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Aequationes Mathematicae

Invariant vector means and complementability of Banach spaces in their second duals

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Abstract. Let X be a Banach space. Fix a torsion-free commutative and cancellative semigroup S whose torsion-free rank is the same as the density of X^{**}. We then show that X is complemented in X^{**} if and only if there exists an invariant mean $M\colon \ell_\infty(S,X) \to X$. This improves upon previous results due to Bustos Domecq (J Math Anal Appl 275(2):512–520, 2002), Kania (J Math Anal Appl 445:797–802, 2017), Goucher and Kania (Studia Math 260:91–101, 2021).

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1. Introduction

Invariant means on amenable groups are an important tool in many parts of Mathematics, especially in Harmonic analysis (see [8,9]). For the basic properties of invariant means, we refer the reader to [8]. Invariant means and their generalizations for vector-valued functions also play an important role in the stability of functional equations and selections of set-valued functions (see [1,5,6,16]).

The space of all bounded functions from a set S into a Banach space X is denoted by $\ell_\infty(S,X)$. Let us recall the definition of an amenable semigroup (see [3]).

Birkhäuser

Definition 1.1. A semigroup $(S,+)$ is called *left* [resp. *right*] *amenable* if and only if there exists a linear map $L\colon \ell_\infty(S,\mathbb{R}) \to \mathbb{R}$ such that

$$\begin{aligned} &\inf f(S) \le L(f) \le \sup f(S),\ \ f \in \ell_\infty(S,\mathbb{R})\\ &L({}_af) = L(f),\ \ a \in S,\ \ f \in \ell_\infty(S,\mathbb{R}),\\ &[L(f_a) = L(f),\ \ a \in S,\ \ f \in \ell_\infty(S,\mathbb{R})], \end{aligned}$$

where

$$\begin{aligned} &{}_af(x) = f(a+x),\ \ a,x \in S,\ \ f \in \ell_\infty(S,\mathbb{R}),\\ &[f_a(x) = f(x+a),\ \ a,x \in S,\ \ f \in \ell_\infty(S,\mathbb{R})]. \end{aligned}$$

If both left and right invariant means exist, then S is called *amenable.*

Remark 1.2. In the above definition the first condition

$$\inf f(S) \le L(f) \le \sup f(S)$$

is equivalent to conditions $L(\mathbb{1}_S) = 1$ and $|L(f)| \le \|f\| := \sup |f(S)|$.

It is known that every commutative semigroup is amenable (an easy consequence of the Markov–Kakutani fixed point theorem, see [15, Theorem 5.23]).

Certain generalizations of invariant means were investigated for vector-valued functions in [5] and the existence thereof appears to be related to properties such as reflexivity.

Some generalized definition of an invariant mean has been used by many mathematicians as a folklore (e.g. by Pełczyński [14]). The explicit form of this definition can be found e.g. in the work of Ger [6].

Definition 1.3. Let $(S,+)$ be a left [right] amenable semigroup and X be a Banach space. A linear map $M\colon \ell_\infty(S,X) \to X$ is called a *left* [*right*] *X-valued invariant mean* if

$$\begin{aligned} &\|M\| \le 1,\\ &M(c\mathbb{1}_S) = c,\ \ c \in X,\\ &M({}_af) = M(f),\ \ a \in S, f \in \ell_\infty(S,X),\\ &[M(f_a) = M(f),\ \ a \in S, f \in \ell_\infty(S,X),] \end{aligned}$$

where

$$\begin{aligned} &{}_af(x) = f(a+x),\ \ a,x \in S,\ \ f \in \ell_\infty(S,X),\\ &[f_a(x) = f(x+a),\ \ a,x \in S,\ \ f \in \ell_\infty(S,X).] \end{aligned}$$

If M is a left and right invariant mean, then M is called an *X-valued invariant mean.*

If in the above definition the norm of map M is equal to at most $\lambda \ge 1$, then M is called an *X-valued invariant λ-mean.*

The existence of such invariant means for a fixed Banach space and for all amenable semigroups has been studied by Domecq [4, Theorem 1 and 2] and by the author in [12]. However, as observed by Lipecki in his Mathematical Review (MR1943762) of Bustos Domecq's paper, the proof of Theorem 2 contains a gap (a flawed choice of the semigroup, so we cannot use the Principle of Local Reflexivity). This gap was corrected by Kania [10].

