On optimal inequalities between three-point quadratures

ANDRZEJ KOMISARSKI AND SZYMON WĄSOWICZ

Abstract. We examine the family of all (at most) three-point symmetric quadratures on $[-1, 1]$ which are exact on polynomials of order 3 to find all possible inequalities between them in the class of 3-convex functions. Next we optimise them by using convex combinations of the quadratures considered. We find the optimal quadrature and use it to construct the adaptive method of approximate integration. An effective method to estimate the error of this method is also given. It needs a considerably fewer number of subdivisions of the interval of integration than the classical adaptive methods as well as the method developed by the second-named author in his recent paper (Wąsowicz in Aequ Math 94(5):887–898, 2020).

Mathematics Subject Classification. Primary: 41A55, 41A80, 65D30, 65D32; Secondary: 26A51, 26D15.

Keywords. Approximate integration, Quadratures, Adaptive methods, Chebyshev's quadrature, Simpson's rule, 3-Convexity.

1. Symmetric three-point quadratures

The object of our investigation is the family of all symmetric (at-most) three-point quadratures on $[-1, 1]$ of the general form $Q_z[f] = A_z f(0) + B_z (f(z) + f(-z))$ for some $z \in (0, 1]$, which are exact on Π_3 (the space of all polynomials of order 3), i.e.

$$Q[p] = \int_{-1}^1 p(x) \, dx$$

for any $p \in \Pi_3$. We additionally require the operator Q_z to be positive, which means that the weights A_z, B_z are nonnegative. To determine them it is enough to check the above equation for monomials $1, x, x^2, x^3$. Then we simply arrive at

$$Q_z[f] = 2 \left(1 - \frac{1}{3z^2} \right) f(0) + \frac{1}{3z^2} (f(-z) + f(z)), \quad z \in \left[\frac{\sqrt{3}}{3}, 1 \right].$$

The classical quadratures are members of this family:

$$Q_{\frac{\sqrt{3}}{3}}[f] = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \\ \text{(the two-point Gauss-Legendre quadrature),}$$

$$Q_{\frac{\sqrt{2}}{2}}[f] = \frac{2}{3}\left(f(0) + f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right) \quad \text{(the Chebyshev quadrature),}$$

$$Q_{\frac{\sqrt{15}}{5}}[f] = \frac{8}{9}f(0) + \frac{5}{9}\left(f\left(-\frac{\sqrt{15}}{5}\right) + f\left(\frac{\sqrt{15}}{5}\right)\right) \\ \text{(the three-point Gauss-Legendre quadrature),}$$

$$Q_1[f] = \frac{4}{3}f(0) + \frac{1}{3}(f(-1) + f(1)) \quad \text{(the simple Simpson's Rule).}$$

Our first goal is to find all possible inequalities of the form

$$Q_{z_1}[f] \leq \int_{-1}^1 f(x) dx \leq Q_{z_2}[f] \quad (1)$$

in the class of all $\mathcal{B}$ -convex functions $f: [-1, 1] \rightarrow \mathbb{R}$, i.e. the functions satisfying the inequality

$$[x_0, x_1, x_2, x_3, x_4; f] := \frac{\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ f(x_0) & f(x_1) & f(x_2) & f(x_3) & f(x_4) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 & x_4 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_0^4 & x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{vmatrix}} \geq 0$$

for all distinct points $x_0, x_1, x_2, x_3, x_4 \in [-1, 1]$. Of course the 3-convexity of a function defined on any interval $\mathcal{I} \subset \mathbb{R}$ is defined analogously, by replacing $[-1, 1]$ with $\mathcal{I}$. If f is four times differentiable on $\mathcal{I}$, then f is 3-convex on $\mathcal{I}$ if and only if $f^{(4)} \geq 0$ on $\mathcal{I}$. Nevertheless, there exist 3-convex functions which are not differentiable enough, e.g. $f(x) = x_+^3$ (where $x_+ = \max\{x, 0\}$) is 3-convex and $f'''(0)$ does not exist. A function f is $\mathcal{B}$ -concave, if $-f$ is 3-convex. Recall that the concept of higher-order convexity was introduced by Hopf [2] and further investigated by Popoviciu [5]. A nice exposition can be found in Kuczma's book [4], which is easily available.

It follows from Bojanić and Roulier's result ([1], see also [7]) that any 3-convex function could be uniformly approximated by (3-convex) splines of the form

$$x \mapsto ax^3 + bx^2 + cx + d + \sum_{i=1}^n a_i(x - x_i)_+^3$$

with $a, b, c, d \in \mathbb{R}$ and $a_i > 0$, $i = 1, \dots, n$. Hence to prove the inequality of the form (1) it is enough to check it for any spline $x \mapsto (x - u)_+^3$ for any $u \in (-1, 1)$.

In the next lemmas we record the basic properties of the quadratures Q_z .

Lemma 1. *Let $f: [-1, 1] \rightarrow \mathbb{R}$ be a 3-convex function, $z \in \left[\frac{\sqrt{3}}{3}, 1\right]$. The function*

$$[-1, 1] \ni a \mapsto Q_z[(\cdot - a)_+^3] - \int_{-1}^1 (x - a)_+^3 dx$$

is even.

