-p)e^{-q\kappa x},(q+p)e^{-q\kappa y})$ otherwise, furthermore putting $\mu:=\mu_1+\mu_2$ and $\Lambda:=\lambda+\lambda_1+\lambda_2$, we get that

$$\begin{aligned}G(qD(\xi+\eta)+\mu) &= Ae^{-q\kappa(x+y)}+B(x+y)+\lambda\\ &\quad -A_1e^{-2q\kappa x}-Bx-qC_1e^{-q\kappa x}+\lambda_1\\ &\quad -A_2e^{-2q\kappa y}-By-qC_2e^{-q\kappa y}+\lambda_2\\ &=\begin{cases}\frac{1}{2}A(\xi+\eta)^2+qC(\xi+\eta)+\Lambda & \text{if } \psi_1\equiv 0,\\ \frac{1}{2(1-p^2)}A(\xi+\eta)^2+qC(\xi+\eta)+\Lambda & \text{otherwise.}\end{cases}\end{aligned}$$

Consequently, for $u:=qD(\xi+\eta)+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=\frac{1}{2p^*}A\left(\frac{u-\mu}{D}\right)^2+C\frac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (H1.2).

(1.2) *Now assume that* $\omega_0 \neq 0$ *holds.* By Case 1. and condition $ad \neq bc$, we have $a \neq -qb$ and $d \neq qc$. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = D \ln |\alpha e^{-q\kappa x} + \beta_k| + \mu_k \quad \text{and}$$
$$f_k(x) = qA_k e^{-q\kappa x} - Bx + C \ln |\alpha e^{-q\kappa x} + \beta_k| + \lambda_k$$

hold for all $x \in I$, where $\alpha{:=}2\lambda^* \neq 0$, $D{:=} - q\frac{2}{\kappa\lambda^*} \neq 0$,

$$C{:=}\frac{q}{4\kappa\lambda^*} \cdot \begin{cases} 4\mu^* - 4\lambda^* B - \frac{1}{2\lambda^*}(a+qb)(c-qd) & \text{if } \psi_1 \equiv 0, \\ 4\mu^* - 4\lambda^* B - \frac{1}{2\lambda^*}(p^2-1)(b+qa)(d-qc) & \text{otherwise,} \end{cases}$$

$$0 \neq A_k{:=}\frac{1}{4\kappa\lambda^*} \cdot \begin{cases} c - qd & \text{if } \psi_1 \equiv 0, \\ ((-1)^{k-1}p - q)(d-qc) & \text{otherwise,} \end{cases}$$

and

$$0 \neq \beta_k{:=} \begin{cases} a + qb & \text{if } \psi_1 \equiv 0, \\ ((-1)^{k-1}p + q)(b+qa) \neq 0 & \text{otherwise.} \end{cases}$$

Note that, regardless of the behavior of the function ψ_1, the identities $\beta_2 A = q\alpha A_1$ and $\beta_1 A = q\alpha A_2$ hold. Hence, substituting $\xi{:=}\alpha e^{-q\kappa x} + \beta_1$, $\eta{:=}\alpha e^{-q\kappa y} + \beta_2$, $\mu{:=}\mu_1 + \mu_2$, and $\Lambda{:=}\lambda + \lambda_1 + \lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(D \ln |\xi\eta| + \mu) &= Ae^{-q\kappa(x+y)} + B(x+y) + \lambda \\ &\quad + qA_1 e^{-q\kappa x} - Bx + C \ln |\alpha e^{-q\kappa x} + \beta_1| + \lambda_1 \\ &\quad + qA_2 e^{-q\kappa y} - By + C \ln |\alpha e^{-q\kappa y} + \beta_2| + \lambda_2 \\ &= \tfrac{1}{\alpha^2} A(\xi\eta - \beta_1\beta_2) + C \ln |\xi\eta| + \Lambda. \end{aligned}$$

Consequently, for $u{:=}D \ln |\xi\eta| + \mu \in g_1(I) + g_2(I)$, we get

$$G(u) = \alpha^{-2} A(\exp(\tfrac{u-\mu}{D}) - \beta_1\beta_2) + C\tfrac{u-\mu}{D} + \Lambda.$$

Thus we obtained the solutions listed in (H1.1).

Case 2. Assume now that $|a| \neq |b|$ *holds.* Here we will distinguish two further cases: either $|p| = 1$ or $|p| \neq 1$.

(2.1) *Suppose that* $|p| = 1$ *holds.* Then, it is easy to see, that the shape of the corresponding solutions still depends on the value of ω_0.