Goucher and Kania [7] consider the following question (communicated privately to T. Kania by J.M.F. Castillo).

> Suppose that a Banach space X admits an invariant mean with respect to every/some commutative group. Must X be complemented in X^{**}?

They proved the following (see [7, Theorem A], [10, Theorem 1.2])

Theorem 1.4. *Let X be a Banach space and $\lambda \geq 1$. Then the following assertions are equivalent.*

1. *X is complemented in X^{**} by a projection of norm at most λ;*
2. *for every amenable semigroup S there exists an X-valued invariant λ-mean on S;*
3. *for every commutative semigroup S there exists an X-valued invariant λ-mean on S;*
4. *for every free commutative group G of rank $|X^{**}|$ there exists an X-valued invariant λ-mean on G;*
5. *there exists an X-valued invariant λ-mean on the additive group of X^{**}.*

It is also demonstrated ([7, Remark 1.1]) that there exists a commutative noncancellative semigroup S (that could be chosen as large as one wishes) such that there exists an X-valued invariant mean on S.

In this paper we will prove that if X is a Banach space and there exists an invariant X-valued mean on any arbitrary commutative cancellative semigroup S of torsion-free rank $\operatorname{dens} X^{**}$, then X is λ-complemented in X^{**}.

2. Preliminaries

First we recall the definition of torsion-free rank (see [2]).

Definition 2.1. Let S be a commutative cancellative semigroup. A set $A \subset S$ is *independent* if $\sum_{i=1}^{n} k_i a_i = \sum_{i=1}^{n} m_i a_i$ for any $n \in \mathbb{N}$ and $a_i \in A$, $k_i, m_i \in \mathbb{N}_0$, $i \in \{1, \ldots, n\}$ implies $k_i = m_i$ for $i \in \{1, \ldots, n\}$.

Let further $\mathcal{A}_0$ be the family of all independent sets L in S consisting only of elements whose order is infinite and such that L is maximal with respect to these properties. The cardinal number of any set in $\mathcal{A}_0$ is called a *torsion-free rank of S* and is denoted by $r_0(S)$ (all the sets in $\mathcal{A}_0$ have the same cardinal number).

The *density character* of a Banach space X, denoted $\operatorname{dens} X$, is the smallest cardinal κ for which X has a dense subset of cardinality κ.

Lemma 2.2. *Let X be an infinite-dimensional Banach space. Then*

1. *if $\mathcal{B}$ is a linearly independent subset of X and $\mathbb{F}$ is a countable dense subfield of a scalar field of X, then $|span_{\mathbb{F}}\mathcal{B}| = |\mathcal{B}|$;*
2. *for every closed subspace Y of X there exists a linearly independent subset $\mathcal{B}$ of X such that $|\mathcal{B}| = \operatorname{dens} X$, $\overline{span}\,\mathcal{B} = X$, $\overline{span}\,(\mathcal{B}\cap Y) = Y$. Moreover, we can assume that the norm of each $x \in \mathcal{B}$ is equal to 1.*

Proof. 1. We observe that

$$|\mathcal{B}| \leq |\mathrm{span}_{\mathbb{F}}\mathcal{B}| = |\bigcup_{n\in\mathbb{N}} (\mathbb{F}\cdot\mathcal{B})^n| \leq |\mathbb{N}| \cdot \sup_{n\in\mathbb{N}} |(\mathbb{F}\cdot\mathcal{B})^n|$$
$$= |\mathbb{N}| \cdot |\mathbb{F}\cdot\mathcal{B}| = |\mathbb{N}| \cdot |\mathbb{F}| \cdot |\mathcal{B}| = |\mathcal{B}|.$$

2. Let Y be a closed subspace of X, D be a dense subset of X such that $|D| = \operatorname{dens} X$ and K be a dense subset of Y such that $|K| \leq \operatorname{dens} X$. Let further

$$D_1 := \Big\{\frac{x}{\|x\|} : x \in K \setminus \{0\}\Big\},$$
$$D_2 := \Big\{\frac{x}{\|x\|} : x \in D \setminus \{0\}\Big\}.$$

