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Aequationes Mathematicae

On linear functional equations modulo $\mathbb{Z}$

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Dedicated to Professor Ludwig Reich on the occasion of his 80th birthday.

Abstract. In this paper, we consider the condition $\sum_{i=0}^{n+1} \varphi_i(r_i x + q_i y) \in \mathbb{Z}$ for real valued functions defined on a linear space V. We derive necessary and sufficient conditions for functions satisfying this condition to be decent in the following sense: there exist functions $f_i : V \to \mathbb{R}$, $g_i : V \to \mathbb{Z}$ such that $\varphi_i = f_i + g_i$, $(i = 0, \dots, n+1)$ and $\sum_{i=0}^{n+1} f_i(r_i x + q_i y) = 0$ for all $x, y \in V$.

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Introduction

The topic of this paper is connected to linear functional equations of the form

$$\sum_{i=0}^{n+1} f_i(r_i x + q_i y) = 0, \qquad (x, y \in V) \tag{1}$$

where V is a linear space, n is a positive integer, $r_0, \dots, r_{n+1}$ and $q_0, \dots, q_{n+1}$ are real numbers and $f_0, \dots, f_{n+1} : V \to \mathbb{R}$ are unknown functions.

It is easy to see that this class contains several fundamental functional equations (e.g., the Cauchy, the Jensen, the square-norm, and the Pexider equations) as a special case. Its investigation goes back (at least) to the beginning of the twentieth century (cf., e.g., [9,26]). Its solutions, in a general case, were determined by Székelyhidi [22,23]. A computer program presenting

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the solutions of functional equations of type (1) was described in [7] (cf., also, [5,6,10,11]). Problems connected to class (1), its generalizations and its applications have been studied by several authors during the last more than 100 years. (Recent related results can be found, among others, in [13,15–17,24].)

In this paper, using the notation above, we consider linear functional equations modulo $\mathbb{Z}$, i.e., we investigate the property

$$\sum_{i=0}^{n+1} \varphi_i(r_i x + q_i y) \in \mathbb{Z} \qquad (x, y \in V). \tag{2}$$

Our aim is to describe the form of the functions satisfying (2). Our investigations were mainly motivated by results of J. A. Baker, K. Baron, J. Brzdęk, M. Sablik, P. Volkmann published in the papers [1,3,4,8], respectively, connected to the Cauchy equation modulo $\mathbb{Z}$, as well as, by studies of A. Lewicka on polynomial functional equations modulo $\mathbb{Z}$ [20].

1. Preliminaries

In this section, we give some definitions and present some preliminary results we need to formulate and to prove our statements.

We start with the well-known concept of the difference operator. In its definition and in the remaining part of the paper V denotes a linear space over $\mathbb{Q}$ (or $\mathbb{R}$ or $\mathbb{C}$). Let $f : V \to \mathbb{R}$ be a function, let, furthermore,

$$\Delta_y^0 f(x) = f(x) \qquad (x, y \in V)$$

and, for a non-negative integer n,

$$\Delta_y^{n+1} f(x) = \Delta_y^n f(x+y) - \Delta_y^n f(x) \qquad (x, y \in V).$$

It is easy to prove that, with the notation above,

$$\Delta_y^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+ky) \qquad (x, y \in V) \tag{3}$$

for each non-negative integer n.

If n is a non-negative integer, a function $f : V \to \mathbb{R}$ is said to be a *polynomial function of degree n* (in another terminology a *polynomial function of degree at most n*) if it satisfies

$$\Delta_y^{n+1} f(x) = 0 \qquad (x, y \in V), \tag{4}$$

f is called a *monomial function of degree n* if it fulfils

$$\Delta_y^n f(x) - n! f(y) = 0 \qquad (x, y \in V). \tag{5}$$

Remark 1. A simple application of property (3) yields that Eqs. (4) and (5) can be obtained with properly chosen functions $f_0, \ldots, f_{n+1}$ and numbers $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ in (1), thus, the classes of polynomial and monomial equations are sub-classes of the set of linear functional equations of type (1).

Remark 2. There is a notable connection between polynomial and monomial functions. With the notation above, if $a_k : V \to \mathbb{R}$, $(k = 0, \ldots, n)$ are monomial functions of degree k, then the function $f : V \to \mathbb{R}$, defined by

$$f = \sum_{k=0}^{n} a_k$$

is a polynomial function of degree n.

