Complementary means with respect to a nonsymmetric invariant mean

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Abstract. It is known that if a bivariate mean K is symmetric, continuous and strictly increasing in each variable, then for every mean M there is a unique mean N such that K is invariant with respect to the mean-type mapping (M, N) , which means that $K \circ (M, N) = K$ and N is called a K -complementary mean for M (Matkowski in Aequ Math 57(1):87–107, 1999). This paper extends this result for a large class of nonsymmetric means. As an application, the limits of the sequences of iterates of the related mean-type mappings are determined, which allows us to find all continuous solutions of some functional equations.

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1. Introduction

Let K, M and N be bivariate means in an interval I . The mean K is called *invariant with respect to the mean-type mapping* $(M, N) : I^2 \rightarrow I^2$, briefly (M, N) -invariant, if $K \circ (M, N) = K$ (cf. [3]). This K is sometimes called the Gauss composition of the means M and N (cf. [2]), and N is referred to as a *complementary mean to M* with respect to K (briefly, *K -complementary mean to M*) ([3], see also [6]).

It is known that, under general conditions, the invariance relation guarantees the convergence of the sequence $((M, N)^n : n \in \mathbb{N})$ of iterates of the mean-type mapping (M, N) to the mean-type (K, K) . This fact has important applications in effectively solving some functional equations.

In [3] it was shown that if the mean K is symmetric, continuous and strictly increasing in each variable, then for every mean M there is a unique K -complementary mean.

The main results of this paper, Theorems 1 and 2, omitting the symmetry condition, give essential generalizations of this result. In particular, Theorem 1, where the symmetry of K is replaced by the implication “ $x < y \implies K(x, y) \geq K(y, x)$ for all $x, y \in I$ ”, says that for every bivariate mean M in I there is a unique mean $M_{[K]}$ such that K is $(M, M_{[K]})$ -invariant. Moreover, $M_{[K]}$ is strict or continuous if so is M , and both M and $M_{[K]}$ are symmetric if so is K . Theorem 2 is a dual counterpart of Theorem 1. Here the symmetry of K is replaced by the implication “ $x < y \implies K(x, y) \leq K(y, x)$ for all $x, y \in I$ ”. For every mean M it provides a unique mean $M^{[K]}$ such that K is $(M^{[K]}, M)$ -invariant, and has similar additional properties as $M_{[K]}$.

A simple example shows that the assumed implications in Theorems 1 and 2 are essential. Moreover it is shown (Remark 3 and Example 2) that in these results the assumption that K is strictly increasing cannot be replaced by the strictness of K .

Applying Theorems 1 and 2 we give conditions under which the sequences of mean-type mappings $(M, M_{[K]})$ and $(M^{[K]}, M)$ converge to (K, K) (Theorem 3 and Theorem 4). This permits, in particular, to determine effectively all continuous functions $F : I^2 \rightarrow \mathbb{R}$ satisfying each of the functional equations

$$\begin{aligned} F(M(x, y), M_{[K]}(x, y)) &= F(x, y) \quad x, y \in I, \\ F(M^{[K]}(x, y), M(x, y)) &= F(x, y), \quad x, y \in I. \end{aligned}$$

2. Complementary means

In the sequel $I \subset \mathbb{R}$ stands for an interval.

A function $M : I \times I \rightarrow I$ is called a *mean* in I , if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

A mean M is called *strict*, if for $x \neq y$, both these inequalities are sharp; and *symmetric*, if $M(x, y) = M(y, x)$ for all $x, y \in I$.

Every mean is *reflexive*, that is $M(x, x) = x$ for all $x \in I$.

Remark 1. If a function $M : I \times I \rightarrow \mathbb{R}$ is reflexive and (strictly) increasing in each of the variables, then it is a (strict) mean in I .

Let $K, M, N : I^2 \rightarrow I$ be means. The mean K is called *invariant with respect to the mean-type mapping* $(M, N) : I^2 \rightarrow I^2$, briefly (M, N) -invariant, if $K \circ (M, N) = K$ (cf. [3]). In the case when K is unique, it is sometimes called the Gauss composition of the means M and N (cf. [2]). If K is a unique (M, N) -invariant mean we say that N is a *complementary mean to M* with respect to K (briefly, N is *K -complementary to M*).

Recall that if M, N are continuous and strict means, then there exists a unique (M, N) -invariant mean (cf. [1], also [4], [2], [5]).