Let further $\mathcal{B}_1$ be a maximal linearly independent subset of D_1 and $\mathcal{B}$ be a maximal linearly independent subset of $D_1 \cup D_2$ such that $\mathcal{B}_1 \subset \mathcal{B}$. We have $\overline{\mathrm{span}}\,\mathcal{B}_1 = Y$, $\overline{\mathrm{span}}\,\mathcal{B} = X$ and $|\mathcal{B}| \leq |D_1| + |D_2| = \operatorname{dens} X$. Let $\mathbb{F} = \mathbb{Q}$ when X is a real space or $\mathbb{F} = \mathbb{Q}(i)$ when X is a complex space. Since $\mathrm{span}_{\mathbb{F}}\mathcal{B}$ is dense in X, $|\mathcal{B}| = |\mathrm{span}_{\mathbb{F}}\mathcal{B}| \geq \operatorname{dens} X$. We also note that the norm of each $x \in \mathcal{B}$ is equal 1.

□

We will also require the version of the principle of local reflexivity due to Lindenstrauss and Rosenthal [11]. We denote by $\kappa\colon X \to X^{**}$ the canonical embedding from a Banach space X into the second dual.

Theorem 2.3. *Let X be a Banach space. Then for every finite-dimensional subspace $F \subset X^{**}$ and each $\varepsilon \in (0, 1]$ there exists a linear map $P_F^{\varepsilon}\colon F \to \kappa(X)$ such that*

1. *$(1-\varepsilon)\|x\| \leq \|P_F^{\varepsilon}(x)\| \leq (1+\varepsilon)\|x\|$, $x \in F$;*
2. *$P_F^{\varepsilon}(x) = x$ for $x \in F \cap \kappa(X)$.*

It is a standard fact that subgroups and quotients of amenable groups are amenable. Using exactly the same ideas one can prove that if a Banach space admits an invariant mean with respect to a group, then it also does so with respect to subgroups and quotients of the group (see [12, Theorem 3.12] and [7, Lemma 2.3]). We would like to get a similar result for quotients of semigroups (subsemigroups of an amenable group need not be amenable) but first we must say something about normal semigroups and quotients of semigroups (see also [17]). Let $(S,+)$ be a semigroup, G be a subsemigroup of S. Then G is called *a normal subsemigroup* if $x+G=G+x$ for every $x\in S$. Of course in a commutative semigroup each subsemigroup is normal.

Let further S be a semigroup and G be a normal subsemigroup of S. We define the quotient semigroup $S/G := S/\overset{G}{\sim}$, where $x \overset{G}{\sim} y$ iff $(x+G)\cap(y+G)\neq\emptyset$. It is easy to notice that for any $g\in G$ the set $[g]_{\overset{G}{\sim}}$ is a neutral element of S/G. Moreover, if G is a group, then G is a neutral element of S/G.

Lemma 2.4. *Let S be an amenable semigroup and G be a normal subsemigroup of S. If there exists an X-valued invariant λ-mean $M\colon \ell_\infty(S,X)\to X$, then there exists an X-valued invariant λ-mean $M\colon \ell_\infty(S/G,X)\to X$.*

Proof. We define a map $M_1\colon \ell_\infty(S/G,X)\to X$ by the formula

$$M_1(f) := M(\psi(f)),\ f\in \ell_\infty(S/G,X),$$

where $\psi(f)(s)=f([s]_{\overset{G}{\sim}})$ for $s\in S$ and $f\in\ell_\infty(S/G,X)$. Since ψ is linear, $\|\psi(f)\|=\|f\|$ and

$$\psi({}_{[t]_{\overset{G}{\sim}}}f)(s) = {}_{[t]_{\overset{G}{\sim}}}f([s]_{\overset{G}{\sim}}) = f([t+s]_{\overset{G}{\sim}}) = \psi(f)(t+s) = ({}_t\psi(f))(s),$$
$$\psi(f_{[t]_{\overset{G}{\sim}}})(s) = f_{[t]_{\overset{G}{\sim}}}([s]_{\overset{G}{\sim}}) = f([s+t]_{\overset{G}{\sim}}) = \psi(f)(s+t) = (\psi(f)_t)(s),$$

for all $s,t\in S$, $f\in\ell_\infty(S/G,X)$, then M_1 is an X-valued invariant λ-mean on S/G. □

3. Main results

Throughout this section we fix an infinite-dimensional Banach space X, $\lambda\geq 1$. Let γ be a cardinal number. We denote by S_γ the commutative semigroup comprising all finite subsets of γ endowed with the operation of taking the union of sets. It is easy to observe that $|S_\gamma|=\gamma$.

