K -subadditive and K -superadditive set-valued functions bounded on “large” sets

ELIZA JABŁOŃSKA AND KAZIMIERZ NIKODEM

Dedicated to Professor Dr. Ludwig Reich on his 80th birthday.

Abstract. We prove that every K -subadditive set-valued map weakly K -upper bounded on a “large” set (e.g. not null-finite, not Haar-null or not a Haar-meager set), as well as any K -superadditive set-valued map K -lower bounded on a “large” set, is locally K -lower bounded and locally weakly K -upper bounded at every point of its domain.

Mathematics Subject Classification. Primary 39B62, Secondary 54C60, 26B25.

Keywords. K -subadditive function, K -superadditive function, Set-valued map, Null-finite set, Haar-null set, Haar-meager set.

1. Introduction and preliminaries

The classical subadditive functions, i.e. functions $f : X \rightarrow \mathbb{R}$ satisfying

$$f(x_1 + x_2) \leq f(x_1) + f(x_2), \quad x_1, x_2 \in X,$$

have many remarkable properties of boundedness discussed, among others, in [11, 16, 17, 19], and recently in [3–6]. For instance, it is known that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is subadditive and upper bounded on a set $T \subset \mathbb{R}^n$ which is of positive Lebesgue measure or is of the second category with the Baire property, then f is locally bounded at every point of $\mathbb{R}^n$ (see [16, Theorem 16.2.3]). This classical result was generalized by Bingham et al. in [3] to the case of others “large” sets in abelian Polish groups, e.g. not null-finite, not Haar-meager, not Haar-null sets.

Recall also that a function f is called superadditive, if $-f$ is subadditive.

In this paper we extend the notions of subadditive and superadditive functions to K -subadditive and K -superadditive set-valued maps. Next, we prove theorems which are far-reaching generalizations of the results mentioned above.

For the concept of K -subadditivity and K -superadditivity we refer to the paper [18].

Let X and Y be abelian metric groups (both with invariant metrics). Assume that K is a subsemigroup of Y (i.e. $K + K \subset K$). Denote by $n(Y)$ the family of all nonempty subsets of Y .

A set-valued map $F : X \rightarrow n(Y)$ is called K -subadditive, if

$$F(x_1) + F(x_2) \subset F(x_1 + x_2) + K \quad (1)$$

for all $x_1, x_2 \in X$, and K -superadditive, if

$$F(x_1 + x_2) \subset F(x_1) + F(x_2) + K \quad (2)$$

for all $x_1, x_2 \in X$.

Note that if F is K -subadditive and single-valued and moreover Y is endowed with the relation $\leq_K$ of partial order defined by

$$x \leq_K y \iff y - x \in K,$$

then conditions (1) and (2) reduce to the following conditions:

$$F(x_1 + x_2) \leq_K F(x_1) + F(x_2)$$

and

$$F(x_1) + F(x_2) \leq_K F(x_1 + x_2),$$

respectively. In particular, if $Y = \mathbb{R}$ and $K = [0, \infty)$, we obtain the standard definitions of subadditive and superadditive functions.

For $K = \{0\}$ K -subadditivity (K -superadditivity) means that

$$F(x_1) + F(x_2) \subset F(x_1 + x_2) \quad (F(x_1 + x_2) \subset F(x_1) + F(x_2))$$

for all $x_1, x_2 \in X$, which is the definition of the superadditivity (subadditivity) of set-valued functions introduced and investigated by Smajdor in [20, 21]. Note, however, that if F is single-valued, then each of the above inclusions means that F is an additive function (i.e. $F(x_1 + x_2) = F(x_1) + F(x_2)$). Thus subadditive and superadditive set-valued maps are extensions of additive functions, whereas K -subadditive and K -superadditive set-valued maps generalize subadditive and superadditive functions, respectively.