Theorem 1. *Let a continuous mean $K : I^2 \rightarrow I$ be increasing in the first variable and strictly increasing in the second one. Suppose that K satisfies the following condition:*

$$x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I. \quad (1)$$

Then

- (i) *for every mean $M : I^2 \rightarrow I$ there is a unique mean $M_{[K]} : I^2 \rightarrow I$ such that K is $(M, M_{[K]})$ -invariant, i.e.*

$$K(M(x, y), M_{[K]}(x, y)) = K(x, y), \quad x, y \in I; \quad (2)$$

- (ii) *if for a symmetric mean M , the mean $M_{[K]}$ is symmetric, then K is symmetric;*
 (iii) *if M is strict then so is $M_{[K]}$;*
 (iv) *if M is continuous then so is $M_{[K]}$.*

Proof. Take an arbitrary mean $M : I^2 \rightarrow I$ and $x, y \in I$. We have

$$\min(x, y) \leq M(x, y) \leq \max(x, y).$$

By setting $z = M_{[K]}(x, y)$, the equation (2) becomes

$$K(M(x, y), z) = K(x, y).$$

If $x = y$ then $M(x, y) = K(x, y) = x$ so this equation simplifies to

$$K(x, z) = x.$$

Since, by the reflexivity of every mean, $K(x, x) = x$, the strict monotonicity of K in the second variable implies that $M_{[K]}(x, y) := z = x$ is the only solution of this equation.

Assume that $x < y$.

In this case we have

$$x \leq M(x, y) \leq y.$$

Define the function $\varphi : [x, y] \rightarrow \mathbb{R}$ by

$$\varphi(t) := K(M(x, y), t) - K(x, y), \quad t \in [x, y].$$

Of course, φ is continuous.

For $t = x$, making use first of the inequality $x \leq M(x, y)$ and implication (1), and then from the inequality $M(x, y) \leq y$ and the monotonicity of K in the second variable, we have

$$\varphi(x) = K(M(x, y), x) - K(x, y) \leq K(x, M(x, y)) - K(x, y) < 0.$$

For $t = y$, the inequality $x \leq M(x, y)$ and the monotonicity of K in the first variable give

$$\varphi(y) = K(M(x, y), y) - K(x, y) \geq K(x, y) - K(x, y) = 0.$$

The Darboux property of φ implies that there is $z \in [x, y]$ such that $\varphi(z) = 0$, i.e. such that

$$K(M(x, y), z) = K(x, y).$$

By the strict monotonicity of K in the second variable, such a z is unique. Setting here $M_{[K]}(x, y) := z$, we get

$$K(M(x, y), M_{[K]}(x, y)) = K(x, y).$$

Since in the case $y < x$ we can argue similarly, the proof of (i) is complete.

(ii) This is a trivial consequence of (2).

(iii) Assume, on the contrary, that $M_{[K]}$ is not strict, so there are $x_0, y_0 \in I$, $x_0 < y_0$, such that $M_{[K]}(x_0, y_0) = x_0$ or $M_{[K]}(x_0, y_0) = y_0$.

In the first case, from (2), we would have

$$K(M(x_0, y_0), x_0) = K(M(x_0, y_0), M_{[K]}(x_0, y_0)) = K(x_0, y_0),$$

and, from (1),

$$K(x_0, y_0) \geq K(y_0, x_0),$$

whence $K(M(x_0, y_0), x_0) \geq K(y_0, x_0)$. But this is a contradiction as K is strictly increasing in the first variable and $x_0 < M(x_0, y_0) < y_0$.

Since in the case when $M_{[K]}(x_0, y_0) = y_0$ we can argue similarly, this completes the proof of (iii).

(iv) We omit the simple argument of this part. $\square$

Definition 1. Under the conditions of Theorem 1, the mean $M_{[K]}$ is called K -complementary (or K -right complementary) to the mean M .

Under the conditions of this theorem, part (i) can be strengthened as follows:

Remark 2. For every mean $M : I^2 \rightarrow I$ there is a unique function $N : I^2 \rightarrow I$ such that

$$K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I,$$

holds and, moreover, N is a mean.

To show that assumption (1) is essential, consider the following

Example 1. Let $I = \mathbb{R}$ and $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a weighted arithmetic mean, i.e. $K(x, y) = ax + (1 - a)y$ for some fixed $a \in (0, 1)$, and let $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a mean. A simple calculation shows that a function $N : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies (2) if and only if

$$N(x, y) = \frac{a(x - M(x, y))}{1 - a} + y, \quad x, y \in \mathbb{R}. \quad (3)$$

