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Aequationes Mathematicae

Quadratic ellipsoids in Minkowski geometries

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Abstract. A Minkowski plane is Euclidean if and only if at least one ellipse is a quadric. We discuss the higher dimensional consequences too.

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1. Introduction

Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^n$, (centrally) symmetric to the origin. Then the function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\lambda > 0 : (\boldsymbol{y} - \boldsymbol{x})/\lambda \in \mathcal{I}\}$$

is a metric on $\mathbb{R}^n$ [1, IV.24], and is called *Minkowski metric on* $\mathbb{R}^n$. It satisfies the strict triangle inequality, i.e. $d(A, B)+d(B, C) = d(A, C)$ is valid if and only if $B \in \overline{AC}$. A pair $(\mathbb{R}^n, d)$, where d is a Minkowski metric, is called *Minkowski geometry*, and $\mathcal{I}$ is called the *indicatrix* of it. In a Minkowski geometry $(\mathbb{R}^n, d)$

(D_1) a set $\mathcal{E}^a_{d;F_1,F_2} := \{E : 2a = d(F_1, E)+d(E, F_2)\}$, where $a > d(F_1, F_2)/2$, is called an *ellipse* if $n = 2$, and an *ellipsoid* in higher dimensions,

where $F_1, F_2 \in \mathcal{M}$ are called the *focuses*, and $a > 0$ is called the *radius*.

A hypersurface in $\mathbb{R}^n$ is called a *quadric* if it is the zero set of an irreducible polynomial of degree two in n variables. We call a hypersurface *quadratic* if it is part of a quadric. Since every isometric mapping between two Hilbert geometries is a restriction of a projectivity, and every projectivity maps quadrics to quadrics, the quadraticity of a metrically defined hypersurface is a geometric property in each Hilbert geometry. Thus the question arises whether the

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metrically defined hypersurfaces are quadrics. In [5] we answered this question for conics.

We prove that (Theorem 4.3) a Minkowski plane is a model of the Euclidean plane if and only if at least one of the ellipses is a quadric, and that (Theorem 4.4) a Minkowski plane is analytic if and only if at least one of the ellipses is analytic.

As for higher dimensions, we prove (Theorem 5.1) that a Minkowski geometry is a model of the Euclidean geometry if and only if every central planar section of at least one quadric is an ellipse.

These results can be regarded as generalizations of [1, IV.25.4].

2. Notations and preliminaries

Points of $\mathbb{R}^n$ are labeled as $A, B, \ldots$, vectors are denoted by $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \ldots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB}$, while $\overline{A}B$ denotes the open ray starting from A passing through B, and $AB = \overline{A}B \cup A\overline{B}$.

On an affine plane the *affine ratio* $(A, B; C)$ of collinear points A, B and C satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$ [1, III.15.10], and the *cross ratio* of collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [1, VI.40.17].

It is easy to observe in (D_1) that an ellipsoid intersects line F_1F_2, the *main axis*, in exactly two points whose distance is twice the radius. Further notions are the *(linear) eccentricity* $c = d(F_1, F_2)/2$, the *numerical eccentricity* $\varepsilon = c/a$. The metric midpoint of the segment $\overline{F_1F_2}$ is called the *center*.

In the plane we use the notations $\boldsymbol{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$. It is worth noting that, by these, we have $\frac{\mathrm{d}}{\mathrm{d}\varphi}\boldsymbol{u}_\varphi = \boldsymbol{u}_\varphi^\perp$.

A *quadric* in the plane has the equation of the form

$$\mathcal{Q}_{\mathfrak{s}}^{\sigma} := \left\{ (x, y) : \begin{array}{ll} 1 = x^2 + \sigma y^2 & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2 & \text{if } \sigma = 0, \end{array} \right\} \tag{D_q}$$

in a suitable affine coordinate system $\mathfrak{s}$, and we call it *elliptic*, *parabolic*, or *hyperbolic* according to whether $\sigma = 1$, $\sigma = 0$, or $\sigma = -1$, respectively.

We usually *polar parameterize* the boundary $\partial\mathcal{D}$ of a non-empty domain $\mathcal{D}$ in $\mathbb{R}^2$ star-like with respect to a point $P \in \mathcal{D}$ so that $\boldsymbol{r}\colon [-\pi, \pi) \to \mathbb{R}^2$ is defined by $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, where r is the *radial function* of $\mathcal{D}$ with *base point* P.