Within sub-case (2.1), *suppose that* $\omega_0 = 0$. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = D_k e^{(-1)^k p\kappa x} + \mu_k \quad \text{and} \quad f_k(x) = C_k e^{(-1)^k p\kappa x} - B_k x + \lambda_k$$

hold for all $x \in I$, with

$$B_k{:=}\frac{d+(-1)^{k-1}pc}{2(b+(-1)^{k-1}pa)}, \quad \text{and}$$
$$\alpha\beta(C,D) = \begin{cases} \beta(C_1,D_1) = \alpha(C_2,D_2) & \text{if } p=-1 \\ \alpha(C_1,D_1) = \beta(C_2,D_2) & \text{if } p=1, \end{cases}$$

where $\alpha{:=}\alpha_0$, $\beta{:=}-\beta_0 \in \mathbb{R}$, $C{:=}\frac{p\mu^*}{\kappa(b^2-a^2)}$, and $D{:=}-\frac{2}{\kappa(b^2-a^2)} \neq 0$.

Observe that $-\frac{p}{2\kappa}(B_2-B_1) = A \neq 0$, and either $B_1 = B$ or $B_2 = B$ if either $p=-1$ or $p=1$, respectively, where $A, B \in \mathbb{R}$ are defined in (4.9). Substituting either $(\xi,\eta){:=}(\alpha e^{\kappa x}, \beta e^{-\kappa y})$ or $(\xi,\eta){:=}(\beta e^{-\kappa x}, \alpha e^{\kappa y})$ if either $p=-1$ or $p=1$, respectively, furthermore putting $\mu{:=}\mu_1+\mu_2$ and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned} G(D(\xi+\eta)+\mu) &= 2A\ln|\alpha e^{\kappa(x+y)}-\beta| + B(x+y)+\lambda \\ &\quad +C_1e^{-p\kappa x} - B_1x + \lambda_1 + C_2e^{p\kappa y} - B_2y + \lambda_2 \\ &= 2A\ln|\alpha e^{\kappa(x+y)}-\beta| \\ &\quad +(B-B_1)x + (B-B_2)y + C(\xi+\eta) + \Lambda. \end{aligned}$$

Thus, depending on the exact value of p, we have

$$\begin{aligned} G(D(\xi+\eta)+\mu) &= \begin{cases} 2A\ln|\alpha e^{\kappa(x+y)}-\beta| - 2A\kappa y + C(\xi+\eta)+\Lambda & \text{if } p=-1 \\ 2A\ln|\alpha e^{\kappa(x+y)}-\beta| - 2A\kappa x + C(\xi+\eta)+\Lambda & \text{if } p=1 \end{cases} \\ &= 2A\ln|\xi+\eta| + C(\xi+\eta)+\Lambda. \end{aligned}$$

This, for $u{:=}D(\xi+\eta)+\mu \in g_1(I)+g_2(I)$, yields that

$$G(u) = 2A\ln|\tfrac{u-\mu}{D}| + C\tfrac{u-\mu}{D} + \Lambda.$$

Thus we obtained the solutions listed in (H2.5).

Within sub-case (2.1), assume that $\omega_0 = \lambda^* \neq 0$. Then there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = (-1)^kD\ln|\gamma e^{(-1)^kp\kappa x} + \beta_k| + \mu_k \quad \text{and}$$
$$f_k(x) = -A_k\ln|\gamma e^{(-1)^kp\kappa x} + \beta_k| - B_kx + \lambda_k$$

hold for all $x \in I$, where $D{:=}\frac{2p}{\kappa\lambda^*} \neq 0$, $\gamma{:=}2\lambda^* \neq 0$,

$$A_k{:=}\tfrac{(-1)^kp}{2\kappa\lambda^*}\left(2\mu^* - \lambda^*\tfrac{d+(-1)^{k-1}pc}{b+(-1)^{k-1}pa}\right),$$
$$B_k{:=}\tfrac{d+(-1)^{k-1}pc}{2(b+(-1)^{k-1}pa)}, \quad \text{and} \quad \beta_k{:=}2((-1)^{k-1}pb+a) \neq 0.$$