Now, let us recall that a subset B of a complete abelian metric group X with an invariant metric is called:

- *Universally Baire* if for each continuous function $f : K \rightarrow X$ mapping a compact metric space K into X the set $f^{-1}(A+x)$ has the Baire property for every $x \in X$ (see [10]);
- *Haar-meager* if there exist a universally Baire set $A \supset B$, a compact metric space K and a continuous function $f : K \rightarrow X$ such that $f^{-1}(A+x)$ is meager in K for each $x \in X$ (see [8] and also [2]);
- *Universally measurable* if it is measurable with respect to each complete Borel probability measure on X (see [7]);

- *Haar-null* if there exist a universally measurable set $A \supset B$ and a σ -additive probability Borel measure μ on X such that $\mu(A+x) = 0$ for each $x \in X$ (see [7]).¹

It was proved in [7] and [8] that in each locally compact abelian Polish group the notions of a Haar-meager set and a Haar-null set are equivalent to the notions of a meager set and a set of Haar measure zero, respectively. Moreover, Haar-meager sets and Haar-null sets have many analogous properties (see, e.g. [1, 9, 14]).

In [2] a new concept of “small” sets was introduced, generalizing (to some extent) the notions of a Haar-meager set and a Haar-null set.

Definition 1. A subset A of an abelian metric group X is called *null-finite* if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ convergent to 0 in X such that the set $\{n \in \mathbb{N} : x + x_n \in A\}$ is finite for every $x \in X$.

The following crucial property of null-finite sets was proved in [2].

Theorem 1. [2][Theorems 5.1 and 6.1] *In a complete abelian metric group with an invariant metric:*

- *Each universally Baire null-finite set is Haar-meager,*
- *Each universally measurable null-finite set is Haar-null.*

In the same paper [2] the authors applied the above result to show that every real-valued additive (midpoint convex) function upper bounded on a set which is universally measurable non-Haar-null or Borel non-Haar-meager in a complete abelian metric group (linear space) with an invariant metric is continuous. Next, Bingham et al. [3] showed that every subadditive real valued function upper bounded on a set which is “large” in the same sense is locally bounded at each point of the domain.

In this paper we generalize results from [3] to K -subadditive and K -superadditive set-valued maps. Our results are also counterparts of some results from [15] concerning K -midconvex and K -midconcave set-valued maps bounded on “large” sets.

2. Main results

Let X and Y be abelian metric groups with invariant metrics. Denote by $B_X(r)$ and $B_Y(r)$ open balls with center 0 and radius r in X and Y , respectively.

A set $B \subset Y$ is called *bounded* in Y , if B is contained in an open ball $B_Y(r)$ for some $r > 0$. This notion generalizes the well-known notion of bounded sets

¹Actually, the mentioned definitions have been introduced in an abelian Polish group, but the separability is not necessary. Hunt et al. highlighted it in [12]–[13].

in a real topological vector space. Clearly, if B_1, B_2 are bounded sets in Y , then the set $B_1 + B_2$ is also bounded in Y .

Denote by $\mathcal{B}(Y)$ the family of all nonempty bounded subsets of Y . A set-valued map $F : X \rightarrow \mathcal{B}(Y)$ is called:

- *K-upper bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \subset B - K \quad \text{for all } x \in A;$$

- *Weakly K-upper bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \cap (B - K) \neq \emptyset \quad \text{for all } x \in A;$$

- *K-lower bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \subset B + K \quad \text{for all } x \in A;$$

- *Weakly K-lower bounded* on a set $A \subset X$, if there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \cap (B + K) \neq \emptyset \quad \text{for all } x \in A.$$

Clearly, K -upper (K -lower) bounded set-valued maps are weakly K -upper (K -lower) bounded. Moreover, in the case when $K = \{0\}$, [weak] K -upper boundedness and [weak] K -lower boundedness are equivalent assumptions, called simply [weak] boundedness.

A set-valued map $F : X \rightarrow \mathcal{B}(Y)$ is called *locally [weakly] K-upper (K-lower) bounded at $x \in X$* , if it is [weakly] K -upper (K -lower) bounded on some neighborhood of x .

First we prove a result which generalizes Theorem 2.2 from [3].