Proof. Because the quadrature Q_z is symmetric, the lemma follows directly from our recent result [3, Lemma 2]). For the sake of completeness we give the proof here. Let $f_a(x) = (x - a)_+^3$ and $g_a(x) = (a - x)_+^3$. It is easy to see that $f_a(x) - g_a(x) = (x - a)^3$ and $g_a(x) = f_{-a}(-x)$. Therefore

$$\begin{aligned} Q_z[f_a] - \int_{-1}^1 f_a(x) dx &= \left(Q_z[f_a - g_a] - \int_{-1}^1 (f_a - g_a)(x) dx \right) \\ &\quad + Q_z[g_a] - \int_{-1}^1 g_a(x) dx. \end{aligned}$$

The term in the bracket vanishes, since Q_z is exact on Π_3 . Because Q_z as well as the integral over $[-1, 1]$ are symmetric functionals, we have

$$Q_z[f_a] - \int_{-1}^1 f_a(x) dx = Q_z[g_a] - \int_{-1}^1 g_a(x) dx = Q_z[f_{-a}] - \int_{-1}^1 f_{-a}(x) dx,$$

which completes the proof. $\square$

Lemma 2. *Let $f: [-1, 1] \rightarrow \mathbb{R}$ be a 3-convex function. The function*

$$\left[\frac{\sqrt{3}}{3}, 1\right] \ni z \mapsto Q_z[f]$$

is nondecreasing.

Proof. Having Bojanić and Roulier's result in mind it is enough to check our assertion only for the functions $(\cdot - a)_+^3$ for all $a \in (-1, 1)$. Observe that $z \mapsto Q_z[(\cdot - a)_+^3]$ is nondecreasing if and only if the function

$$z \mapsto Q_z[(\cdot - a)_+^3] - \int_{-1}^1 (x - a)_+^3 dx$$

is nondecreasing. Indeed, the integral above does not depend on z . By Lemma 1 we should only show that the function $z \mapsto Q_z[(\cdot - a)_+^3]$, where $a \in [0, 1)$, is nondecreasing. This is obviously true because

$$\begin{aligned} \left(Q_z[(\cdot - a)_+^3] \right)'_z &= \left(\frac{(z - a)_+^3}{3z^2} \right)'_z = \frac{3(z - a)_+^2 \cdot 3z^2 - (z - a)_+^3 \cdot 6z}{9z^4} \\ &= \frac{(z - a)_+^2 (3z - 2(z - a)_+)}{3z^3} \geq 0. \end{aligned}$$

□

Lemma 3. *The inequality*

$$\int_{-1}^1 f(x) dx \leq Q_z[f]$$

holds for all 3-convex functions $f: [-1, 1] \rightarrow \mathbb{R}$ if and only if $z = 1$.

Proof. If $z < 1$, then for a 3-convex function $f(x) = (x - z)_+^3$ we have

$$\int_{-1}^1 f(x) dx > 0 = Q_z[f]$$

and the inequality in question does not hold for all 3-convex functions. Take now $z = 1$. Due to Lemma 1 it is enough to check the inequality

$$\int_{-1}^1 (x - a)_+^3 dx \leq Q_1[(\cdot - a)_+^3]$$

for $a \in [0, 1)$. In this case the above inequality is trivial because it reduces to

$$\frac{(1 - a)^4}{4} \leq \frac{(1 - a)^3}{3}.$$

□

Lemma 4. *The inequality*

$$Q_z[f] \leq \int_{-1}^1 f(x) dx$$

holds for all 3-convex functions $f: [-1, 1] \rightarrow \mathbb{R}$ if and only if $\frac{\sqrt{3}}{3} \leq z \leq \frac{3}{4}$.

Proof. By taking $f(x) = x_+^3$ we get the necessity (we omit an easy computation). Now let $\frac{\sqrt{3}}{3} \leq z \leq \frac{3}{4}$. As in the previous proofs, it is enough to check the inequality

$$Q_z[(\cdot - a)_+^3] \leq \int_{-1}^1 (x - a)_+^3 dx$$

for all $a \in [0, 1)$. If $a \geq \frac{3}{4}$, then

$$Q_z[(\cdot - a)_+^3] = 0 \leq \int_{-1}^1 (x - a)_+^3 dx.$$

If $0 \leq a < \frac{3}{4}$, then

$$\begin{aligned} \int_{-1}^1 (x - a)_+^3 dx &= \frac{(1 - a)^4}{4} = \frac{16}{27} \left(\frac{3}{4} - a \right)^3 + \left(\frac{a^2}{2} - \frac{11}{27}a \right)^2 \\ &\quad + \frac{a^2}{2 \cdot 27^2} \geq \frac{16}{27} \left(\frac{3}{4} - a \right)^3 \\ &= \frac{1}{3 \cdot \left(\frac{3}{4} \right)^2} \left(\frac{3}{4} - a \right)^3 = Q_{\frac{3}{4}}[(\cdot - a)_+^3] \geq Q_z[(\cdot - a)_+^3] \end{aligned}$$

because we have $z \leq \frac{3}{4}$ and Lemma 2. $\square$

Lemmas 2, 3, 4 lead us to the following conclusions.

Corollary 5. *The inequality*

$$Q_{z_1}[f] \leq \int_{-1}^1 f(x) dx \leq Q_{z_2}[f]$$

is valid for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$ if and only if $\frac{\sqrt{3}}{3} \leq z_1 \leq \frac{3}{4}$ and $z_2 = 1$.