Theorem 3.1. *Let γ be an infinite cardinal number. If there exists an X-valued invariant λ-mean $M\colon \ell_\infty(S_\gamma,X)\to X$, then for every subspace E of X^{**} such that $\operatorname{dens} E=\gamma$ there exists a linear map $P\colon E\to X$ such that $\|P\|\leq\lambda$ and $P(x)=x$ for $x\in\kappa(X)\cap E$.*

Proof. Let $\mathbb{K}$ be a scalar field of X. In view of Lemma 2.2 there exists a linearly independent subset $\mathcal{B}$ of E such that $\overline{\operatorname{span}}\,\mathcal{B}=E$, $\overline{\operatorname{span}}\,(\mathcal{B}\cap\kappa(X))=\kappa(X)\cap E$, $|\mathcal{B}|=\operatorname{dens} E=\gamma$. Let $T\colon\gamma\to\mathcal{B}$ be a bijection and $M\colon\ell_\infty(S_\gamma,X)\to X$ be an X-valued invariant λ-mean .

For $A\in S_\gamma$ we define $\varepsilon_A := \frac{1}{|A|+1}$ and $P^{\varepsilon_A}_{\operatorname{span} T(A)}$ is a fixed linear operator satisfying the conditions of Theorem 2.3.

We define the map $P\colon E \to X$ in the following way (on the dense subspace span$\mathcal{B}$, the map is simply continuously extended to the closure): for $x \in \operatorname{span}\mathcal{B}$ we put $P(x) := M(\phi_x)$, where

$$\phi_x(A) := \begin{cases} P^{\varepsilon_A}_{\operatorname{span}T(A)}(x), & x \in \operatorname{span}T(A) \\ 0, & x \notin \operatorname{span}T(A) \end{cases}, \; A \in S_\gamma$$

when $x \in \mathcal{B}$ and

$$\phi_x(A) := \sum_{i=1}^{n} \lambda_i \phi_{x_i}(A), \; A \in S_\gamma,$$

when $x = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, $x_1, \dots, x_n \in \mathcal{B}$.

For $x, y \in \operatorname{span}\mathcal{B}$ and $\alpha \in \mathbb{K}$ we notice that $\phi_{\alpha x + y} = \alpha\phi_x + \phi_y$. Thus

$$P(\alpha x + y) = M(\phi_{\alpha x+y}) = \alpha M(\phi_x) + M(\phi_y) = \alpha P(x) + P(y),$$

so P is linear on span$\mathcal{B}$.

Let $x = \sum_{i=1}^{n} \lambda_i x_i$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, $x_1, \dots, x_n \in \mathcal{B}$. Let further $A_0 \in S_\gamma$ be such that $x_1, \dots x_n \in T(A_0)$.
We observe that

$$\begin{aligned}
\|P(x)\| &= \|M(\phi_x)\| = \|M(\phi_x(\cdot \cup A_0))\| \le \lambda \sup_{A \in S_\gamma} \|\phi_x(A \cup A_0)\| \\
&= \lambda \sup_{A \in S_\gamma} \|\sum_{i=1}^{n} \lambda_i \phi_{x_i}(A \cup A_0)\| = \lambda \sup_{A \in S_\gamma} \|\sum_{i=1}^{n} \lambda_i P^{\varepsilon_{A \cup A_0}}_{\operatorname{span}(A \cup A_0)}(x_i)\| \\
&= \lambda \sup_{A \in S_\gamma} \|P^{\varepsilon_{A \cup A_0}}_{\operatorname{span}T(A \cup A_0)}\Big(\sum_{i=1}^{n} \lambda_i x_i\Big)\| = \lambda \sup_{A \in S_\gamma} \|P^{\varepsilon_{A \cup A_0}}_{\operatorname{span}T(A \cup A_0)}(x)\| \\
&\le \lambda \sup_{A \in S_\gamma} (1 + \varepsilon_{A \cup A_0})\|x\| \le \lambda\Big(1 + \frac{1}{1 + |A_0|}\Big)\|x\|.
\end{aligned}$$