We call a curve in the plane *analytic* if the coordinates of its points depend on its arc length analytically.

3. Utilities

In the presented technical lemmas the underlying plane is Euclidean.

Lemma 3.1. *The border of a convex domain is an analytic curve if and only if any one of its radial functions is analytic.*

Proof. Let $\mathcal{D}$ be an open convex domain containing the origin $O = (0,0)$. Let $s \mapsto \boldsymbol{p}(s)$ be an arc length parameterization of $\partial\mathcal{D}$, where $s \geq 0$, and let $\varphi \mapsto \boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$ be a polar parameterization of $\partial\mathcal{D}$ on $[-\pi,\pi)$ such that $\boldsymbol{p}(0) = \boldsymbol{r}(-\pi)$. Then

$$s(\xi) = \int_{-\pi}^{\xi} |\dot{\boldsymbol{r}}(\varphi)| d\varphi = \int_{-\pi}^{\xi} \sqrt{\dot{r}^2(\varphi) + r^2(\varphi)} d\varphi, \tag{3.1}$$

hence the function $s\colon \xi \mapsto s(\xi)$ is strictly monotonously increasing, and therefore its inverse function $\sigma\colon s(\xi) \mapsto \xi$ exists and is strictly monotonously increasing.

First, assume the analyticity of r. Then, as r is bounded from below by a positive number, the integrand on the right-hand side of (3.1) is analytic, and therefore s is analytic. As $\dot{s}(\xi)$ is positive by (3.1), the analyticity of σ follows from the analytic inverse function theorem [3, Theorem 4.2], and this implies the analyticity of $\boldsymbol{p}(s) = \boldsymbol{r}(\sigma(s)) = r(\sigma(s))\boldsymbol{u}_{\sigma(s)}$.

Conversely, assume that $\boldsymbol{p}$ is analytic. As the derivatives of the cosine and sine functions do not vanish simultaneously, $\boldsymbol{u}_{\sigma(s)} = \boldsymbol{p}(s)/|\boldsymbol{p}(s)|$ proves that σ is analytic. As the derivative $\dot{\sigma}(t) = 1/\dot{s}(\sigma(t))$ vanishes nowhere, the analyticity of s follows again by the analytic inverse function theorem [3, Theorem 4.2]. Then the analyticity of $r(\xi) = \langle \boldsymbol{p}(s(\xi)), \boldsymbol{u}_\xi \rangle$ follows.

The lemma is proved. □

Notice that by differentiating the last formula in the proof and then substituting the derivative of (3.1) leads to

$$\dot{r}(\xi) = \langle \dot{\boldsymbol{p}}(s(\xi)), \boldsymbol{u}_\xi \rangle \sqrt{\dot{r}^2(\xi) + r^2(\xi)}. \tag{3.2}$$

Let again $\mathcal{D}$ be an open convex domain containing the origin $O = (0,0)$. Let $\varphi \mapsto \boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$ be a polar parameterization of $\partial\mathcal{D}$ on $[-\pi,\pi)$, and let $A = \boldsymbol{r}(0)$, $B = \boldsymbol{r}(\pi)$. Let $F_1 \in \overline{AO}$ and $F_2 \in \overline{OB}$.

Given $\varphi_0 \in (0,\pi/2)$, let $D_0 = \boldsymbol{r}(\varphi_0)$, $\alpha_0 = \angle BF_1D_0$ and $\beta_0 = \angle BF_2D_0$. Assuming that D_{2i} is defined for an $i \in \mathbb{N}$, we define sequences recursively as follows (See Fig. 1.): $D_{2i+1} := \overline{D_{2i}F_1} \cap \partial\mathcal{D}$, $\alpha_{2i+1} := \alpha_{2i} + \pi$, and $\beta_{2i+1} := \angle BF_2D_{2i+1}$; then $D_{2i+2} := \overline{D_{2i+1}F_2} \cap \partial\mathcal{D}$, $\alpha_{2i+2} = \angle BF_1D_{2i+2}$, and $\beta_{2i+2} := \beta_{2i+1} - \pi$. We clearly have $\varphi_{2i} \in (0,\pi/2)$ and $\varphi_{2i+1} \in (\pi, 3\pi/2)$ for every $i \in \mathbb{N}$.