Corollary 6. *The inequality*

$$Q_{\frac{3}{4}}[f] \leq \int_{-1}^1 f(x) dx \leq Q_1[f]$$

is the best lower bound and the best upper bound of the integral of a 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$ in the class of the quadratures Q_z , where $\frac{\sqrt{3}}{3} \leq z \leq 1$.

Corollary 7. *Let $\frac{\sqrt{3}}{3} \leq z \leq 1$. Then the inequality*

$$Q_{\frac{\sqrt{3}}{3}}[f] \leq Q_z[f] \leq Q_1[f]$$

holds for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$.

The second-named author proved in [8] (see Theorem 14) that if $\mathcal{T}$ is a positive linear functional defined (at least) on a linear subspace of $\mathbb{R}^{[-1,1]}$ generated by a cone of 3-convex functions, which coincides with the integral functional on Π_3 , then the inequality

$$Q_{\frac{\sqrt{3}}{3}}[f] \leq \mathcal{T}[f] \leq Q_1[f]$$

holds for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$. This result is considerably more general than our Corollary 7, however we are convinced that the present results do agree with our earlier research.

2. Improving the inequalities between quadratures Q_z

Let $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$. Corollary 5 gives us the inequality

$$Q_z[f] \leq \int_{-1}^1 f(x) dx \leq Q_1[f] \quad (2)$$

for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$. In the class of all quadratures Q_z it could be improved only by taking $Q_{\frac{3}{4}}$ instead of Q_z . Another possible direction for further improvement seems to be taking the convex combinations of Q_z and Q_1 . Then fix $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ and consider the quadrature $tQ_z + (1-t)Q_1$ with $t \in [0, 1]$. In this section we will try to compare it with the integral functional in the class of 3-convex functions. First we show that it is impossible to improve the lower bound.

Theorem 8. *Let $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ and $t \in [0, 1]$. If the inequality*

$$tQ_z[f] + (1-t)Q_1[f] \leq \int_{-1}^1 f(x) dx$$

holds for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$, then $t = 1$.

Proof. We have

$$\begin{aligned} \frac{1-t}{3} &= \lim_{a \rightarrow 1^-} \frac{t \cdot 0 + (1-t) \cdot \frac{(1-a)^3}{3}}{(1-a)^3} \\ &= \lim_{a \rightarrow 1^-} \frac{t \cdot Q_z[(\cdot - a)_+^3] + (1-t) \cdot Q_1[(\cdot - a)_+^3]}{(1-a)^3} \\ &\leq \lim_{a \rightarrow 1^-} \frac{1}{(1-a)^3} \int_{-1}^1 (x-a)_+^3 dx = \lim_{a \rightarrow 1^-} \frac{(1-a)^4}{4(1-a)^3} = 0. \end{aligned}$$

Hence $t \geq 1$, so, by $t \in [0, 1]$, we arrive at $t = 1$. $\square$

Then we could only expect the inequalities of the form

$$Q_z[f] \leq \int_{-1}^1 f(x) dx \leq tQ_z[f] + (1-t)Q_1[f].$$

If $t = 0$, we have Q_1 on the right and the inequality holds. Then, with fixed $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$, we will increase t to the optimal value t_z , above which the right inequality is not valid.

Theorem 9. Let $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$. There exists $t_z \in [0, 1)$ satisfying the following condition: the inequality

$$Q_z[f] \leq \int_{-1}^1 f(x) dx \leq tQ_z[f] + (1-t)Q_1[f]$$

holds for all 3-convex functions $f: [-1, 1] \rightarrow \mathbb{R}$ if and only if $0 \leq t \leq t_z$. Moreover, the polynomial $s \mapsto 4z^2s^3 - 4zs^2 + (1-z)s + (1-z)$ admits in $[0, 1]$ only one zero s_z and

$$t_z = \frac{1 + 3z - 4zs_z}{4(1 - zs_z)(1 - zs_z^3)}.$$

Proof. We infer by Lemma 4 that the inequality

$$Q_z[f] \leq \int_{-1}^1 f(x) dx$$

holds for any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$. We ask for which $t \in [0, 1]$ the inequality

$$\int_{-1}^1 f(x) dx \leq tQ_z[f] + (1-t)Q_1[f]$$

is fulfilled by any 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$. To find the desired condition for t it is enough to restrict ourselves to the functions $(\cdot - a)_+^3$, where $a \in [0, 1]$ (see Lemma 1 and the result of Bojanić and Roulier discussed just above the formulation of this lemma. Examining the proof of Lemma 2 we observe that $\left(Q_z[(\cdot - a)_+^3]\right)'_z > 0$ for $z \in (a, 1]$, so the function $\left[\frac{\sqrt{3}}{3}, 1\right] \ni z \mapsto Q_z[f]$ (nondecreasing by Lemma 2) is in fact strictly increasing on $(a, 1]$, whence $Q_z[(\cdot - a)_+^3] < Q_1[(\cdot - a)_+^3]$. Then the inequality

$$\int_{-1}^1 f(x) dx \leq t Q_z[(\cdot - a)_+^3] + (1 - t) Q_1[(\cdot - a)_+^3] \quad (3)$$

holds for any $a \in [0, 1]$ if and only if

$$t \leq \inf_{a \in [0, 1]} \frac{Q_1[(\cdot - a)_+^3] - \int_{-1}^1 (x - a)_+^3 dx}{Q_1[(\cdot - a)_+^3] - Q_z[(\cdot - a)_+^3]} = \inf_{a \in [0, 1]} F_z(a),$$

where

$$F_z(a) = \frac{z^2}{4} \cdot \frac{(1 - a)^3(1 + 3a)}{z^2(1 - a)^3 - (z - a)_+^3}.$$

Then $t_z = \inf_{a \in [0, 1]} F_z(a)$. We get by (2) that $F_z(a) \in [0, 1]$, whenever $a \in [0, 1]$.