Note that $\frac{1}{2}(A_1+A_2) = A$, and that the function F defined in (4.8) can be written as

$$F(x) = 2A\ln|\alpha e^{2\kappa x} - \beta| + 2Bx + \lambda,$$

where $\lambda := \lambda_0 - 2A\ln|4\lambda^*|$, and either $(\alpha,\beta) = \gamma(\beta_2, -\beta_1)$ or $(\alpha,\beta) = \gamma(\beta_1, -\beta_2)$ if either $p = -1$ or $p = 1$, respectively. Substituting $\xi := \gamma e^{-p\kappa x} + \beta_1$, $\eta := \gamma e^{p\kappa y} + \beta_2$, $\mu := \mu_1 + \mu_2$, and $\Lambda := \lambda + \lambda_1 + \lambda_2$, Eq. (2.3) reduces to

$$G(D\ln|\tfrac{\eta}{\xi}| + \mu) = \begin{cases} 2A\ln|\alpha e^{\kappa(x+y)} - \beta| + B(x+y) \\ -A_1\ln|\xi| - B_1 x - A_2\ln|\eta| - B_2 y + \Lambda & \text{if } p = -1, \\ 2A\ln|\alpha e^{\kappa(x+y)} - \beta| + B(x+y) \\ -A_1\ln|\xi| - B_1 x - A_1\ln|\eta| - B_2 x + \Lambda & \text{if } p = 1, \end{cases}$$

$$= \begin{cases} 2A\ln|\beta_1\tfrac{\eta}{\xi} - \beta_2| - A_2\ln|\tfrac{\eta}{\xi}| + (B - B_1)x + (2\kappa A + B - B_2)y + \Lambda & \text{if } p = -1, \\ 2A\ln|\beta_1\tfrac{\eta}{\xi} - \beta_2| - A_2\ln|\tfrac{\eta}{\xi}| + (B - B_2)y + (2\kappa A + B - B_1)x + \Lambda & \text{if } p = 1. \end{cases}$$

A short calculation shows that we have either $B_1 = B$ and $B_2 = 2\kappa A + B$ or $B_1 = 2\kappa A + B$ and $B_2 = B$ if either $p = -1$ or $p = 1$, respectively. Consequently, for $u := D\ln|\frac{\eta}{\xi}| + \mu \in g_1(I) + g_2(I)$, we get

$$G(u) = 2A\ln|\beta_1\exp(\tfrac{u-\mu}{D}) - \beta_2| - A_2\tfrac{u-\mu}{D} + \Lambda.$$

Thus we obtained the solution listed in (H2.4).

(2.2) *Suppose that* $|p| \neq 1$ *holds.* Then we will distinguish three sub-cases depending on the sign of the discriminant

$$\Delta := 4(\lambda^*)^2 - 4\cdot\begin{cases} a^2 - b^2 & \text{if } \psi_1 \equiv 0, \\ (p^2-1)(a^2-b^2) & \text{otherwise} \end{cases}$$

of the second degree polynomial in the denominator of g_k.

Within sub-case (2.2), *suppose that* $\Delta < 0$ holds. Then there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$g_k(x) = D\arctan(\alpha_k e^{\kappa x} + \gamma) + \mu_k \quad \text{and}$$
$$f_k(x) = -A\ln|(\alpha_k e^{\kappa x} + \gamma)^2 + 1| - Bx + C\arctan(\alpha_k e^{\kappa x} + \gamma) + \lambda_k$$

hold for all $x \in I$, where $\alpha_k := \frac{2}{\sqrt{-\Delta}}(a+b) \neq 0$ if $\psi_1 \equiv 0$ and $\alpha_k := \frac{2}{\sqrt{-\Delta}}(p + (-1)^{k-1})(a+b) \neq 0$ otherwise, furthermore $\gamma := \frac{2\lambda^*}{\sqrt{-\Delta}} \neq 0$,

$$C := -\frac{1}{\kappa\sqrt{-\Delta}}\Big(2\mu^* - \lambda^*\frac{ac-bd}{a^2-b^2}\Big), \quad \text{and}$$
$$D := \frac{8}{\kappa\sqrt{-\Delta}} \neq 0.$$

Observe that F can be reformulated as

$$F(x) = 2A\ln|\alpha e^{2\kappa x} - \beta| + 2Bx + \lambda,$$

where $\alpha := \alpha_1\alpha_2$, $\beta := \gamma^2 + 1$, and $\lambda := \lambda_0 + 2A\ln\big|\frac{\Delta}{4(a+b)}\big|$ or $\lambda := \lambda_0 + 2A\ln\big|\frac{\Delta}{4(p^2-1)(a+b)}\big|$ depending on whether $\psi_1 \equiv 0$ or not, respectively. Hence, substituting $\xi := \alpha_1 e^{\kappa x} +$