On the other hand, if $f : V \to \mathbb{R}$ is a polynomial function of degree n, then there exist monomial functions $a_k : V \to \mathbb{R}$ of degree k, $(k = 0, \ldots, n)$ such thatfor the functions $f_0, \ldots, f_{n+1}$ for the numbers $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ in (1)

$$f = \sum_{k=0}^{n} a_k.$$

(cf., e.g. [18,19] and, in a general setting, [23]).

According to Remark 2, monomial functions are a kind of 'building blocks' for polynomial functions. As L. Székelyhidi proved (cf. [22], furthermore, [23, 26]), this property, under some circumstances, is also valid for solutions of general linear functional equations of type (1).

Theorem 1. *Let V be a linear space, n be a non-negative integer and let $r_0, \ldots, r_{n+1}$, $q_0, \ldots, q_{n+1}$ be rational numbers satisfying the property*

$$r_i q_j \neq r_j q_i \qquad (i, j = 0, \ldots, n+1,\ i \neq j). \tag{6}$$

The functions $f_0, \ldots, f_{n+1} \colon V \to \mathbb{R}$ solve functional equation (1), if and only if,

$$f_i = \sum_{k=0}^{n} a_k^{(i)}, \qquad (i = 0, \ldots, n+1) \tag{7}$$

where $a_k^{(i)} \colon V \to \mathbb{R}$ $(i = 0, \ldots, n+1,\ k = 0, \ldots, n)$ are monomial functions of degree k such that

$$\sum_{i=0}^{n+1} r_i^j q_i^{k-j} a_k^{(i)}(x) = 0 \qquad (x \in V,\ k = 0, \ldots, n,\ j = 0, \ldots, k).$$

As it was mentioned in the Introduction, in this paper, we investigate functions satisfying property (2). Our study was inspired by several articles (e. g.

[1,3,4] and [8]), on the *Cauchy equation modulo* $\mathbb{Z}$ (in another terminology *Cauchy's congruence*), i.e., on the condition

$$\varphi(x+y)-\varphi(x)-\varphi(y)\in\mathbb{Z}\qquad(x,y\in V),\tag{8}$$

where v is a linear space and $\varphi\colon V\to\mathbb{R}$ is a function.

According to Baker [1], a function $\varphi\colon V\to\mathbb{R}$ is called a *decent solution of* (1) (or a *decent solution of Cauchy's congruence*), if it satisfies (1) and there exist functions $f\colon V\to\mathbb{R}$ and $g\colon V\to\mathbb{Z}$, such that f is a solution of Cauchy's functional equation

$$f(x+y)-f(x)-f(y)=0\qquad(x,y\in V)$$

and φ is of the form $\varphi=f+g$. (Further connected results can be found, among others, in the papers [2,14,21].)

Motivated by the investigations above, in the case when V is a linear space, n is a non-negative integer and $\varphi\colon V\to\mathbb{R}$ is a function, A. Lewicka considered the property

$$\Delta_y^{n+1}\varphi(x)\in\mathbb{Z}\qquad(x,y\in V),\tag{9}$$

which she called *polynomial congruence of degree* n (cf. [20]). Analogously to the case connected to the Cauchy equation, a function $\varphi\colon V\to\mathbb{R}$ is said to be a *decent solution of the polynomial congruence of degree* n if it satisfies (9) and there exist functions $f\colon V\to\mathbb{R}$ and $g\colon V\to\mathbb{Z}$, such that f is a polynomial function of degree n and $\varphi=f+g$ [20].

In her paper [20], A. Lewicka proved, among others, the following 3 Theorems.