Theorem 2. *Let X and Y be abelian metric groups with invariant metrics. Assume that $A \subset X$ is a set which is not null-finite and K is a subsemigroup of Y . If a set-valued map $F : X \rightarrow \mathcal{B}(Y)$ is K -subadditive and weakly K -upper bounded on A , then F is locally weakly K -upper bounded and locally K -lower bounded at each point of X .*

Proof. Let F be K -subadditive and weakly K -upper bounded on A . Then there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \cap (B - K) \neq \emptyset, \quad x \in A. \quad (3)$$

First we prove that F is locally weakly K -upper bounded at 0. So, suppose that it is not true and F is not weakly K -upper bounded on any neighborhood of 0. Consequently, for every

$$n \in \mathbb{N}, \quad U_n := B_X\left(\frac{1}{2^n}\right) \quad \text{and} \quad B_n := B + B_Y(n) \in \mathcal{B}(Y),$$

there exists $x_n \in U_n$ such that

$$F(x_n) \cap (B_n - K) = \emptyset.$$

Hence

$$(F(x_n) + K) \cap B_n = \emptyset. \quad (4)$$

Moreover, by the definition of U_n the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to 0 in X . The set A is not null-finite, so there exists $a \in X$ such that the set $\mathbb{N}_0 := \{n \in \mathbb{N} : a + x_n \in A\}$ is not finite. By (3) we have

$$F(a + x_n) \cap (B - K) \neq \emptyset, \quad n \in \mathbb{N}_0. \quad (5)$$

In view of K -subadditivity,

$$F(x_n) + K \supset F(-a) + F(a + x_n), \quad n \in \mathbb{N}_0. \quad (6)$$

Since $F(-a) \in \mathcal{B}(Y)$, we can find $n_0 \in \mathbb{N}_0$, such that $F(-a) \subset B_Y(n_0)$. Hence $F(-a) \subset B_Y(n)$ for every $n \geq n_0$. Fix $n \in \mathbb{N}_0$, $n \geq n_0$. By (5) there exist $b_n \in B$ and $k_n \in K$ such that

$$b_n - k_n \in F(a + x_n).$$

Then, in view of (6) we have

$$b_n - k_n + F(-a) \subset F(x_n) + K,$$

and hence

$$b_n + F(-a) \subset F(x_n) + K. \quad (7)$$

On the other hand

$$b_n + F(-a) \subset B + B_Y(n) = B_n. \quad (8)$$

From (7) and (8) we obtain

$$(F(x_n) + K) \cap B_n \neq \emptyset,$$

which contradicts (4).

Thus F is locally weakly K -upper bounded at 0, i.e. there are a neighborhood U_0 of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that

$$F(x) \cap (B_0 - K) \neq \emptyset, \quad x \in U_0. \quad (9)$$

In the second step we prove that F is locally weakly K -upper bounded at every point of the domain. For a proof by contradiction suppose that F is not weakly K -upper bounded on any neighborhood U_{x_0} of some point x_0 . Consequently, for

$$U_{x_0} := U_0 + x_0 \quad \text{and} \quad B_0 + F(x_0) \in \mathcal{B}(Y),$$

there exists $x_1 \in U_{x_0}$ such that

$$F(x_1) \cap (B_0 + F(x_0) - K) = \emptyset,$$

and thus

$$(F(x_1) + K) \cap (B_0 + F(x_0)) = \emptyset. \quad (10)$$

Since K -subadditivity implies

$$F(x_1) + K \supset F(x_1 - x_0) + F(x_0), \quad (11)$$

where $x_1 - x_0 \in U_0$, in view of (9) we can find $b \in B_0$ and $k \in K$ such that $b - k \in F(x_1 - x_0)$. Then, by (11),

$$F(x_1) + K \supset b - k + F(x_0)$$

and consequently,

$$F(x_1) + K \supset b + F(x_0),$$

which contradicts (10) and proves that F is locally weakly K -upper bounded at x_0 .

Finally, we will show that F is locally K -lower bounded at every point $x \in X$. We have already proved that there exist a neighborhood U_0 of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that (9) holds. We may assume that U_0 is symmetric with respect to 0. Fix $x \in X$ arbitrarily and take $U_x := U_0 + x$. If $y \in U_x$, then $x - y \in U_0$ and by (9)

$$F(x - y) \cap (B_0 - K) \neq \emptyset, \quad y \in U_x.$$

Hence there exist $z \in F(x - y)$, $b \in B_0$ and $k \in K$ such that $z = b - k$. By the K -subadditivity of F we have

$$F(y) + b - k = F(y) + z \subset F(y) + F(x - y) \subset F(x) + K, \quad y \in U_x$$

and hence

$$F(y) \subset F(x) - b + K \subset (F(x) - B) + K, \quad y \in U_x.$$

Since $F(x) - B \in \mathcal{B}(Y)$, this shows that F is K -lower bounded on U_x and finishes the proof. $\square$

In Theorem 2 a stronger assumption like K -upper boundedness of F on a “large” set A does not strengthen the statement. More precisely, a set-valued map $F : X \rightarrow \mathcal{B}(Y)$ which is K -subadditive and K -upper bounded on A need not be locally K -upper bounded at each point of X .