Moreover, $t_z \in [0, 1]$, because (3) holds for $t = 0$ (cf. Lemma 3) and it fails for $t = 1$ (cf. Lemma 4).

Observe that if $a \geq z$, we have

$$F_z(a) = \frac{1 + 3a}{4} \geq \frac{1 + 3z}{4} = F_z(z),$$

hence $t_z = \inf_{a \in [0, z]} F_z(a)$. The homography

$$a(s) = \frac{z(1 - s)}{1 - zs}$$

is a decreasing bijection between $[0, 1]$ and $[0, z]$. This implies that

$$t_z = \inf_{s \in [0, 1]} F_z(a(s)) = \inf_{s \in [0, 1]} \frac{1 + 3z - 4zs}{4(1 - zs)(1 - zs^3)}$$

(we omit the long computations). For $s \in [0, 1]$ the sign of a derivative $(F_z(a(s)))'_s$ is the same as the sign of the polynomial

$$\begin{aligned} P_z(s) &= (1 + 3z - 4zs)'_s (4(1 - zs)(1 - zs^3)) \\ &\quad - (1 + 3z - 4zs)(4(1 - zs)(1 - zs^3))'_s \\ &= -4z^2s^3 + 4zs^2 + (z - 1)s + (z - 1). \end{aligned}$$

Furthermore,

$$\begin{aligned}\lim_{s \rightarrow -\infty} P_z(s) &= +\infty, \\ P_z(0) &= z - 1 < 0, \\ P_z(1) &= 2(1 - 2z)(z - 1) > 0, \\ \lim_{s \rightarrow +\infty} P_z(s) &= -\infty.\end{aligned}$$

Therefore the polynomial P_z admits in $(0, 1)$ exactly one zero (denote it by s_z) and the function $[0, 1] \ni s \mapsto F_z(a(s))$ reaches the minimum at s_z . Thus

$$t_z = \inf_{s \in [0, 1]} F_z(a(s)) = F_z(s_z) = \frac{1 + 3z - 4zs_z}{4(1 - zs_z)(1 - zs_z^3)}$$

and the proof is finished. $\square$

The Theorem 9 asserts that if $f: [-1, 1] \rightarrow \mathbb{R}$ is 3-convex, then

$$\int_{-1}^1 f(x) \, dx \in [Q_z[f], t_z Q_z[f] + (1 - t_z)Q_1[f]].$$

Now we show that for a fixed 3-convex function f the length of this interval, i.e. the function

$$\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto (1 - t_z)(Q_1[f] - Q_z[f]),$$

is nonincreasing. This tells us that the thinnest interval is obtained for $z = \frac{3}{4}$ and the best approximation of the integral is achieved by its midpoint, i.e. by the quadrature

$$\frac{1 + t_{\frac{3}{4}}}{2} Q_{\frac{3}{4}}[f] + \frac{1 - t_{\frac{3}{4}}}{2} Q_1[f].$$

Lemma 10. *The function $\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto t_z$ is strictly increasing (the notation of Theorem 9).*

Proof. We use the notation of the proof of Theorem 9. We showed there that for any $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ the polynomial $F(z, s) = 4z^2s^3 - 4zs^2 + (1 - z)s + (1 - z)$ has exactly one root $s_z \in [0, 1]$. Then the equation $F(z, s) = 0$ uniquely determines the function $\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto s_z$. For a fixed $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ the point (z, s_z) fulfils the assumptions of the implicit function theorem. Of course $F(z, s_z) = 0$. If $\left.\frac{\partial F(z, s)}{\partial s}\right|_{s=s_z} = 0$, then s_z would be the multiple root of a polynomial $s \mapsto F(z, s)$, which is impossible, because (as it was shown in the proof of

Theorem 9) this polynomial has three distinct real roots. Thus the function $z \mapsto s_z$ is differentiable at any $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$. This implies that the function

$$\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto t_z = \frac{1 + 3z - 4zs_z}{4(1 - zs_z)(1 - zs_z^3)},$$

is also differentiable. By the chain rule

$$\frac{dt_z}{dz} = \frac{\partial t_z}{\partial z} \Big|_{s=s_z} + \frac{\partial t_z}{\partial s} \Big|_{s=s_z} \cdot \frac{ds_z}{dz}.$$