Theorem 2. *Let V be a linear space, n be a non-negative integer and assume that $\varphi\colon V\to\mathbb{R}$ fulfils the polynomial congruence of degree n. Then the following conditions are equivalent:*

(a) *The function φ is a decent solution of the polynomial congruence of degree n.*
(b) *For every $v\in V$, there exists a polynomial p_v of degree smaller than $n+1$ with real coefficients such that $\varphi(\xi v)-p_v(\xi)\in\mathbb{Z}$ for all $\xi\in\mathbb{Q}$.*
(c) *For every vector $v\in V$, there exist $\varepsilon>0$ and a polynomial p_v of degree smaller than $n+1$ with real coefficients such that $\varphi(\xi v)-p_v(\xi)\in\mathbb{Z}$ for all $\xi\in\mathbb{Q}\cap(0,\varepsilon)$.*
(d) *For every vector $v\in V$, there exist $\varepsilon>0$ and a polynomial p_v of degree smaller than $n+1$ with real coefficients such that $\tilde{\varphi}(\xi v)-\tilde{p}_v(\xi)\in\mathbb{Z}$ for all $\xi\in\mathbb{Q}\cap(0,\varepsilon)$.*
(e) *For every vector $v\in V$, there exist $\varepsilon>0$ and $\alpha\in[0,1]$ such that for every $\xi\in\mathbb{Q}\cap(0,\varepsilon)$ we have $\tilde{\varphi}(\xi v)\in\left(\alpha,\alpha+\frac{1}{2^{n+1}}\right)$.*
(f) *For every vector $v\in V$, there exists $\varepsilon>0$ such that the function $\xi\ni\mathbb{Q}\to\tilde{\varphi}(\xi v)$ is monotone on $\mathbb{Q}\cap(0,\varepsilon)$.*

Theorem 3. *Let V be a linear space and let $E \subseteq X$ such that $intH(E) \neq \emptyset$. If $\varphi\colon V \to \mathbb{R}$ fulfils the polynomial congruence of degree n so that $\varphi(x) \in \mathbb{Z} + (-\alpha, \alpha)$ for $x \in E$ with some $0 < \alpha < \frac{1}{2^{n+1}(2^{n+1}-1)}$, then φ is a decent solution of the polynomial congruence of degree n. Moreover, $\varphi = f + g$, where f is a continuous polynomial function of degree n and g is an integer valued function.*

Theorem 4. *Let V be a linear space and let $\varphi\colon V \to \mathbb{R}$ be a solution of the polynomial congruence of degree n. Assume that one of the following two hypotheses is valid*

(a) *$X = \mathbb{R}^m$ with some positive m and φ is Lebesgue measurable,*
(b) *X is a real Fréchet space and φ is a Baire measurable function.*

Then φ is a decent solution of the polynomial congruence of degree n. Moreover, $\varphi = f + g$ with f being a continuous polynomial function of degree n and g being an integer-valued and Lebesgue (resp. Baire) measurable function.

Similarly to the statements formulated in Remark 1, it is easy to see that property (9) can be obtained with properly chosen functions $f_0, \ldots, f_{n+1}$ and numbers $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ in (2). This means that, for each non-negative integer n, a solution of the polynomial congruence of degree n also satisfies a certain linear congruence of type (2).

Based on this fact, we may introduce the following generalization of the concepts above. If V is a linear space, n is a non-negative integer, $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ are rational numbers, the functions $\varphi_0, \ldots, \varphi_{n+1}\colon V \to \mathbb{R}$ are called *decent solutions of* (2), if they satisfy (2), furthermore, there exist functions $f_0, \ldots, f_{n+1}\colon V \to \mathbb{R}$ and $g_0, \ldots, g_{n+1}\colon V \to \mathbb{Z}$, such that $f_0, \ldots, f_{n+1}$ are solutions of (1) and $\varphi_i = f_i + g_i$ for $i = 0, \ldots, n+1$.

It is easy to see that, if $\varphi_i = f_i + g_i$, such that $f_i\colon V \to \mathbb{R}$, $g_i\colon V \to \mathbb{Z}$ and $f_0, \ldots, f_{n+1}$ satisfy (1), then the system of functions $\varphi_0, \ldots, \varphi_{n+1}$ fulfils (2). In the remaining part of the paper, we investigate the question, which conditions imply that functions, satisfying (2) are its decent solutions.

We note that, as a consequence of G. Godini's example in [12], generally, it is not true that each function fulfilling (1) is a decent solution of (1) (cf., also, [25]). Since the Cauchy equation is a special case of (1), it is also not always true that functions satisfying (2) are decent.

2. Main results

We start our investigations with the examination of the connection between polynomial congruences and linear congruences. Our first theorem reads as follows.