Lemma 3.2. *If $i \to \infty$, then α_{2i}, β_{2i} and φ_{2i} tend to zero, α_{2i+1}, β_{2i+1}, and φ_{2i+1} tend to π, and $\alpha_{2i+2}/\alpha_{2i}$ tends to $(F_1,F_2;A,B)$.*

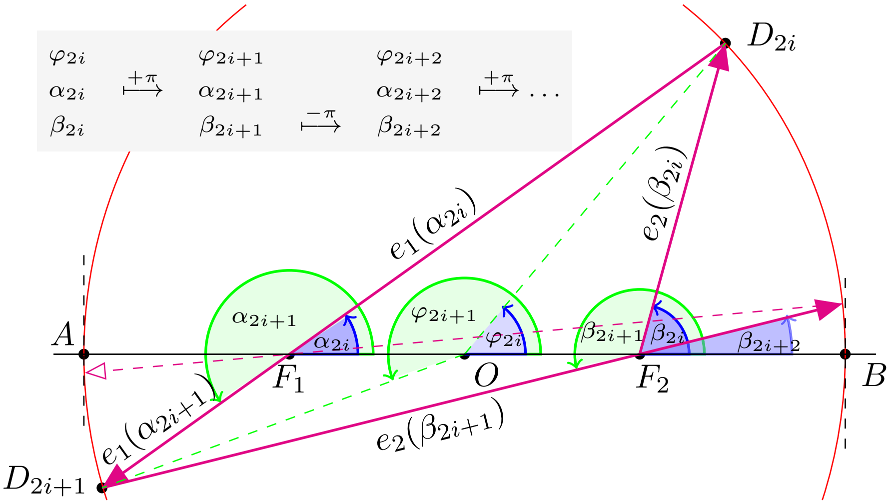

Figure 1. Sequence of angles

Proof. Observe that $\varphi_{2i} \in (0, \pi/2)$, $\varphi_{2i+1} \in (\pi, 3\pi/2)$, and

$$\beta_{2i+2} < \alpha_{2i} < \beta_{2i} \quad \text{and} \quad \beta_{2i+1} < \alpha_{2i+1} < \beta_{2i-1} \ (i \in \mathbb{N}),$$

hence the sequences β_{2i}, β_{2i+1}, φ_{2i}, φ_{2i+1}, α_{2i} and α_{2i+1} decrease monotonously, hence they are convergent.

From $\lim_{i\to\infty} \beta_{2i} > 0$, $\lim_{i\to\infty} \beta_{2i+1} > \pi$, $\lim_{i\to\infty} \alpha_{2i} > 0$, and $\lim_{i\to\infty} \alpha_{2i+1} > \pi$ follow, so $\lim_{i\to\infty} \frac{\beta_{2i+2}}{\beta_{2i}} = 1$, hence $\lim_{i\to\infty} \frac{\alpha_{2i}}{\beta_{2i}} = 1$. By the sine law the last limit gives $\lim_{i\to\infty} \frac{d(F_2,D_{2i})}{d(D_{2i},F_1)} = 1$, which, by the continuity of the functions involved, implies $d(F_2, \lim_{i\to\infty} D_{2i}) = d(\lim_{i\to\infty} D_{2i}, F_1)$, and consequently $\lim_{i\to\infty} \varphi_{2i} = \pi/2$ which contradicts $\varphi_0 \in (0, \pi/2)$ as φ_i is monotonously decreasing. Thus, the assumption was false, so α_{2i}, β_{2i}, and φ_{2i} tend to zero, and α_{2i+1}, β_{2i+1}, and φ_{2i+1} tend to π.

So, observing Fig. 1, we see that

$$\begin{aligned} e_1(\alpha_{2i}) &:= d(F_1,D_{2i}) \to d(F_1,B),\ e_1(\alpha_{2i+1}) := d(F_1,D_{2i+1}) \to d(F_1,A),\\ e_2(\beta_{2i}) &:= d(F_2,D_{2i}) \to d(F_2,B),\ e_2(\beta_{2i+1}) := d(F_2,D_{2i+1}) \to d(F_2,A), \end{aligned} \tag{3.3}$$

and, by the sine law,

$$\frac{\tan\alpha_{2i}}{\tan\beta_{2i+2}} = \frac{e_2(\beta_{2i+1})\cos\beta_{2i+1}}{e_1(\alpha_{2i+1})\cos\alpha_{2i+1}} \quad \text{and} \quad \frac{\tan\alpha_{2i+2}}{\tan\beta_{2i+2}} = \frac{e_2(\beta_{2i+2})\cos\beta_{2i+2}}{e_1(\alpha_{2i+2})\cos\alpha_{2i+2}}.$$