Since A_0 is arbitrary, we get $\|P(x)\| \le \lambda\|x\|$.
Moreover, if $x \in \kappa(X)$, then from the properties of $\mathcal{B}$ we get $x_1, \dots, x_n \in \kappa(X)$ and

$$\begin{aligned}
\phi_x(A \cup A_0) &= \sum_{i=1}^{n} \lambda_i \phi_{x_i}(A \cup A_0) = \sum_{i=1}^{n} \lambda_i P^{\varepsilon_{A \cup A_0}}_{\operatorname{span}T(A \cup A_0)}(x_i) \\
&= \sum_{i=1}^{n} \lambda_i x_i = x, \;\; A \in S_\gamma.
\end{aligned}$$

Hence

$$P(x) = M(\phi_x) = M(\phi_x(\cdot \cup A_0)) = x.$$

□

Theorem 3.2. *Let S be a commutative cancellative semigroup of torsion-free rank δ, $\gamma = \max(\delta, \omega)$. If there exists an X-valued invariant λ-mean $M_S\colon \ell_\infty(S, X) \to X$, then there exists an X-valued invariant λ-mean $M\colon \ell_\infty(S_\gamma, X) \to X$.*

Proof. First we observe that we can assume that S contains only elements of infinite order. Indeed the set G of all elements of finite order is a group and a torsion-free rank of S/G is equal to γ. In view of Lemma 2.4 there exists an X-valued invariant λ-mean on S/G.

Let $A \subset S$ be a maximal linearly independent set. Hence $|A| = \delta$.

- First assume that $|A| = \gamma$ and let $A = \{x_\alpha : \alpha < \gamma\}$. For each $x \in S$ we define a set

$$D_x := \{x_1, \ldots, x_n \in A : \exists_{k,k_1,\ldots,k_n \in \mathbb{N}}\, \exists_{I \subset \{1,\ldots,n\}}\, kx + \sum_{i\in I} k_i x_i = \sum_{i \notin I} k_i x_i\}.$$

First, we show that the above set is well-defined. If there exist $k, m \in \mathbb{N}$, $k_1, \ldots k_n, m_1, \ldots m_n \in \mathbb{N} \cup \{0\}$, $x_1, \ldots, x_n \in A$, and $I, J \subset \{1, \ldots, n\}$ such that $k_i \neq 0$ for $i \in I$, $m_i \neq 0$ for $i \in J$ and

$$kx + \sum_{i\in I} k_i x_i = \sum_{i\notin I} k_i x_i,$$
$$mx + \sum_{i\in J} m_i x_i = \sum_{i\notin J} m_i x_i,$$

then

$$mkx + \sum_{i\in I} mk_i x_i + \sum_{i\notin J} km_i x_i = kmx + \sum_{i\in J} km_i x_i + \sum_{i\notin I} mk_i x_i,$$

whence

$$\sum_{i\in I} mk_i x_i + \sum_{i\notin J} km_i x_i = \sum_{i\in J} km_i x_i + \sum_{i\notin I} mk_i x_i,$$

so

$$\sum_{i\in I\cap J} mk_i x_i + \sum_{i\in I\setminus J} (mk_i + km_i) x_i + \sum_{i\notin I\cup J} km_i x_i$$
$$= \sum_{i\in I\cap J} km_i x_i + \sum_{i\in J\setminus I} (km_i + mk_i) x_i + \sum_{i\notin I\cup J} mk_i x_i.$$

As A is linearly independent, we have $I \setminus J = J \setminus I = \emptyset$, which means that $I = J$. Thus we get that $km_i = mk_i$ for $i \in \{1, \ldots, n\}$, so D_x is well-defined.