Theorem 5. *Let V be a linear space, n be a non-negative integer, let $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ be rational numbers with the property (6) and assume that* *the functions* $\varphi_0, \dots, \varphi_{n+1} \colon V \to \mathbb{R}$ *satisfy* (2). *Then each of the functions* $\varphi_0, \dots, \varphi_{n+1}$ *fulfils the polynomial congruence of degree* n.

Proof. At first, we show that φ_{n+1} satisfies (9).

Assume that $r_{n+1} \neq 0$ and $q_{n+1} \neq 0$. Taking arbitrary $u, w \in V$ and writing $x = \frac{u-w}{r_{n+1}}$ and $y = \frac{w}{q_{n+1}}$ in (2), we get

$$\varphi_{n+1}(u) + \sum_{i=0}^{n} \varphi_i \left(\frac{r_i}{r_{n+1}} u + \left(\frac{q_i}{q_{n+1}} - \frac{r_i}{r_{n+1}} \right) w \right) \in \mathbb{Z} \qquad (u, w \in V). \quad (10)$$

Denote $R_i = \frac{r_i}{r_{n+1}}$ and $Q_i = \frac{q_i}{q_{n+1}} - \frac{r_i}{r_{n+1}}$ for $i \in \{0, \dots, n\}$. Then $R_i, Q_i \in \mathbb{Q}$, $R_i, Q_i \neq 0$, $R_i Q_j \neq R_j Q_i$ for $i, j \in \{0, \dots, n\}, i \neq j$ and

$$\varphi_{n+1}(u) + \sum_{i=0}^{n} \varphi_i \left(R_i u + Q_i w \right) \in \mathbb{Z} \qquad (u, w \in V). \quad (11)$$

By property (6), r_{n+1} and q_{n+1} cannot be 0 simultaneously. If $q_{n+1} = 0$ and $r_{n+1} \neq 0$, then put $x = \frac{u}{r_{n+1}}, y = w$ in (2) to get (11) with $R_i = \frac{r+i}{r_{n+1}}, Q_i = q_i$.

Replacing u by $u + s$ and w by $w + t$, we obtain

$$\varphi_{n+1}(u+s) + \sum_{i=0}^{n} \varphi_i \left((R_i u + Q_i w) + (R_i s + Q_i t) \right) \in \mathbb{Z} \qquad (u, w, s, t \in V). \quad (12)$$

Substracting the left hand side of (11) from the left hand side of (12), we obtain

$$\Delta_s \varphi_{n+1}(u) + \sum_{i=0}^{n} \Delta_{R_i s + Q_i t} \varphi_i \left(R_i u + Q_i w \right) \in \mathbb{Z} \qquad (u, w, s, t \in V). \quad (13)$$

Writing $t = -\frac{R_n}{Q_n} s$ in (13), we get

$$\Delta_s \varphi_{n+1}(u) + \sum_{i=0}^{n-1} \Delta_{\frac{R_i Q_n - R_n Q_i}{Q_n} s} \varphi_i \left(R_i u + Q_i w \right) \in \mathbb{Z} \qquad (u, w, s \in V). \quad (14)$$

Considering this for the function $\Delta_s \varphi_{n+1}$ instead of φ_{n+1} and for $\Delta_{\frac{R_i Q_n - R_n Q_i}{Q_n} s} \varphi_i$ instead of φ_i for $i \in \{0, \dots, n-1\}$, we obtain

$$\Delta_{s', s} \varphi_{n+1}(u) + \sum_{i=0}^{n-2} \Delta_{\frac{R_i Q_{n-1} - R_{n-1} Q_i}{Q_{n-1}} s', \frac{R_i Q_n - R_n Q_i}{Q_n} s} \varphi_i \left(R_i u + Q_i w \right) \in \mathbb{Z} \qquad (u, w, s, s' \in V). \quad (15)$$

A continuation of this argumentation yields that

$$\Delta_s^{n+1} \varphi_{n+1}(u) \in \mathbb{Z} \qquad (u, s \in V).$$

□

Due to Theorem 1, we can characterize decent solutions of (2) in terms of decent solutions of polynomial congruences.