But in the proof of Theorem 9 we showed that the function $s \mapsto F_z(a(s))$ takes the minimum at $s = s_z$, whence $\frac{\partial t_z}{\partial s} \Big|_{s=s_z} = 0$. Thus

$$\frac{dt_z}{dz} = \frac{\partial t_z}{\partial z} \Big|_{s=s_z}.$$

The sign of this derivative is the same as the sign of the expression

$$16z^2s_z^5 - (12z^2 + 8z)s_z^4 + 4s_z^3 - 12s_z + 12,$$

which is the same (by subtracting $F(z, s_z)(4s_z^2 - 3s_z) = 0$) as the sign of

$$zs_z(8s_z^3 - 8s_z^2 + s_z - 3) - s_z^2 - 9s_z + 12. \quad (4)$$

Consider this as an affine function of z . Since $\left[z \in \frac{\sqrt{3}}{3}, \frac{3}{4}\right] \subset (0, 1)$ and $s_z \in [0, 1]$, the value of (4) lies between

$$0 \cdot s_z(8s_z^3 - 8s_z^2 + s_z - 3) - s_z^2 - 9s_z + 12 = -s_z^2 - 9s_z + 12 \geq 2$$

and

$$\begin{aligned} & 1 \cdot s_z(8s_z^3 - 8s_z^2 + s_z - 3) - s_z^2 - 9s_z + 12 \\ &= 8s_z^4 - 8s_z^3 - 12s_z + 12 = (1 - s_z)(12 - 8s_z^3) \geq 0. \end{aligned}$$

This shows that $\frac{dt_z}{dz} > 0$ and the proof is finished. $\square$

Corollary 11. *If $f: [-1, 1] \rightarrow \mathbb{R}$ is 3-convex then the function*

$$\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto (1 - t_z)(Q_1[f] - Q_z[f])$$

is nonincreasing (the notation of Theorem 9).

Proof. The Corollary follows immediately by Lemmas 2 and 10, because the functions $z \mapsto 1 - t_z$ and $z \mapsto Q_1[f] - Q_z[f]$ are positive and nondecreasing. $\square$

Remark 12. Let $f: [-1, 1] \rightarrow \mathbb{R}$ be 3-concave. Applying Corollary 11 to the 3-convex function $-f$ we get that the function

$$\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto (1 - t_z)(Q_1[-f] - Q_z[-f]) = (1 - t_z)(Q_z[f] - Q_1[f])$$

is nonincreasing. Then the function

$$\left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right] \ni z \mapsto (1 - t_z)|Q_1[f] - Q_z[f]|$$

is nonincreasing, whenever $f: [-1, 1] \rightarrow \mathbb{R}$ is either 3-convex or 3-concave.

3. Inequalities between the quadratures on the interval $[a, b]$

In this short section we transform the quadratures Q_z into the interval $[a, b]$. Let $f: [a, b] \rightarrow \mathbb{R}$. The affine substitution

$$[-1, 1] \ni x \mapsto \frac{a+b}{2} + \frac{b-a}{2}x = u \in [a, b]$$

transforms f to a function $g: [-1, 1] \rightarrow \mathbb{R}$ given by $g(u) = f(x)$. If f is 3-convex on $[a, b]$, then g is 3-convex on $[-1, 1]$ (cf. [5, p.18]). Furthermore, the relations

$$\int_{-1}^1 g(x) dx \stackrel{\leq}{\geq} Q_z[g] = 2 \left(1 - \frac{1}{3z^2}\right) g(0) + \frac{1}{3z^2} (g(-z) + g(z))$$

lead us to

$$\begin{aligned} \int_a^b f(x) dx &\stackrel{\leq}{\geq} \frac{b-a}{2} \left[2 \left(1 - \frac{1}{3z^2}\right) f\left(\frac{a+b}{2}\right) + \frac{1}{3z^2} \left(f\left(\frac{a+b}{2} - \frac{b-a}{2}z\right) \right. \right. \\ &\quad \left. \left. + f\left(\frac{a+b}{2} + \frac{b-a}{2}z\right) \right) \right]. \end{aligned}$$

Denote the quadrature on the right by $Q_{z;a,b}[f]$. Then by Theorem 9 we get

Corollary 13. *For any $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ there exists $t_z \in [0, 1]$ (given by Theorem 9) satisfying the following condition: the inequality*

$$Q_{z;a,b}[f] \leq \int_{-1}^1 f(x) dx \leq t Q_{z;a,b}[f] + (1-t) Q_{1;a,b}[f]$$

holds for all 3-convex functions $f: [a, b] \rightarrow \mathbb{R}$ if and only if $0 \leq t \leq t_z$.

In the context of the above inequality the best approximation of the integral is the arithmetic mean of its lower and upper bounds. The error bound is not greater than one half of the distance between these bounds. In this way we arrive at

Corollary 14. *For any $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ there exists $t_z \in [0, 1]$ (given by Theorem 9) satisfying the following condition: the inequality*

$$\left| \frac{1+t}{2} Q_{z;a,b}[f] + \frac{1-t}{2} Q_{1;a,b}[f] - \int_a^b f(x) dx \right| \leq \frac{1-t}{2} (Q_{1;a,b}[f] - Q_{z;a,b}[f]) \quad (5)$$

holds for all 3-convex functions $f: [a, b] \rightarrow \mathbb{R}$ if and only if $0 \leq t \leq t_z$.