Example 1. Let $K = [0, \infty)$ and $F : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{R})$ be given by

$$F(x) = \begin{cases} \left[0, \frac{1}{|x|}\right], & x \neq 0, \\ \{0\}, & x = 0. \end{cases}$$

Such a set-valued mapping is K -subadditive and K -upper bounded e.g. on the set $[1, 2]$ (it is enough to choose $B = [0, 1]$). But F is not K -upper bounded at 0.

Now, we will prove an analogous result for K -superadditive set-valued maps. Note, however, that this result can not be obtained as a consequence of Theorem 2. Namely, the K -superadditivity of a set-valued map F does not imply the K -subadditivity of $-F$.

Example 2. Let $\mathbb{Q}$ be the set of all rational numbers. Let $F : \mathbb{R} \rightarrow n(\mathbb{R})$ be given by $F(x) = [0, |x|]$ for $x \in \mathbb{R}$. Clearly, F is $\mathbb{Q}$ -superadditive. Moreover, the set-valued map $-F$ given by $-F(x) = [-|x|, 0]$ for $x \in \mathbb{R}$ is not $\mathbb{Q}$ -subadditive (but is $\mathbb{Q}$ -superadditive).

Similarly, we can find an example where K is not a group.

Example 3. Let $K = [0, \infty)$ and $F : \mathbb{R} \rightarrow n(\mathbb{R})$ be given by $F(x) = [\sin x - 3, \sin x + 3]$ for $x \in \mathbb{R}$. Clearly, F is K -superadditive, but the set-valued map $-F$ given by $-F(x) = [-3 - \sin x, 3 - \sin x]$ for $x \in \mathbb{R}$ is not K -subadditive.

Theorem 3. *Let X and Y be abelian metric groups with invariant metrics. Assume that $A \subset X$ is a set which is not null-finite and K is a subsemigroup of Y . If a set-valued map $F : X \rightarrow \mathcal{B}(Y)$ is K -superadditive and K -lower bounded on A , then F is locally K -lower bounded and locally weakly K -upper bounded at each point of X .*

Proof. Since F is K -lower bounded on A , there exists a set $B \in \mathcal{B}(Y)$ such that

$$F(x) \subset B + K, \quad x \in A. \quad (12)$$

First we will prove that F is locally K -lower bounded at 0. So, suppose that it is not true and F is not K -lower bounded on any neighborhood of 0. Consequently, for every

$$n \in \mathbb{N}, \quad U_n := B_X \left(\frac{1}{2^n} \right) \quad \text{and} \quad B_n := B + B_Y(n) \in \mathcal{B}(Y),$$

there exists $x_n \in U_n$ such that

$$F(x_n) \not\subset B_n + K. \quad (13)$$

Moreover, by the definition of U_n the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to 0 in X . The set A is not null-finite, so there exists $a \in X$ such that the set $\mathbb{N}_0 := \{n \in \mathbb{N} : a + x_n \in A\}$ is not finite. Then, by (12) we have

$$F(a + x_n) \subset B + K, \quad n \in \mathbb{N}_0. \quad (14)$$

Since $F(-a) \in \mathcal{B}(Y)$, we can find $n_0 \in \mathbb{N}_0$, such that $F(-a) \subset B_Y(n_0)$. Consequently $F(-a) \subset B_Y(n)$ for every $n \geq n_0$. Fix $n \in \mathbb{N}_0$, $n \geq n_0$. In view of K -superadditivity and (12),

$$F(x_n) \subset F(-a) + F(a + x_n) + K \subset B_Y(n) + B + K = B_n + K,$$

which contradicts (13).