Theorem 6. *Let V be a linear space, n be a non-negative integer, let $r_0, \dots, r_{n+1}$ and $q_0, \dots, q_{n+1}$ be rational numbers satisfying (6) and assume that $\varphi_0, \dots, \varphi_{n+1} \colon V \to \mathbb{R}$ fulfil (2). The functions $\varphi_0, \dots, \varphi_{n+1}$ are decent solutions of (2), if and only if, each of the functions $\varphi_0, \dots, \varphi_{n+1}$ is a decent solution of the polynomial congruence of degree n.*

Proof. Assume that $\varphi_i = f_i + g_i$, for $i = 0, \dots, n+1$ with some functions $f_1, \dots, f_{n+1} \colon V \to \mathbb{R}$ and $g_1, \dots, g_{n+1} \colon V \to \mathbb{Z}$, such that $f_1, \dots, f_{n+1}$ solve (1). Then Theorem 1 implies that each of the functions f_i is of the form (7), thus, it is a polynomial function of degree n. Therefore, each of the functions φ_i is a decent solution of the polynomial congruence of degree n.

Now, assume that $\varphi_i = f_i + g_i$ for $i = 0, \dots, n+1$, where $f_1, \dots, f_{n+1} \colon V \to \mathbb{R}$ and $g_1, \dots, g_{n+1} \colon V \to \mathbb{Z}$ such that $f_1, \dots, f_{n+1}$ are polynomial functions of degree n. According to Remark 2, there exist monomial functions $a_k^{(i)} : V \to \mathbb{R}$ of degree k, $(i = 0, \dots, n+1,\ k = 0, \dots, n)$, such that

$$f_i = \sum_{k=0}^{n} a_k^{(i)}.$$

Taking an arbitrary $\xi \in \mathbb{Q}$ and putting $x = \xi u$, $y = \xi w$ for $u, w \in V$ in (2), we obtain

$$\sum_{i=0}^{n+1} \varphi_i(\xi(r_i u + q_i w)) \in \mathbb{Z} \qquad (u, w \in V),$$

thus,

$$\sum_{i=0}^{n+1} f_i(\xi(r_i u + q_i w)) \in \mathbb{Z} \qquad (u, w \in V).$$

On the other hand,

$$\begin{aligned}\sum_{i=0}^{n+1} f_i(\xi(r_i u + q_i w)) &= \sum_{i=0}^{n+1} \left(\sum_{k=0}^{n} a_k^{(i)}(\xi(r_i u + q_i w)) \right) \\ &= \sum_{k=0}^{n} \xi^k \left(\sum_{i=0}^{n+1} a_k^{(i)}(r_i u + q_i w) \right) \qquad (u, w \in V).\end{aligned}$$

Since the equation above is valid for all rational numbers ξ, we get that

$$\sum_{i=0}^{n+1} a_k^{(i)}(r_i u + q_i w) = 0 \qquad (u, w \in V,\ k = 1, \dots, n)$$

and

$$\sum_{i=0}^{n+1} a_0^{(i)}(r_i u + q_i w) \in \mathbb{Z} \qquad (u, w \in V).$$

Therefore,

$$\sum_{i=0}^{n+1} f_i(\xi(r_i u + q_i w)) = \sum_{i=0}^{n+1} a_0^{(i)}(r_i u + q_i w) \qquad (u, w \in V).$$

Writing $\hat{f}_i = f_i$ and $\hat{g}_i = g_i$ for $i = 0, \ldots, n$, furthermore,

$$\hat{f}_{n+1} = f_{n+1} - \sum_{i=0}^{n+1} a_0^{(i)}(r_i u + q_i w) \qquad (u, w \in V)$$

and

$$\hat{g}_{n+1} = g_{n+1} + \sum_{i=0}^{n+1} a_0^{(i)}(r_i u + q_i w), \qquad (u, w \in V)$$

we obtain that $\varphi_i = \hat{f}_i + \hat{g}_i$ and $\hat{g}_i \colon V \to \mathbb{Z}$ for $i = 0, \ldots, n+1$ and

$$\sum_{i=0}^{n+1} \hat{f}_i(\xi(r_i u + q_i w)) = \sum_{i=0}^{n+1} f_i(\xi(r_i u + q_i w)) - \sum_{i=0}^{n+1} a_0^{(i)}(r_i u + q_i w) = 0 \qquad (u, w \in V),$$

which proves that $\varphi_0, \ldots, \varphi_{n+1}$ are decent solutions of (2). $\square$

Using Theorems 2 and 6, we can prove the following statement.