4. Adaptive integration

For $n \in \mathbb{N}$ let $a = x_0 < x_1 < \dots < x_n = b$ with

$$x_k = a + k \cdot \frac{b-a}{n}, \quad k = 0, 1, \dots, n$$

be a partition of $[a, b]$ into n subintervals of equal lengths. Define the emph-composite quadrature

$$Q_{z;a,b}^n[f] = \sum_{k=0}^{n-1} Q_{z;x_k,x_{k+1}}[f],$$

where $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$. We will approximate the integral of a 3-convex (or a 3-concave) function $f: [a, b] \rightarrow \mathbb{R}$ by the quadratures $\frac{1+t}{2} Q_{z;a,b}^n + \frac{1-t}{2} Q_{1;a,b}^n$ (with $0 \leq t \leq t_z$, where t_z is given by Theorem 9) up to a given tolerance $\varepsilon > 0$. For such an approach (i.e. *adaptive integration*, which is widely known in numerical analysis) one needs to choose n large enough. In this section we introduce the *stopping inequality*, which, as the name suggests, stops the integration algorithm and assures the goodness of approximation. One of the classical stopping inequalities in this spirit, given in 1972 by Rowland and Varol [6], concerns the composite Simpson's Rule (in our terminology $Q_{1;a,b}^n$): if f is either 3-convex, or 3-concave on $[a, b]$, then

$$\left| Q_{1;a,b}^{2n}[f] - \int_a^b f(x) dx \right| \leq |Q_{1;a,b}^{2n}[f] - Q_{1;a,b}^n[f]|. \quad (6)$$

Now, if the right hand side of (6) does not exceed ε , then nor does the left hand side and the approximation of the integral by $Q_{1;a,b}^{2n}$ is accurate enough. Recall that another well known approach to estimate the error bound is to use the derivative of some degree. This could cause some inconvenience because of difficulties in computing its supremum norm. The method presented in this paper needs only to compute the nodes (simple task for a computer since they form an arithmetic sequence) and the values of the integrand at these nodes.

For a more extensive discussion on adaptive integration see the second-named author's recent paper [9].

Theorem 15. Let $n \in \mathbb{N}$, $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ and $0 \leq t \leq t_z$, where t_z is given by Theorem 9. If $f: [a, b] \rightarrow \mathbb{R}$ is either a 3-convex or a 3-concave function, then

$$\left| \frac{1+t}{2} Q_{z;a,b}^n[f] + \frac{1-t}{2} Q_{1;a,b}^n[f] - \int_a^b f(x) dx \right| \leq \frac{1-t}{2} |Q_{1;a,b}^n[f] - Q_{z;a,b}^n[f]|$$

□

Proof. The proof uses the triangle inequality and inequality (5) of Corollary 14. First assume that f is 3-convex. Then

$$\begin{aligned} & \left| \frac{1+t}{2} Q_{z;a,b}^n[f] + \frac{1-t}{2} Q_{1;a,b}^n[f] - \int_a^b f(x) dx \right| \\ &= \left| \sum_{k=0}^{n-1} \left(\frac{1+t}{2} Q_{z;x_k,x_{k+1}}^n[f] + \frac{1-t}{2} Q_{1;x_k,x_{k+1}}^n[f] - \int_{x_k}^{x_{k+1}} f(x) dx \right) \right| \\ &\leq \sum_{k=0}^{n-1} \left| \left(\frac{1+t}{2} Q_{z;x_k,x_{k+1}}^n[f] + \frac{1-t}{2} Q_{1;x_k,x_{k+1}}^n[f] - \int_{x_k}^{x_{k+1}} f(x) dx \right) \right| \\ &\leq \sum_{k=0}^{n-1} \frac{1-t}{2} (Q_{1;x_k,x_{k+1}}^n[f] - Q_{z;x_k,x_{k+1}}^n[f]) \\ &= \frac{1-t}{2} (Q_{1;a,b}^n[f] - Q_{z;a,b}^n[f]) = \frac{1-t}{2} |Q_{1;a,b}^n[f] - Q_{z;a,b}^n[f]|. \end{aligned}$$

If f is 3-concave, then we apply the above inequality to the 3-convex function $-f$. Both sides remain unchanged. Thus the proof is finished. □

Let us present our stopping inequality. The result below is of course an immediate consequence of Theorem 15.

Corollary 16. Let $\varepsilon > 0$, $n \in \mathbb{N}$, $z \in \left[\frac{\sqrt{3}}{3}, \frac{3}{4}\right]$ and $0 \leq t \leq t_z$, where t_z is given by Theorem 9. If $f: [a, b] \rightarrow \mathbb{R}$ is either a 3-convex or a 3-concave function and

$$|Q_{1;a,b}^n[f] - Q_{z;a,b}^n[f]| < \frac{2\varepsilon}{1-t}, \quad (7)$$

then

$$\left| \frac{1+t}{2} Q_{z;a,b}^n[f] + \frac{1-t}{2} Q_{1;a,b}^n[f] - \int_a^b f(x) dx \right| < \varepsilon. \quad (8)$$

The adaptive method we develop goes by the following steps.

1. Take a function $f: [a, b] \rightarrow \mathbb{R}$ which is either 3-convex, or 3-concave.
2. Fix a tolerance $\varepsilon > 0$ and take $n = 1$.
3. Increase n until the inequality (7) is fulfilled.
4. By Corollary 16 the inequality (8) is fulfilled, so we take

$$\int_a^b f(x) dx \approx \frac{1+t}{2} Q_{z;a,b}^n[f] + \frac{1-t}{2} Q_{1;a,b}^n[f].$$

Then the inequality (7) constitutes the stopping criterion for our method. If $\varepsilon > 0$ is arbitrarily chosen, then this inequality is fulfilled for n large enough and the algorithm is stopped. Indeed, by a standard argument we have $\lim_{n \rightarrow \infty} |Q_{1;a,b}^n[f] - Q_{z;a,b}^n[f]| = 0$.