We define a map $\varphi\colon \ell_\infty(S_\gamma, X) \to \ell_\infty(S, X)$ by the formula

$$\varphi(f)(x) := f(\{\alpha < \gamma : x_\alpha \in D_x\}), \; x \in S, \; f \in \ell_\infty(S_\gamma, X).$$

It is easy to observe that φ is linear, $\|\varphi(f)\| = \|f\|$ for $f \in \ell_\infty(S_\gamma, X)$ and $\varphi(c\mathbb{1}_{S_\gamma}) = c\mathbb{1}_S$ for $c \in X$.

Let $M_S\colon \ell_\infty(S,X) \to X$ be an X-valued invariant λ-mean. We define $M\colon \ell_\infty(S_\gamma,X) \to X$ by the formula

$$M(f) := M_S(\varphi(f)),\ f \in \ell_\infty(S_\gamma,X).$$

From the properties of φ we obtain that M is linear, $M(c\mathbb{1}_{S_\gamma}) = c$ for $c \in X$, and $\|M\| \le \|M_S\| \le \lambda$.

Now we show that M is invariant. Let $f \in \ell_\infty(S_\gamma,X)$ and $A \in S_\gamma$. Since $A = \{\alpha_1,\dots,\alpha_n\}$, from the invariance on each singleton $\{\alpha_i\}$ we obtain

$$\begin{aligned} M({}_A f) &= M({}_{\{\alpha_1\}}({}_{\{\alpha_2,\dots,\alpha_n\}}f)) = M({}_{\{\alpha_2,\dots,\alpha_n\}}f) = \dots \\ &= M({}_{\{\alpha_n\}}f) = M(f),\ f \in \ell_\infty(S_\gamma,X). \end{aligned}$$

Hence we need to prove the invariance on each singleton, so we can assume that $A = \{\beta\}$ for some $\beta < \gamma$. Let $Z := \{x \in S:\ x_\beta \notin D_{x+x_\beta}\}$. We show that

$$Z \cap (mx_\beta + Z) = \emptyset,\ m \in \mathbb{N}. \tag{3.1}$$

Suppose that $x \in Z \cap (mx_\beta + Z)$ for some $m \in \mathbb{N}$. Then there exists $y \in Z$ such that $x = mx_\beta + y$. Hence $x_\beta \notin D_{y+x_\beta} \cup D_{y+(m+1)x_\beta}$ but on the other hand, if $x_\beta \notin D_{y+x_\beta}$, then $x_\beta \in D_{y+(m+1)x_\beta}$, so we have a contradiction.

Since S is cancellative, from (3.1) we obtain that

$$(nx_\beta + Z) \cap (mx_\beta + Z) = \emptyset,\ m,n \in \mathbb{N}_0,\ m > n. \tag{3.2}$$

Let $g \in \ell_\infty(S,X)$ be such that $g(x) = 0$ for $x \in S \setminus Z$. From (3.2) we get

$$\begin{aligned} n\|M_S(g)\| &= \Big\|\sum_{i=1}^n M_S({}_{ix_\beta}g)\Big\| = \Big\|M_S\Big(\sum_{i=1}^n {}_{ix_\beta}g\Big)\Big\| \\ &\le \lambda\Big\|\sum_{i=1}^n {}_{ix_\beta}g\Big\| \le \lambda\|g\|,\ n \in \mathbb{N}, \end{aligned}$$

so $M_S(g) = 0$.

For each $y \in S$ we have

– if $x_\beta \notin D_y$, then $D_{y+x_\beta} = D_y \cup \{x_\beta\}$, so

$$\begin{aligned} \varphi({}_{\{\beta\}}f)(y) &= f(\{\alpha < \gamma:\ x_\alpha \in D_y\} \cup \{\beta\}) \\ &= f(\{\alpha < \gamma:\ x_\alpha \in D_{y+x_\beta}\}) = \big({}_{x_\beta}\varphi(f)\big)(y); \end{aligned}$$

– if $x_\beta \in D_y$ and $x_\beta \in D_{y+x_\beta}$, then $D_{y+x_\beta} = D_y$, so

$$\begin{aligned} \varphi({}_{\{\beta\}}f)(y) &= f(\{\alpha < \gamma:\ x_\alpha \in D_y\} \cup \{\beta\}) = f(\{\alpha < \gamma:\ x_\alpha \in D_y\}) \\ &= f(\{\alpha < \gamma:\ x_\alpha \in D_{y+x_\beta}\}) = \big({}_{x_\beta}\varphi(f)\big)(y); \end{aligned}$$

– if $x_\beta \notin D_{y+x_\beta}$, then $y \in Z$.