The quotient of these is

$$\frac{\tan\alpha_{2i+2}}{\tan\alpha_{2i}} = \frac{e_2(\beta_{2i+2})\cos\beta_{2i+2}}{e_1(\alpha_{2i+2})\cos\alpha_{2i+2}}\frac{e_1(\alpha_{2i+1})\cos\alpha_{2i+1}}{e_2(\beta_{2i+1})\cos\beta_{2i+1}}$$

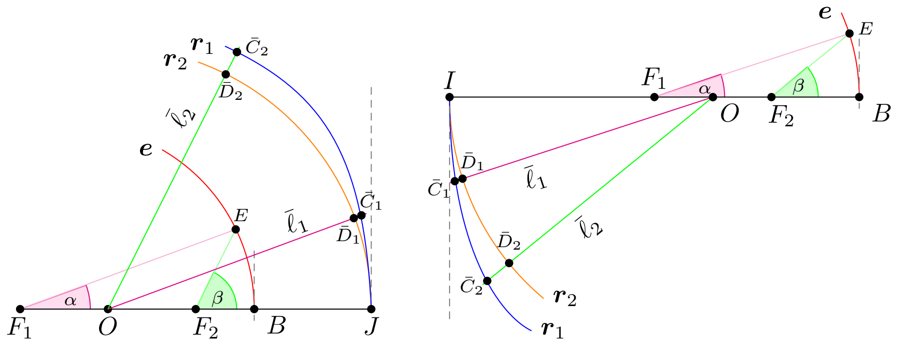

FIGURE 2. Example configurations, where $\dot{\boldsymbol{r}}_1(0) = \boldsymbol{u}_{\pi/2}$ and $J = \boldsymbol{r}_1(0)$, and $\dot{\boldsymbol{r}}_1(0) = \boldsymbol{u}_{-\pi/2}$ and $I = \boldsymbol{r}_1(0)$

which, through (3.3), immediately implies the last statement of the lemma. □

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be analytic arc length parametrizations of curves, such that $\boldsymbol{r}_1(0) = \boldsymbol{r}_2(0)$ and $\dot{\boldsymbol{r}}_1(0) = \dot{\boldsymbol{r}}_2(0)$. Let ℓ be the line through $\boldsymbol{r}_1(0)$ that is orthogonal to $\dot{\boldsymbol{r}}_1(0)$, and let F_1, F_2, and B be different points on ℓ such that $B \notin \overline{F_1F_2}$ and $\boldsymbol{r}_1(0) \notin \{B, F_1, F_2\}$. Let $\boldsymbol{e}$ be an analytic arc length parametrization of a curve, such that $B = \boldsymbol{e}(0)$ and $\dot{\boldsymbol{e}}(0) = \boldsymbol{u}_{\pi/2}$. Every point $E = \boldsymbol{e}(s)$ determines two straight lines $\ell_1 := F_1E$ and $\ell_2 := F_2E$ forming small angles α and β with ℓ, respectively. Let the straight line $\bar{\ell}_j$ ($j = 1, 2$) through the midpoint O of the segment $\overline{F_1F_2}$ be parallel to ℓ_j. See Fig. 2.

Label the intersections of $\bar{\ell}_j$ with $\boldsymbol{r}_i$ ($i, j = 1, 2$) as $\bar{C}_1 = \boldsymbol{r}_1(s_1(\alpha))$, $\bar{D}_1 = \boldsymbol{r}_2(s_2(\alpha))$ and $\bar{C}_2 = \boldsymbol{r}_1(s_1(\beta))$, $\bar{D}_2 = \boldsymbol{r}_2(s_2(\beta))$, where $s_i(\alpha)$ and $s_i(\beta)$ are the corresponding arc length parameters of $\boldsymbol{r}_i$ at the intersection of $\boldsymbol{r}_i$ with $\bar{\ell}_j$ ($i, j = 1, 2$).