5. Numerical experiments

In the last section we present a few numerical experiments with the quadratures

$$\frac{1+t}{2} Q_{z;a,b}^n[f] + \frac{1-t}{2} Q_{1;a,b}^n[f].$$

We apply the adaptive method which we described above. Our aim is to compare their results with the results of the experiments with quadratures based on the Chebyshev quadrature and Simpson's rule (in the notation of our paper this is the case of $z = \frac{\sqrt{2}}{2}$ and $t = \frac{1}{2}$) performed in [9]. If for $z = \frac{\sqrt{2}}{2}$ we apply the optimal t_z given by Theorem 9, the number of needed subdivisions of the interval of integration is less than the analogous number obtained for $t = \frac{1}{2}$. But the smallest number of subdivisions we achieve for $z = \frac{3}{4}$ with the optimal $t_{\frac{3}{4}}$. This agrees with the comment given just before the formulation of Lemma 10 telling us that the best approximation of the integral of a 3-convex function $f: [-1, 1] \rightarrow \mathbb{R}$ is achieved by the quadrature

$$\frac{1+t_{\frac{3}{4}}}{2} Q_{\frac{3}{4};-1,1}[f] + \frac{1-t_{\frac{3}{4}}}{2} Q_{1;-1,1}[f].$$

We get a similar conclusion on any interval $[a, b]$. Then the results of the present paper improve on the results of [9].

In our experiments we use the same functions and intervals of integration as in [9]. We also use (in fact) the same computer program which we created in the Python programming language for the purposes of [9] (of course some slight modifications needed for the present paper were done).

We perform the experiments with some specially chosen values of z and t .

1. For $z = \frac{\sqrt{3}}{3}$ we get as $Q_{z;a,b}^n$ the adaptive quadrature based on the 2-point Gauss quadrature. We apply the optimal value of t , i.e. $t_z = t_{\frac{\sqrt{3}}{3}} \approx 0.5829014$.
2. For $z = \frac{\sqrt{2}}{2}$ we get as $Q_{z;a,b}^n$ the adaptive quadrature based on the Chebyshev quadrature. Taking $t = \frac{1}{2}$ we arrive at the quadrature which was investigated in [9], i.e. $\frac{3}{4}Q_{\frac{\sqrt{2}}{2};a,b}^n + \frac{1}{4}Q_{1;a,b}^n$ (denoted in [9] by $\frac{3}{4}\mathcal{C} + \frac{1}{4}\mathcal{S}$). Here we recall the results.
3. Again we take $z = \frac{\sqrt{2}}{2}$, but we apply the optimal $t_z = t_{\frac{\sqrt{2}}{2}} \approx 0.7240841$.
4. Next we examine the highest possible $z = \frac{3}{4}$ with $t = \frac{3}{4}$.
5. Then we take $z = \frac{3}{4}$ with the optimal $t_{\frac{3}{4}} \approx 0.7689878$.
6. Finally we consider the method of Rowland and Varol with the stopping criterion given by (6). In fact it was done in [9], so we recall the results here.

Because the program is not too long, below we present its listing.

```
# adaptive_Qz.py
from math import sqrt, exp
from decimal import Decimal

def dec(x):
    return Decimal(str(x))

def Q_simple(z, func, a, b):
    a, b, z = dec(a), dec(b), dec(z)
    return (2*(1-1/(3*z*z))*func((a+b)/2)+(func((a+b)/2-(b-a)*z/2)+func((a+b)/2+(b-a)*z/2))/(3*z*z))*(b-a)/2

def Q(z, func, a, b, subdivisions):
    a, b, z, total = dec(a), dec(b), dec(z), 0
    for k in range(subdivisions):
        total += Q_simple(z, func, a=a+k*(b-a)/subdivisions, b=a+(k+1)*(b-a)/subdivisions)
    return total

def n(z=sqrt(2)/2, t=0.5, func=lambda x: 1/x, a=1, b=2, epsilon=0.0001):
    a, b, z, t, epsilon, subdivisions = dec(a), dec(b), dec(z), dec(t), dec(epsilon), 1
    while abs(Q(z, func, a, b, subdivisions)-Q(1, func, a, b, subdivisions)) > 2*epsilon/(1-t):
        subdivisions += 1
    return subdivisions

def n_Rowland_Varol(func=lambda x: 1/x, a=1, b=2, epsilon=0.0001):
    a, b, epsilon, subdivisions = dec(a), dec(b), dec(epsilon), 1
    while abs(Q(1, func, a, b, 2*subdivisions)-Q(1, func, a, b, subdivisions)) > epsilon:
        subdivisions += 1
    return 2*subdivisions

z = sqrt(3)/3, sqrt(2)/2, sqrt(2)/2, 0.75, 0.75
t = 0.5829014, 0.5, 0.7240841, 0.75, 0.7689878

output = ''
for k in range(1, 17):
    epsilon = 10 ** -k
    output += f'{k:2}'
    for j in range(5):
        output += f'{n(z=z[j], t=t[j], epsilon=epsilon):5}'
        output += f'{n_Rowland_Varol(epsilon=epsilon):5}\n'
print('\nExperiment 1\n'+output)

output = ''
```

```

for k in range(1, 11):
    epsilon = 10 ** -8
    output += f'{k:2}'
    for j in range(5):
        output += f'{n(func=lambda x: dec(exp(x)), z=z[j],t=t[j], a=0, b=k, epsilon=
            epsilon): 5}'
    output += f'{n_Rowland_Varol(func=lambda x: dec(exp(x)), a=0, b=k, epsilon=epsilon):
        5}\n'
print('\nExperiment 2\n'+output)

```

Experiment 1. We start with the integral $\int_1^2 \frac{dx}{x} = \ln 2$. The function $f(x) = \frac{1}{x}$ is 3-convex as $f^{(4)}(x) = \frac{24}{x^5} > 0$ for $x \in [1, 2]$. In the table below we put the numbers of subdivisions of the interval $[1, 2]$ needed to approximate the integral with the desired tolerance $\varepsilon \in \{10^{-1}, 10^{-2}, \dots, 10^{-16}\}$.