Thus F is locally K -lower bounded at 0, i.e. there are a neighborhood U_0 of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that

$$F(x) \subset B_0 + K, \quad x \in U_0. \quad (15)$$

Now, we will prove that F is locally K -lower bounded at every point of the domain. For a proof by contradiction suppose that F is not K -lower bounded on any neighborhood U_{x_0} of some point x_0 . Consequently, for

$$U_{x_0} := U_0 + x_0 \quad \text{and} \quad B_0 + F(x_0) \in \mathcal{B}(Y),$$

there exists $x_1 \in U_{x_0}$ such that

$$F(x_1) \not\subset B_0 + F(x_0) + K. \quad (16)$$

Since $x_1 - x_0 \in U_0$, K -superadditivity and (15) implies

$$F(x_1) \subset F(x_1 - x_0) + F(x_0) + K \subset B_0 + F(x_0) + K,$$

which contradicts (16) and proves that F is locally K -lower bounded at x_0 .

Finally, we will show that F is locally weakly K -upper bounded at each point $x \in X$. Since F is locally K -lower bounded at 0, there are a symmetric neighborhood U_0 of 0 and a set $B_0 \in \mathcal{B}(Y)$ such that (15) holds. Fix an arbitrary $x \in X$ and take $U_x := U_0 + x$. Take any $y \in U_x$. Then $y = z + x$, where $z \in U_0$. Since U_0 is symmetric, also $-z \in U_0$. By the K -superadditivity of F and (15) we have

$$F(x) \subset F(x + z) + F(-z) + K \subset F(y) + B_0 + K. \quad (17)$$

Fix any $z_0 \in F(x)$. By (17) there exist $v \in F(y)$, $b \in B_0$ and $k \in K$ such that $z_0 = v + b + k$. Hence $v = z_0 - b - k$ and, consequently,

$$F(y) \cap ((z_0 - B) - K) \neq \emptyset.$$

Since this condition holds for any $y \in U_x$, it proves that F is locally weakly K -upper bounded at x . The proof is finished. $\square$

The next example shows that in Theorem 3 a weaker assumption like weakly K -lower boundedness of $F : X \rightarrow \mathcal{B}(Y)$ on a “large” set A does not imply even a weaker conclusion. More precisely, a set-valued map F which is K -superadditive and weakly K -lower bounded on A need not be locally weakly K -lower bounded at each point of X .

Example 4. Let $K = [0, \infty)$, $a : \mathbb{R} \rightarrow \mathbb{R}$ be a discontinuous additive function and $F : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{R})$ be given by

$$F(x) = [-2|a(x)|, -|a(x)|], \quad |x| < 1, \\ [-2|a(x)|, 0], \quad |x| \geq 1.$$

Such a set-valued mapping is K -superadditive and weakly K -lower bounded e.g. on the set $[1, 2]$. Moreover, F is not weakly K -lower bounded at 0. Indeed, if $F(x) \cap (B + K) \neq \emptyset$ for each $x \in U_0$ a neighbourhood U_0 and a bounded set B , then $|a(x)| \leq -\inf B$ for each $x \in U_0$ which contradicts the discontinuity of a .

Remark 1. In Theorems 2 and 3 we can not obtain that F is locally K -upper bounded (instead of locally weakly K -upper bounded) at each point. Consider, for example, the set-valued map $F : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{R})$ defined by

$$F(x) = [0, |a(x)|], \quad x \in \mathbb{R},$$

where $a : \mathbb{R} \rightarrow \mathbb{R}$ is an additive discontinuous function, and take $K = [0, \infty)$. Then F is K -subadditive as well as K -superadditive. Moreover, F is locally weakly K -upper bounded at every point (it is even weakly K -upper bounded on the whole $\mathbb{R}$ because $F(x) \cap ([0, 1] - K) \neq \emptyset$ for every $x \in \mathbb{R}$). F is also locally K -lower bounded at every point (it is even K -lower bounded on the whole $\mathbb{R}$ because $F(x) \subset [0, 1] + K$ for every $x \in \mathbb{R}$). However, F is not locally K -upper bounded at any point. Indeed, if for some open set $U \subset \mathbb{R}$ and some $[m, M] \subset \mathbb{R}$ we had $F(x) \subset [m, M] - K$ for $x \in U$, then $a(x) \leq M$ for $x \in U$, which is impossible because a is discontinuous (by [16][Lemma 9.3.1]).

As an immediate consequence of Theorems 2, 3 and 1, we obtain the following generalization of [3][Corollary 2.4].