γ, $\eta{:=}\alpha_2 e^{\kappa y}+\gamma$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned}
&G(D\arctan(\tfrac{\xi+\eta}{1-\xi\eta})+\mu)\\
&\quad=2A\ln|\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma^2-1|+B(x+y)+\lambda\\
&\qquad-A\ln|(\alpha_1 e^{\kappa x}+\gamma)^2+1|-Bx+C\arctan(\alpha_1 e^{\kappa x}+\gamma)+\lambda_1\\
&\qquad-A\ln|(\alpha_2 e^{\kappa y}+\gamma)^2+1|-By+C\arctan(\alpha_2 e^{\kappa y}+\gamma)+\lambda_2\\
&\quad=A\ln|(\gamma\tfrac{\xi+\eta}{1-\xi\eta}+1)^2((\tfrac{\xi+\eta}{1-\xi\eta})^2+1)^{-1}|+C\arctan(\tfrac{\xi+\eta}{1-\xi\eta})+\Lambda,
\end{aligned}$$

where in the last step, we used the identity

$$\begin{aligned}
&\frac{(\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma^2-1)^2}{(\xi^2+1)(\eta^2+1)}\\
&\quad=\frac{(1-\xi\eta+\gamma(\xi+\eta))^2}{(1-\xi\eta)^2+(\xi+\eta)^2}=\Big(\gamma\frac{\xi+\eta}{1-\xi\eta}+1\Big)^2\Big(\Big(\frac{\xi+\eta}{1-\xi\eta}\Big)^2+1\Big)^{-1}.
\end{aligned}$$

Consequently, for $u{:=}D\arctan(\frac{\xi+\eta}{1-\xi\eta})+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln|\gamma\sin(\tfrac{u-\mu}{D})+\cos(\tfrac{u-\mu}{D})|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (H2.2).

Within sub-case (2.2), *suppose that* $\Delta=0$ holds. Then there exist $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$\begin{aligned}
g_k(x)&=D(\alpha_k e^{\kappa x}+\gamma)^{-1}+\mu_k\quad\text{and}\\
f_k(x)&=-2A\ln|\alpha_k e^{\kappa x}+\gamma|-Bx+C(\alpha_k e^{\kappa x}+\gamma)^{-1}+\lambda_k
\end{aligned}$$

hold for all $x\in I$, where $A,B\in\mathbb{R}$ are defined in (4.9),

$$C{:=}\frac{1}{\kappa}\Big(2\mu^*-\lambda^*\frac{ac-bd}{a^2-b^2}\Big),\quad D{:=}-\frac{4}{\kappa}\neq 0,\quad \gamma{:=}\lambda^*\neq 0,$$

furthermore either $\alpha_k{:=}a+b\neq 0$ or $\alpha_k{:=}(p+(-1)^{k-1})(a+b)\neq 0$ if $\psi_1\equiv 0$ or not, respectively.

Observe that, due to our assumption $\Delta=0$, the function F can be written as

$$F(x)=2A\ln|\alpha e^{2\kappa x}-\beta|+2Bx+\lambda,$$

where $\alpha{:=}\alpha_1\alpha_2$, $\beta{:=}\gamma^2$, and either $\lambda{:=}\lambda_0-2A\ln|a+b|$ or $\lambda{:=}\lambda_0-A\ln|(p^2-1)(a+b)|$ if $\psi_1\equiv 0$ or not, respectively. Hence, substituting $\xi{:=}\alpha_1 e^{\kappa x}+\gamma$ and $\eta{:=}\alpha_2 e^{\kappa y}+\gamma$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, Eq. (2.3) reduces to

$$\begin{aligned}
G\big(D\tfrac{\xi+\eta}{\xi\eta}+\mu\big)&=2A\ln|\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma^2|+B(x+y)+\lambda\\
&\quad-2A\ln|\alpha_1 e^{\kappa x}+\gamma|-Bx+C(\alpha_1 e^{\kappa x}+\gamma)^{-1}+\lambda_1\\
&\quad-2A\ln|\alpha_2 e^{\kappa y}+\gamma|-By+C(\alpha_2 e^{\kappa y}+\gamma)^{-1}+\lambda_2\\
&=2A\ln|1-\gamma\tfrac{\xi+\eta}{\xi\eta}|+C\tfrac{\xi+\eta}{\xi\eta}+\Lambda.
\end{aligned}$$

Consequently, for $u{:=}D\frac{\xi+\eta}{\xi\eta}+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln|1-\gamma\tfrac{u-\mu}{D}|+C\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (H2.1).