Hence

$$\big(\varphi({}_{\{\beta\}}f) -_{x_\beta} \varphi(f)\big)(y) = 0,\ y \in S \setminus Z,$$

so

$$M({}_{\{\beta\}}f) = M_S(\varphi({}_{\{\beta\}}f) = M_S({}_{x_\beta}\varphi(f)) = M_S(\varphi(f)) = M(f).$$

- Now assume that $|A| < \gamma$. Hence $\gamma = \omega$. Let $N = |A|$, $A = \{x_1, \ldots, x_N\}$. Since S can be embedded in a group, for each $x \in S$ there exist $k(x) \in \mathbb{N}$, $k_1(x), \ldots, k_N(x) \in \mathbb{Z}$ such that $k(x)x = \sum_{i=1}^{N} k_i(x)x_i$. We define a map $\varphi \colon \ell_\infty(S_\omega, X) \to \ell_\infty(S, X)$ by the formula

$$\varphi(f)(x) := f\Big(\big\{\alpha \in \omega :\ \alpha k(x) \leq |k_1(x)|\big\}\Big),\ x \in S,\ f \in \ell_\infty(S_\omega, X).$$

It is easy to observe that φ is linear, $\|\varphi(f)\| \leq \|f\|$ for $f \in \ell_\infty(S_\omega, X)$ and $\varphi(c\mathbb{1}_{S_\omega}) = c\mathbb{1}_S$ for $c \in X$.
Let $M_S \colon \ell_\infty(S, X) \to X$ be an X-valued invariant λ-mean. We define $M \colon \ell_\infty(S_\omega, X) \to X$ by the formula

$$M(f) := M_S(\varphi(f)),\ f \in \ell_\infty(S_\omega, X).$$

From the properties of φ we obtain that M is linear, $M(c\mathbb{1}_{S_\gamma}) = c$ for $c \in X$, and $\|M\| \leq \lambda$.
Now we show that M is invariant. Let $f \in \ell_\infty(S_\omega, X)$ and $A \in S_\omega$. Similarly as in the previous case we need only to prove the invariance on each singleton, so we can assume that $A = \{\beta\}$ for some $\beta \in \omega$. Let

$$Z := \Big\{x \in S :\ |k_1(x)| < \beta k(x)\Big\}.$$

We show that

$$Z \cap (2m\beta x_1 + Z) = \emptyset,\ m \in \mathbb{N}. \tag{3.3}$$

Suppose that $x \in Z \cap (mx_\beta + Z)$ for some $m \in \mathbb{N}$. Then there exists $y \in Z$ such that $x = 2m\beta x_1 + y$. Hence

$$k(y)[y + 2m\beta x_1] = [k_1(y) + 2mk(y)\beta]x_1 + \sum_{i=2}^{N} k_i(y)x_i,$$

which gives us

$$\beta k(y) > k_1(y) + 2m\beta k(y) > -\beta k(y) + 2\beta k(y) = \beta k(y),$$

so we have a contradiction.
Since S is cancellative, from (3.3) we obtain that

$$(2n\beta x_1 + Z) \cap (2m\beta x_1 + Z) = \emptyset,\ m, n \in \mathbb{N}_0,\ m > n. \tag{3.4}$$

Now observe that for $x \in S \setminus Z$ we have

$$\begin{aligned}\varphi(f_{\{\beta\}}(x)) &= f_{\{\beta\}}\Big(\big\{\alpha \in \omega :\ \alpha k(x) \leq |k_1(x)|\big\}\Big)\\ &= f\Big(\big\{\alpha \in \omega :\ \alpha k(x) \leq |k_1(x)|\big\} \cup \{\beta\}\Big)\\ &= f\Big(\big\{\alpha \in \omega :\ \alpha k(x) \leq |k_1(x)|\big\}\Big) = \varphi(f(x)),\end{aligned}$$