Assume that for every weighted arithmetic mean $M(x, y) = bx + (1 - b)y$, $b \in [0, 1]$, the function N is a mean in $\mathbb{R}$, that is that

$$N(x, y) = \frac{a(1-b)}{1-a}x + \left(1 - \frac{a(1-b)}{1-a}\right)y, \quad x, y \in \mathbb{R},$$

is a mean in $\mathbb{R}$. Clearly, N is a mean for every $b \in [0, 1]$ if, and only if,

$$\frac{a(1-b)}{1-a} \leq 1, \quad b \in [0, 1].$$

Since the function $[0, 1] \ni b \mapsto \frac{a(1-b)}{1-a}$ is decreasing, this inequality holds true iff $\frac{a(1-0)}{1-a} \leq 1$, that is iff

$$a \leq \frac{1}{2}.$$

We can also argue as follows. The function N given by (3) is a mean for every mean M if and only if

$$\min(x, y) \leq \frac{a(x - M(x, y))}{1-a} + y \leq \max(x, y), \quad x, y \in \mathbb{R},$$

or, equivalently, if and only if, for every mean M , and for all $x, y \in \mathbb{R}$,

$$\begin{aligned} ax + (1-a)y - (1-a)\max(x, y) \\ \leq aM(x, y) \leq ax + (1-a)y - (1-a)\min(x, y). \end{aligned}$$

Taking here first $M = \min$ and then $M = \max$ we obtain, respectively, the inequalities

$$\begin{aligned} ax + (1-a)y &\leq a\min(x, y) + (1-a)\max(x, y), \quad x, y \in \mathbb{R}, \\ (1-a)\min(x, y) + a\max(x, y) &\leq ax + (1-a)y, \quad x, y \in \mathbb{R}. \end{aligned}$$

It is easy to see that each of these inequalities holds, if and only if $a \leq \frac{1}{2}$.

Note that with $K(x, y) = ax + (1-a)y$, the inequality $a \leq \frac{1}{2}$ is equivalent to implication (1).

If K satisfies the “opposite” condition to (1), interchanging the roles of variables in K and applying Theorem 1, we obtain the following

Theorem 2. *Let a continuous mean $K : I^2 \rightarrow I$ be strictly increasing in the first variable and increasing in the second one. Suppose that K satisfies the following condition:*

$$x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I.$$

Then

- (i) *for every mean $M : I^2 \rightarrow I$ there is a unique mean $M^{[K]} : I^2 \rightarrow I$ such that K is $(M, M^{[K]})$ -invariant, i.e.*

$$K\left(M^{[K]}(x, y), M(x, y)\right) = K(x, y), \quad x, y \in I;$$

- (ii) if for some symmetric mean M , the mean $M^{[K]}$ is symmetric, then K is symmetric;
- (iii) if M is strict then so is $M^{[K]}$;
- (iv) if M is continuous then so is $M^{[K]}$.

Definition 2. Under the conditions of Theorem 2, the mean $M^{[K]}$ can be referred to as K -complementary (or K -left complementary) to the mean M .

From Theorems 1 and 2 we obtain the following improvement of Remark 1 in [3]:

Corollary 1. If $K : I^2 \rightarrow I$ is a continuous strictly increasing in each variable and symmetric mean, then for every mean $M : I^2 \rightarrow I$ there is a unique function $N : I^2 \rightarrow I$ such that $K \circ (M, N) = K$, moreover N is a mean and

$$M_{[K]} = N = M^{[K]}.$$

We end this section with

Remark 3. In Theorems 1 and 2 (as well as in [3], Remark 1), the strict increasing monotonicity in each variable of the invariant mean $K : I^2 \rightarrow I$, cannot be weakened by the assumption that K is a strict mean.

To show it consider the following

Example 2. It is known that the contra-harmonic mean $K : (0, \infty)^2 \rightarrow (0, \infty)$,

$$K(x, y) = \frac{x^2 + y^2}{x + y},$$

is not strictly increasing. The symmetry of K implies that condition (1) of Theorem 1 is satisfied. Taking for M the arithmetic mean $A(x, y) = \frac{x+y}{2}$ for $x, y > 0$, by a simple calculation, we get

$$K_{[A]}(x, y) = \frac{x^2 + y^2 + \sqrt{2}\sqrt{x^4 + y^4}}{2(x + y)}, \quad x, y > 0.$$

Of course, $K_{[A]} : (0, \infty)^2 \rightarrow (0, \infty)$ and $K_{[A]}$ is reflexive, i.e. $K_{[A]}(x, x) = x$; so $K_{[A]}$ is a bivariate pre-mean in $(0, \infty)$. But, as

$$K_{[A]}(1, 10) > 11,$$

$K_{[A]}$ is not a mean.