Lemma 3.3. *If the curves $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are different in every neighborhood of the point $K := \boldsymbol{r}_1(0)$, and E tends to B on the curve $\boldsymbol{e}$, then*

$$\frac{\delta(\alpha)}{\delta(\beta)} \to (F_2, F_1; B)^k, \quad \textit{for an integer } k \geq 2, \tag{3.4}$$

where $\delta\colon \varphi \mapsto \langle \boldsymbol{r}_1(s_1(\varphi)) - \boldsymbol{r}_2(s_2(\varphi)), \boldsymbol{u}_\varphi \rangle$.

Proof. If $\boldsymbol{r}_1^{(i)}(0) = \boldsymbol{r}_2^{(i)}(0)$ for every $i \in \mathbb{N}$, then, by the analyticity of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, $\boldsymbol{r}_1 = \boldsymbol{r}_2$ in a neighborhood of K, so $k := \min\{i \in \mathbb{N} : \boldsymbol{r}_1^{(i)}(0) \neq \boldsymbol{r}_2^{(i)}(0)\}$ is well defined and is at least two.

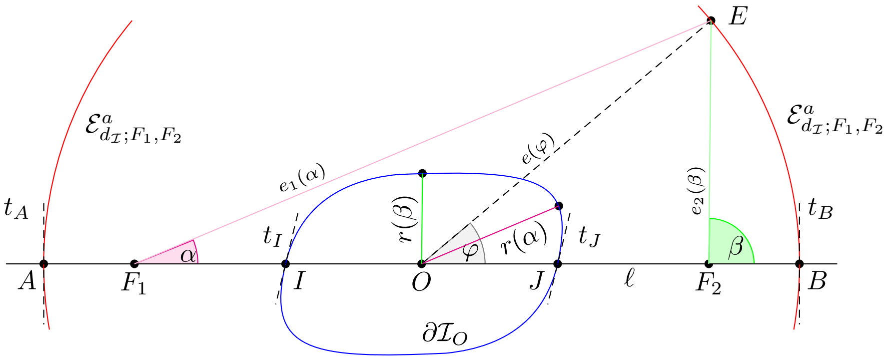

Figure 3. Ellipse $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ in a Minkowski plane

Let $E^\perp$ be the orthogonal projection of E onto ℓ. Using L'Hôpital's rule we get

$$\frac{|F_2 - B|}{|F_1 - B|} = \lim_{s\to 0} \frac{|F_2 - E^\perp|}{|F_1 - E^\perp|} = \lim_{s\to 0} \frac{\tan\alpha}{\tan\beta} = \lim_{s\to 0} \frac{(1+\tan^2\alpha)\frac{d\alpha}{ds}}{(1+\tan^2\beta)\frac{d\beta}{ds}} = \lim_{s\to 0} \frac{\dot\alpha}{\dot\beta}. \tag{3.5}$$

If $\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\beta)}$ exists, then L'Hôpital's rule can be used, so we obtain

$$\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\beta)} = \lim_{s\to 0} \frac{\dot\delta(\alpha)\dot\alpha}{\dot\delta(\beta)\dot\beta} = \lim_{s\to 0} \frac{\dot\delta(\alpha)}{\dot\delta(\beta)} \lim_{s\to 0} \frac{\dot\alpha}{\dot\beta} = \cdots = \lim_{s\to 0} \frac{\delta^{(k)}(\alpha)}{\delta^{(k)}(\beta)} \Big(\lim_{s\to 0} \frac{\dot\alpha}{\dot\beta}\Big)^k = \Big(\lim_{s\to 0} \frac{\dot\alpha}{\dot\beta}\Big)^k,$$

which proves the lemma. □

4. One ellipse in a Minkowski plane

We start by considering the Minkowski plane $(\mathbb{R}^2, d_{\mathcal{I}})$ with indicatrix $\mathcal{I}$.

Since every ellipse is bounded, if an ellipse is a quadric, then it is an elliptic quadric.

Take an ellipse $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$, and let ℓ be a line through F_1 that passes F_2. Let A, B be the intersections of ℓ with $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ such that $F_1 \in \overline{AF_2}$ and $F_2 \in \overline{F_1B}$. Let $\mathcal{I}_O$ be the translate of the indicatrix centered at the midpoint O of $\overline{F_1F_2}$, and let I, J be the intersections of ℓ with $\partial\mathcal{I}_O$, so that $I \in \overline{O}A$ and $J \in \overline{O}B$. Furthermore, let t_A, t_B and t_I, t_J, respectively, denote the tangents of the appropriate curve $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ or $\partial\mathcal{I}_O$ at A, B and I, J. See Fig. 3.