so from (3.4) we obtain that

$$\begin{aligned}&n\|M(f - f_{\{\beta\}})\| = \|nM_S\big(\varphi(f) - \varphi(f_{\{\beta\}})\big)\|\\ &= \|\sum_{i=1}^{n} M_S\big((\varphi(f) - \varphi(f_{\{\beta\}}))_{2i\beta x_1}\big)\|\\ &= \|M_S\Big(\sum_{i=1}^{n}\big(\varphi(f) - \varphi(f_{\{\beta\}})\big)_{2i\beta x_1}\Big)\| \leq \lambda\|\varphi(f) - \varphi(f_{\{\beta\}})\|\end{aligned}$$

for every $n \in \mathbb{N}$, which means that $M(f_{\{\beta\}}) = M(f)$.

□

Using Theorems 1.4, 3.1 and 3.2 we obtain the following

Corollary 3.3. *The following assertions are equivalent:*

1. *X is complemented in X^{**} by a projection of norm at most λ;*
2. *for every amenable semigroup S there exists an X-valued invariant λ-mean on S;*
3. *for any cancellative semigroup S of torsion-free rank δ,* $\operatorname{dens} X^{**} = \max(\delta, \omega)$, *there exists an X-valued invariant λ-mean on S.*

The following example shows that in general in the third assertion of the previous corollary the torsion-free rank of semigroup S cannot be less than the density of X.

Example 3.4. Let Γ be an uncountable set such that $|\Gamma|$ is a regular cardinal number. We define the set

$$X := \{f \in \ell_\infty(\Gamma) :\ |\{\alpha \in \Gamma :\ f(\alpha) \neq 0\}| < |\Gamma|\}.$$

It is easy too see that X is a Banach space. Since $\mathbb{1}_{\{\alpha\}} \in X$ for $\alpha \in \Gamma$, $\operatorname{dens} X = |\Gamma|$.

Let S be an amenable semigroup, $|S| < \operatorname{dens} X$ and $L\colon \ell_\infty(S, \mathbb{R}) \to \mathbb{R}$ be an invariant mean. We define $M\colon \ell_\infty(S, X) \to X$ by the formula

$$M(g)(\alpha) := L(g(\cdot)(\alpha)),\ g \in \ell_\infty(S, X),\ \alpha \in \Gamma.$$

First, we observe that

$$\begin{aligned}\{\alpha \in \Gamma :\ M(g)(\alpha) \neq 0\} &= \{\alpha \in \Gamma :\ L(g(\cdot)(\alpha)) \neq 0\}\\ &\subset \bigcup_{s \in S}\{\alpha \in \Gamma :\ g(s)(\alpha) \neq 0\}\end{aligned}$$

and since $|\Gamma|$ is regular, we have

$$|\{\alpha \in \Gamma :\ M(g)(\alpha) \neq 0\}| \leq |S| \cdot \sup_{s\in S} |\{\alpha \in \Gamma :\ g(s)(\alpha) \neq 0\}| < |\Gamma|,$$

so M is well-defined.

It is easy to see that M is linear. We have also

$$\begin{aligned} \|M(g)\| &= \sup_{\alpha\in\Gamma} |M(g)(\alpha)| = \sup_{\alpha\in\Gamma} |L(g(\cdot)(\alpha))| \\ &\leq \sup_{\alpha\in\Gamma} \sup_{s\in S} |g(s)(\alpha)| = \sup_{s\in S} \|g(s)\| = \|g\|,\ g \in \ell_\infty(S, X), \end{aligned}$$

and

$$M(c\mathbb{1}_S)(\alpha) = L(c(\alpha)\mathbb{1}_S) = c(\alpha),\ c \in X,\ \alpha \in \Gamma.$$

Finally, we observe that

$$\begin{aligned} M({}_a g)(\alpha) &= L(g(a + \cdot)(\alpha)) = L(g(\cdot)(\alpha)) \\ &= M(g)(\alpha),\ g \in \ell_\infty(S, X),\ a \in S,\ \alpha \in \Gamma, \end{aligned}$$

so M is an X-valued invariant mean.

In the paper of Pełczyński and Sudakov [13, Theorem 1] it is shown that X isn't complemented in its bidual.

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