Corollary 4. *Let X be a complete abelian metric group and Y be an abelian metric group, both with invariant metrics. Assume that $A \subset X$ is a universally measurable non-Haar-null or universally Baire non-Haar-meager set and K is a subsemigroup of Y . If a set-valued map $F : X \rightarrow \mathcal{B}(Y)$ is K -subadditive and weakly K -upper bounded on A or K -superadditive and K -lower bounded on A , then F is locally K -lower bounded and locally weakly K -upper bounded at each point of X .*

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Banach, T., Głab, S., Jabłońska, E., Swaczyna, J.: Haar- $\mathcal{I}$ sets: looking at small sets in Polish groups through compact glasses. *Diss. Math.* **564**, 1–105 (2021)
- [2] Banach, T., Jabłońska, E.: Null-finite sets in metric groups and their applications. *Israel J. Math.* **230**, 361–386 (2019)

- [3] Bingham, N.H., Jabłońska, E., Jabłoński, W., Ostaszewski, A.J.: On subadditive functions bounded above on a large set. *Results Math.* **75**, 58 (2020)
- [4] Bingham, N.H., Ostaszewski, A.J.: Generic subadditive functions. *Proc. Am. Math. Soc.* **136**(12), 4257–4266 (2008)
- [5] Bingham, N.H., Ostaszewski, A.J.: Automatic continuity: subadditivity, convexity, uniformity. *Aequationes Math.* **78**(3), 257–270 (2009)
- [6] Bingham, N.H., Ostaszewski, A.J.: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening. *Indag. Math. (N.S.)* **29**, 687–713 (2018)
- [7] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. *Israel J. Math.* **13**, 255–260 (1972)
- [8] Darji, U.B.: On Haar meager sets. *Topology Appl.* **160**, 2396–2400 (2013)
- [9] Elekes, M., Nagy, D.: Haar null and Haar meager sets: a survey and new results. *Bull. London Math. Soc.* **52**, 561–619 (2020)
- [10] Feng, Q., Magidor, M., Woodin, H.: Universally baire sets of reals. In: Judah, H., Just, W., Woodin, H. (eds.) *Set Theory of the Continuum*, pp. 203–242. Springer, New York (1992)
- [11] Hille, E., Phillips, R.S.: *Functional Analysis and Semigroups*. American Mathematical Soc, vol. 31. (1957)
- [12] Hunt, B.R., Sauer, T., Yorke, J.A.: Prevalence: a translation-invariant almost every on infinite-dimensional spaces. *Bull. Am. Math. Soc.* **27**, 217–238 (1992)
- [13] Hunt, B.R., Sauer, T., Yorke, J.A.: Prevalence: an addendum. *Bull. Am. Math. Soc.* **28**, 306–307 (1993)
- [14] Jabłońska, E.: Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. *J. Math. Anal. Appl.* **421**, 1479–1486 (2015)
- [15] Jabłońska, E., Nikodem, K.: K -midconvex and K -midconcave set-valued maps bounded on large sets. *J. Convex Anal.* **26**, 563–572 (2019)
- [16] Kuczma, M.: *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN – Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985, 2nd edn. Birkhäuser, Basel–Boston–Berlin (2009)
- [17] Matkowski, J.: Subadditive functions and a relaxation of the homogeneity condition of semigroups. *Proc. Am. Math. Soc.* **117**, 991–1001 (1993)
- [18] Nikodem, K.: K -Convex and K -Concave Set-valued Functions, *Zeszyty Nauk. Politechniki Łódzkiej Mat.* 559; *Rozprawy Mat.* 114, Łódź (1989)
- [19] Rosenbaum, R.A.: Sub-additive functions. *Duke Math. J.* **17**, 227–242 (1950)
- [20] Smajdor, W.: Subadditive and subquadratic set-valued functions. *Uniwersytet Śląski, Katowice* (1987)
- [21] Smajdor, W.: Superadditive set-valued functions and Banach-Steinhaus theorem. *Radovi Mat.* **3**, 203–214 (1987)

Eliza Jabłońska
Faculty of Applied Mathematics
AGH University of Science and Technology
Mickiewicza 30
30-059 Kraków
Poland
e-mail: elizajab@agh.edu.pl

Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2
43-309 Bielsko-Biala
Poland
e-mail: knikodem@ath.bielsko.pl

Received: October 26, 2020

Revised: September 27, 2021

Accepted: September 28, 2021