Tolerance	$z = \frac{\sqrt{3}}{3}$ $t \frac{\sqrt{3}}{3} \approx$ 0.5829014	$z = \frac{\sqrt{2}}{2}$ $t = \frac{1}{2}$	$z = \frac{\sqrt{2}}{2}$ $t \frac{\sqrt{2}}{2} \approx$ 0.7240841	$z = \frac{3}{4}$ $t = \frac{3}{4}$	$z = \frac{3}{4}$ $t \frac{3}{4} \approx$ 0.7689878	Rowland and Varol [6]
10^{-1}	1	1	1	1	1	2
10^{-2}	1	1	1	1	1	2
10^{-3}	1	1	1	1	1	4
10^{-4}	2	2	2	2	2	4
10^{-5}	3	3	3	3	3	8
10^{-6}	6	5	5	5	4	14
10^{-7}	10	9	8	8	8	24
10^{-8}	17	16	14	13	13	42
10^{-9}	29	28	25	23	23	74
10^{-10}	52	50	43	41	40	132
10^{-11}	91	89	77	72	71	234
10^{-12}	162	158	136	128	126	414
10^{-13}	288	280	241	228	223	736
10^{-14}	511	498	429	405	397	1310
10^{-15}	908	884	762	719	705	2328
10^{-16}	1615	1572	1355	1279	1254	4138

Experiment 2. Next we experiment with the 3-convex function $f(x) = e^x$ over the intervals $[0, b]$ with $b \in \{1, 2, \dots, 10\}$. We choose the tolerance $\varepsilon = 10^{-8}$.

The right endpoint b	$z = \frac{\sqrt{3}}{3}$ $t \frac{\sqrt{3}}{3} \approx$ 0.5829014	$z = \frac{\sqrt{2}}{2}$ $t = \frac{1}{2}$	$z = \frac{\sqrt{2}}{2}$ $t \frac{\sqrt{2}}{2} \approx$ 0.7240841	$z = \frac{3}{4}$ $t = \frac{3}{4}$	$z = \frac{3}{4}$ $t \frac{3}{4} \approx$ 0.7689878	Rowland and Varol [6]
1	12	12	11	10	10	32
2	34	33	28	27	26	86
3	66	64	56	53	52	170
4	114	111	96	90	89	292
5	183	178	154	145	142	470
6	282	275	237	224	219	722
7	423	412	355	335	328	1082
8	620	604	520	491	482	1588
9	895	872	752	709	695	2294
10	1277	1244	1072	1012	992	3274

Acknowledgements

The authors thank the anonymous referee for very careful reading of the manuscript, which resulted in detailed language corrections significantly improving the text of the paper.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Bojanić, R., Roulier, J.: Approximation of convex functions by convex splines and convexity preserving continuous linear operators. *Rev. Anal. Numér. Théorie Approx.* **3**(2), 143–150 (1975)
- [2] Hopf, E.: Über die Zusammenhänge zwischen gewissen höheren Differenzenquotienten reeller Funktionen einer reellen Variablen und deren Differenzierbarkeitseigenschaften. Ph.D. dissertation, Friedrich-Wilhelms-Universität, Berlin (1926)
- [3] KomisarSKI, A., Wąsowicz, S.: Inequalities between remainders of quadratures. *Aequ. Math.* **91**(6), 1103–1114 (2017)
- [4] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities, 2nd edn. Birkhäuser Verlag, Basel. Cauchy's equation and Jensen's inequality, Edited and with a preface by Attila Gilányi (2009)

- [5] Popoviciu, T.: Sur quelques propriétés des fonctions d'une ou de deux variables réelles. *Mathematica Cluj* **8**, 1–85 (1934)
- [6] Rowland, J.H., Varol, Y.L.: Exit criteria for Simpson's compound rule. *Math. Comp.* **26**, 699–703 (1972)
- [7] Toader, Gh.: Some generalizations of Jessen's inequality. *Anal. Numér. Théor. Approx.* **16**(2), 191–194 (1987)
- [8] Wąsowicz, S.: On some extremalities in the approximate integration. *Math. Inequal. Appl.* **13**(1), 165–174 (2010)
- [9] Wąsowicz, S.: On a certain adaptive method of approximate integration and its stopping criterion. *Aequ. Math.* **94**(5), 887–898 (2020)

Andrzej Komisarski
Faculty of Mathematics and Computer Science
University of Lodz
Banacha 22
90-238 Łódź
Poland
e-mail: andkom@math.uni.lodz.pl

Szymon Wąsowicz
Department of Mathematics
University of Bielsko-Biala
Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: swasowicz@ath.bielsko.pl

Received: February 18, 2021

Revised: December 30, 2021

Accepted: January 1, 2022