For any fixed Euclidean metric d_e let r be the radial function of $\mathcal{I}_O$ with respect to O, $\alpha = \angle(EF_1O)$, $\beta = \angle(EF_2B)$ and $\varphi = \angle(EOB)$ for the points E of $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$. We consider these angles as functions $\alpha(\varphi)$, $\beta(\varphi)$, and also define the functions $e_1(\alpha) = d_e(F_1, E)$, $e_2(\beta) = d_e(F_2, E)$ and $e(\varphi) = d_e(O, E)$. Then $d_{\mathcal{I}}(F_1, E) = e_1(\alpha)/r(\alpha)$, and $d_{\mathcal{I}}(F_2, E) = e_2(\beta)/r(\beta)$, hence

$$2a = \frac{e_1(\alpha)}{r(\alpha)} + \frac{e_2(\beta)}{r(\beta)}. \tag{4.1}$$

Lemma 4.1. *If the ellipse $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ is a quadric, then $t_A \parallel t_B \parallel t_I \parallel t_J$.*

Proof. Since $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ is a quadric, φ and E are bijectively related, hence the functions $\alpha(\varphi)$, $\beta(\varphi)$ are well defined.

The symmetry of $\mathcal{I}$ entails that $t_I \parallel t_J$, and it also follows that the affine center of the quadric $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ coincides with its metric center O, hence $t_A \parallel t_B$ too.

Choose a Euclidean metric d_e so that $t_A \perp \ell \perp t_B$.

Differentiation of (4.1) with respect to φ gives

$$0 = \frac{\dot{e}_1(\alpha)r(\alpha) - e_1(\alpha)\dot{r}(\alpha)}{r^2(\alpha)}\dot{\alpha} + \frac{\dot{e}_2(\beta)r(\beta) - e_2(\beta)\dot{r}(\beta)}{r^2(\beta)}\dot{\beta}.$$

As $\dot{e}_1(0) = \dot{e}_2(0) = 0 = \dot{e}_1(\pi) = \dot{e}_2(\pi)$, this gives at $\varphi = 0$ and at $\varphi = \pi$ that

$$\begin{aligned} 0 &= \dot{r}(0)(e_1(0)\dot{\alpha}(0) + e_2(0)\dot{\beta}(0)), \\ 0 &= \dot{r}(\pi)(e_1(\pi)\dot{\alpha}(\pi) + e_2(\pi)\dot{\beta}(\pi)), \end{aligned} \tag{4.2}$$

respectively. Applying (3.5) for the present configuration, we obtain

$$e_1(0)\dot{\alpha}(0) = e_2(0)\dot{\beta}(0) \neq 0 \quad \text{and} \quad e_1(\pi)\dot{\alpha}(\pi) = e_2(\pi)\dot{\beta}(\pi) \neq 0,$$

which prove $\dot{r}(0) = \dot{r}(\pi) = 0$ in (4.2), hence the lemma. □

Lemma 4.2. *If the ellipse $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ is an analytic curve in a neighborhood of A and B, then the curve $\partial\mathcal{I}_O$ is analytic in a neighborhood of I and J.*

Proof. By Lemma 3.1 and its proof, the functions $\alpha \mapsto e_1$, $\beta \mapsto e_2$, $s \mapsto \alpha$ and $s \mapsto \beta$, and the inverses $\alpha \mapsto s$ and $\beta \mapsto s$ are analytic. Specifically these imply that $\beta(\alpha)$ is also an analytic function.

As $x \mapsto 1/x$ is analytic in a neighborhood of $r(0) > 0$, $r(\beta)$ is analytic in a neighborhood of the zero too, so we only need to prove that $\bar{r}(\beta) := 1/r(\beta)$ is analytic in some neighborhood of the zero. With this in mind (4.1) becomes

$$\bar{r}(\beta) = \frac{e_1(\alpha(\beta))}{-e_2(\beta)}\bar{r}(\alpha(\beta)) + \frac{2a}{e_2(\beta)}. \tag{4.3}$$