(It can be verified similarly, that $K_{[G]}$, $K_{[H]}$ where G and H stand, respectively, for the geometric and harmonic mean, are pre-means, but not means.)

3. Some applications

Applying Theorem 1 and the main result of [5] (see also [4]) we obtain the following

Theorem 3. *Suppose that a mean $K : I^2 \rightarrow I$ is continuous and strictly increasing in each variable.*

(i) *If K satisfies the condition*

$$x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I,$$

then for every continuous and strict mean $M : I^2 \rightarrow I$, the sequence $((M, M_{[K]})^n : n \in \mathbb{N})$ of iterates of the mean-type mapping $(M, M_{[K]}) : I^2 \rightarrow I^2$ converges uniformly on compact sets to the mean-type map (K, K) .

(ii) *If K satisfies the condition*

$$x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I,$$

then for every continuous and strict mean $M : I^2 \rightarrow I$, the sequence $((M^{[k]}, M)^n : n \in \mathbb{N})$ of iterates of the mean-type mapping $(M^{[k]}, M) : I^2 \rightarrow I^2$ converges uniformly on compact sets to the mean-type map (K, K) .

Using this result we prove the following

Theorem 4. *Let a mean $K : I^2 \rightarrow I$ be continuous and strictly increasing in each variable and $M : I^2 \rightarrow I$ be an arbitrary strict and continuous mean.*

(i) *Suppose that K satisfies the condition*

$$x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I.$$

Then a function $F : I^2 \rightarrow \mathbb{R}$ continuous at every point of the diagonal $\Delta(I^2) := \{(x, x) : x \in I\}$ satisfies the functional equation

$$F(M(x, y), M_{[K]}(x, y)) = F(x, y), \quad x, y \in I, \quad (4)$$

if and only if there is a single variable continuous function $\varphi : I \rightarrow \mathbb{R}$ such that

$$F(x, y) = \varphi(K(x, y)), \quad x, y \in I.$$

(ii) *Suppose that K satisfies the condition*

$$x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I.$$

Then a function $F : I^2 \rightarrow \mathbb{R}$ continuous at every point of the diagonal $\Delta(I^2)$ satisfies the functional equation

$$\begin{aligned} F(M^{[K]}(x, y), M(x, y)) \\ = F(x, y), \quad x, y \in I, \end{aligned}$$

if and only if there is a single variable continuous function $\varphi : I \rightarrow \mathbb{R}$ such that

$$F(x, y) = \varphi(K(x, y)), \quad x, y \in I.$$

Proof. Assume first that $F : I^2 \rightarrow \mathbb{R}$ is continuous on the diagonal $\Delta(I^2)$ and satisfies equation (4), that is

$$F \circ (M, M_{[K]}) = F.$$

Hence, by induction,

$$F = F \circ (M, M_{[K]})^n, \quad n \in \mathbb{N}, \quad (5)$$

where $(M, M_{[K]})^n$ is the n th iterate of $(M, M_{[K]})$. By Theorem 3 the sequence of mean-type mappings $(M, M_{[K]})^n$ converges to the mean-type mapping $(K, K) : I^2 \rightarrow I^2$; that is

$$\lim_{n \rightarrow \infty} (M, M_{[K]})^n(x, y) = (K(x, y), K(x, y)), \quad (x, y) \in I^2.$$

Since $(K(x, y), K(x, y))$ belongs to the diagonal $\Delta(I^2)$ for every $(x, y) \in I^2$, by (5) and the continuity of F on $\Delta(I^2)$ implies that for every $(x, y) \in I^2$,

$$\begin{aligned} F(x, y) &= \lim_{n \rightarrow \infty} F((M, M_{[K]})^n(x, y)) = F\left(\lim_{n \rightarrow \infty} (M, M_{[K]})^n(x, y)\right) \\ &= F(K(x, y), K(x, y)). \end{aligned}$$

Setting

$$\varphi(t) := F(t, t), \quad t \in I,$$

we conclude that $F(x, y) = \varphi(K(x, y))$ for all $(x, y) \in I^2$.

To prove the converse implication, take an arbitrary function $\varphi : I \rightarrow \mathbb{R}$ and put $F := \varphi \circ K$. Then, for all $x, y \in I$, making use of the K -invariance with respect to $M, M_{[K]}$, we have

$$\begin{aligned} F(M(x, y), M_{[K]}(x, y)) &= (\varphi \circ K)(M(x, y), M_{[K]}(x, y)) \\ &= \varphi(K(M(x, y), M_{[K]}(x, y))) = \varphi(K(x, y)) \\ &= F(x, y), \end{aligned}$$

which completes the proof of (i).

We omit the similar proof of (ii). □

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