Corollary 1. *Let V be a linear space, n be a non-negative integer, let $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ be rational numbers satisfying* (6) *and assume that*
$\varphi_0, \ldots, \varphi_{n+1} \colon V \to \mathbb{R}$ fulfil (2). *Then the following conditions are equivalent.*

- (a) *The functions $(\varphi_0, \ldots, \varphi_{n+1})$ are decent solutions of* (2).
- (b) *For each $v \in V$ and $i \in \{0, \ldots, n{+}1\}$, there exist a polynomial $p_v^i \in \mathbb{R}_n[X]$ and $\varepsilon > 0$, such that $\varphi_i(\xi v) - p_v^i(\xi) \in \mathbb{Z}$ for $\xi \in \mathbb{Q} \cap (0, \varepsilon)$.*
- (c) *For each $v \in V$, there exist $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, such that $\tilde{\varphi}_i(\xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}})$ for $\xi \in \mathbb{Q} \cap (0, \varepsilon)$.*
- (d) *For each $v \in V$, there exists $\varepsilon > 0$ such that the function $\mathbb{Q} \ni \xi \to \tilde{\varphi}_i(\xi v) \in [0, 1)$ is monotone on $\mathbb{Q} \cap (0, \varepsilon)$.*

Proof. According to Theorem 2, each of the conditions (b), (c) and (d) is equivalent to the decency of each of the functions $\varphi_0, \ldots, \varphi_{n+1}$ as solutions of the polynomial congruence of degree n, which implies the decency of the functions $\varphi_0, \ldots, \varphi_{n+1}$ as solutions of (2) (cf. Theorem 6). $\square$

Finally, we formulate and prove two corollaries of our theorems as well as some results of Lewicka [20] on solutions of linear congruences satisfying certain regularity properties.

Corollary 2. *Let V be a linear space, n be a non-negative integer, let $r_0, \ldots, r_{n+1}$ and $q_0, \ldots, q_{n+1}$ be rational numbers satisfying* (6) *and assume that*

$\varphi_0, \ldots, \varphi_{n+1} \colon V \to \mathbb{R}$ *fulfil* (2)*. If there exists a set* $E_{v,i} \subseteq \mathbb{R}$ *for every* $v \in V$ *and* $i \in \{0, \ldots, n+1\}$ *such that* $\mathrm{int}H(E_{v,i}) \neq \emptyset$ *and*

$$\varphi_i(\xi v) \in \mathbb{Z} + \left(-\frac{1}{2^n(2^{n-2}-1)}, \frac{1}{2^n(2^{n-2}-1)}\right) \qquad (\xi \in E_{v,i}),$$

then $\varphi_0, \ldots, \varphi_{n+1}$ *are decent solutions of* (2)*.*

Proof. Since $\varphi_0, \ldots, \varphi_{n+1}$ fulfil (2), Theorem 5 yields that each of the functions above satisfy (9). Thus, according to Theorem 3, each of the functions $\varphi_0, \ldots, \varphi_{n+1}$ is a decent solution of (9). Therefore, as a consequence of Theorem 6, we obtain that $\varphi_0, \ldots, \varphi_{n+1}$ are decent solutions of (2). □

Corollary 3. *Let* V *be a linear space,* n *be a non-negative integer, let* $r_0, \ldots, r_{n+1}$ *and* $q_0, \ldots, q_{n+1}$ *be rational numbers satisfying* (6) *and assume that* $\varphi_0, \ldots, \varphi_{n+1} \colon V \to \mathbb{R}$ *fulfil* (2)*. If each of the functions* $\varphi_0, \ldots, \varphi_{n+1}$ *is either Lebesgue measurable or Baire measurable or* $\mathbb{Q}$*-radial continuous at some point, then* $\varphi_0, \ldots, \varphi_{n+1}$ *are decent solutions of* (2)*.*

Proof. Since $\varphi_0, \ldots, \varphi_{n+1}$ fulfil (2), from Theorem 5 it follows that each of these functions fulfils (9). Thus, according to Theorem 4, each of the functions $\varphi_0, \ldots, \varphi_{n+1}$ is a decent solution of (9). Therefore, Theorem 6 implies that $\varphi_0, \ldots, \varphi_{n+1}$ are decent solutions of (2). □

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