Finally, suppose that $\Delta>0$ *holds.* Then there exist constants $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ such that

$$\begin{aligned}g_k(x)&=D\ln\left|\tfrac{\alpha_k e^{\kappa x}+\gamma_1}{\alpha_k e^{\kappa x}+\gamma_2}\right|+\mu_k\quad\text{and}\\ f_k(x)&=-A_1\ln|\alpha_k e^{\kappa x}+\gamma_1|-A_2\ln|\alpha_k e^{\kappa x}+\gamma_2|-Bx+\lambda_k\end{aligned}$$

hold for all $x\in I$, where either $\alpha_k=2(a+b)$ or $\alpha_k{:=}2(p+(-1)^{k-1})(a+b)$ if $\psi_1\equiv 0$ or not, respectively, furthermore $\gamma_k{:=}2\lambda^*+(-1)^k\sqrt{\Delta}$,

$$A_k{:=}A+\frac{(-1)^{k-1}}{\kappa\sqrt{\Delta}}\Big(2\mu^*-\lambda^*\frac{ac-bd}{a^2-b^2}\Big),\quad\text{and}\quad D{:=}\frac{4}{\kappa\sqrt{\Delta}}\neq 0.$$

Observe that $\gamma_1\gamma_2$ cannot be zero, $\frac{1}{2}(A_1+A_2)=A\neq 0$, and that the function F can be written as

$$F(x)=2A\ln|\alpha e^{2\kappa x}-\beta|+2Bx+\lambda,\quad x\in I,$$

where $\alpha{:=}\alpha_1\alpha_2$, $\beta{:=}\gamma_1\gamma_2$, and either $\lambda{:=}\lambda_0-2A\ln|4(a+b)|$ or $\lambda{:=}\lambda_0-2A\ln|4(p^2-1)(a+b)|$ if $\psi_1\equiv 0$ or not, respectively. Hence, substituting $\xi_k{:=}\alpha_1 e^{\kappa x}+\gamma_k$, $\eta_k{:=}\alpha_2 e^{\kappa y}+\gamma_k$, $\mu{:=}\mu_1+\mu_2$, and $\Lambda{:=}\lambda+\lambda_1+\lambda_2$, and using that $\gamma_2-\gamma_1=2\sqrt{\Delta}\neq 0$, Eq. (2.3) reduces to

$$\begin{aligned}&G(D\ln|\tfrac{\xi_1\eta_1}{\xi_2\eta_2}|+\mu)\\ &\quad=2A\ln|\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma_1\gamma_2|+B(x+y)+\lambda\\ &\qquad-A_1\ln|\alpha_1 e^{\kappa x}+\gamma_1|-A_2\ln|\alpha_1 e^{\kappa x}+\gamma_2|-Bx+\lambda_1\\ &\qquad-A_1\ln|\alpha_2 e^{\kappa y}+\gamma_1|-A_2\ln|\alpha_2 e^{\kappa y}+\gamma_2|-By+\lambda_2\\ &\quad=2A\ln|\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma_1\gamma_2|\\ &\qquad-A_1\ln|\xi_1|+(A_1-2A)\ln|\xi_2|-A_1\ln|\eta_1|+(A_1-2A)\ln|\eta_2|+\Lambda\\ &\quad=2A\ln\left|\tfrac{\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma_1\gamma_2}{\xi_2\eta_2}\right|-A_1\ln\left|\tfrac{\xi_1\eta_1}{\xi_2\eta_2}\right|+\Lambda\\ &\quad=2A\ln\left|\tfrac{\gamma_2}{\gamma_2-\gamma_1}\tfrac{\xi_1\eta_1}{\xi_2\eta_2}-\tfrac{\gamma_1}{\gamma_2-\gamma_1}\right|-A_1\ln\left|\tfrac{\xi_1\eta_1}{\xi_2\eta_2}\right|+\Lambda,\end{aligned}$$

where, in the last step, we used the identity

$$\alpha_1\alpha_2 e^{\kappa(x+y)}-\gamma_1\gamma_2=\tfrac{\gamma_2}{\gamma_2-\gamma_1}\xi_1\eta_1-\tfrac{\gamma_1}{\gamma_2-\gamma_1}\xi_2\eta_2.$$

Consequently, for $u{:=}D\ln|\frac{\xi_1\eta_1}{\xi_2\eta_2}|+\mu\in g_1(I)+g_2(I)$, we get

$$G(u)=2A\ln\left|\tfrac{\gamma_2}{\gamma_2-\gamma_1}\exp(\tfrac{u-\mu}{D})-\tfrac{\gamma_1}{\gamma_2-\gamma_1}\right|-A_1\tfrac{u-\mu}{D}+\Lambda.$$

Thus we obtained the solutions listed in (H2.3). □

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