Introduce the functions $f(\beta) := \alpha(\beta)$, $g(\beta) := \frac{e_1(\alpha(\beta))}{-e_2(\beta)}$, and $h(\beta) := \frac{2a}{e_2(\beta)}$. Then f, g, and h are analytic in a neighborhood of 0, $f(0)=0$, $\dot{f}(0)=\frac{e_2(0)}{e_1(0)}<1$, $g(0) = \frac{-e_1(0)}{e_2(0)}$, and $h(0) = \frac{2a}{e_2(0)}$. Furthermore, by (4.3), the function $\phi(\beta) := \bar{r}(\beta)$ solves the functional equation $\phi(\alpha) = g(\alpha)\phi(f(\alpha)) + h(\alpha)$. However, by [3, Theorem 4.6], such a functional equation has a unique solution, which additionally is analytic in a neighborhood of 0. Consequently, $r(\beta)$ is the reciprocal of that unique analytic solution, so $\partial\mathcal{I}_O$ is analytic around I, and, by its symmetry, around J too. □

Theorem 4.3. *A Minkowski plane is a model of the Euclidean plane if and only if at least one ellipse is a quadric.*

Proof. Every ellipse is a quadric in the Euclidean plane, so we only have to prove that a Minkowski plane is Euclidean if at least one ellipse is a quadric.

Assume that $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ is a quadric.

If $F_1 = F_2$, then $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ is a homothetic image of $\partial\mathcal{I}$, hence $\partial\mathcal{I}$ is a quadric, and therefore $d_\mathcal{I}$ is Euclidean by [1, 25.4].

From now on we assume that $F_1 \neq F_2$.

We have $t_A \parallel t_I \parallel t_J \parallel t_B$ by Lemma 4.1, and, as every (planar) quadric is an analytical curve, the border $\partial\mathcal{I}_O$ is analytic in a neighborhood of I and J by Lemma 4.2, where O is the midpoint of $\overline{F_1F_2}$. Furthermore, by the central symmetry of $\mathcal{I}_O$ and the definition of $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$, we have $c = d_\mathcal{I}(F_1, O)$, $\overrightarrow{AF_1} = \overrightarrow{F_2B}$ and $\overrightarrow{IA} = \overrightarrow{BJ}$, so O is the (affine) midpoint of both $\overline{IJ}$ and $\overline{AB}$. Additionally, we have $a \cdot d_\mathcal{I}(O, J) = d_\mathcal{I}(O, B)$, because the definition of $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ implies

$$\begin{aligned}2d_\mathcal{I}(O,B) &= 2d_\mathcal{I}(O,F_2) + 2d_\mathcal{I}(F_2,B)\\ &= d_\mathcal{I}(F_1,O) + d_\mathcal{I}(O,F_2) + d_\mathcal{I}(F_2,B) + 2a - d_\mathcal{I}(F_1,B) = 2a.\end{aligned}$$

Denote by C_1 and C_2 the points where the line through O parallel to t_I intersects $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$. Fix the affine coordinate system, where $O = (0,0)$, $J = (1,0)$ and $C_1 = (0, \sqrt{a^2 - c^2})$, and choose the Euclidean metric d_e such that $\{(1,0),(0,1)\}$ is an orthonormal basis. Let $\mathcal{C}$ denote the unit circle of d_e. See Fig. 4.

Then both $\mathcal{E}^a_{d_e;F_1,F_2}$ and $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ are quadrics, and they have four common points A, B and C_1, C_2, and two common tangents t_A and t_B, hence they coincide.

By the definition of $\mathcal{E}^a_{d_e;F_1,F_2}$ we have $e_1(\alpha) + e_2(\beta) = 2a$. Using this fact and using $\delta\colon \varphi \mapsto r(\varphi) - 1$, which corresponds to Lemma 3.3, in (4.1) we have

$$\delta(\alpha) = -\delta(\beta)\frac{e_2(\beta)}{e_1(\alpha) + 2a\delta(\beta)}. \tag{4.4}$$

Taking the limit of this as $\varphi \to 0$, we obtain

$$\lim_{\varphi\to 0}\frac{\delta(\alpha)}{\delta(\beta)} = -\frac{a-c}{a+c} = -(F_2, F_1; B).$$

Comparing this to (3.4), we obtain $(F_2, F_1; B) = -1$, which contradicts $F_2 \in \overline{F_1B}$. This contradiction implies that $\delta(\beta) = 0$ in a neighborhood of J.

On the other hand, (4.4) implies that if $\delta(\beta_0) \neq 0$ for some β_0, then $\delta(\beta_{2i})$ never vanishes in the process described in Lemma 3.2, meanwhile β_{2i} tends to

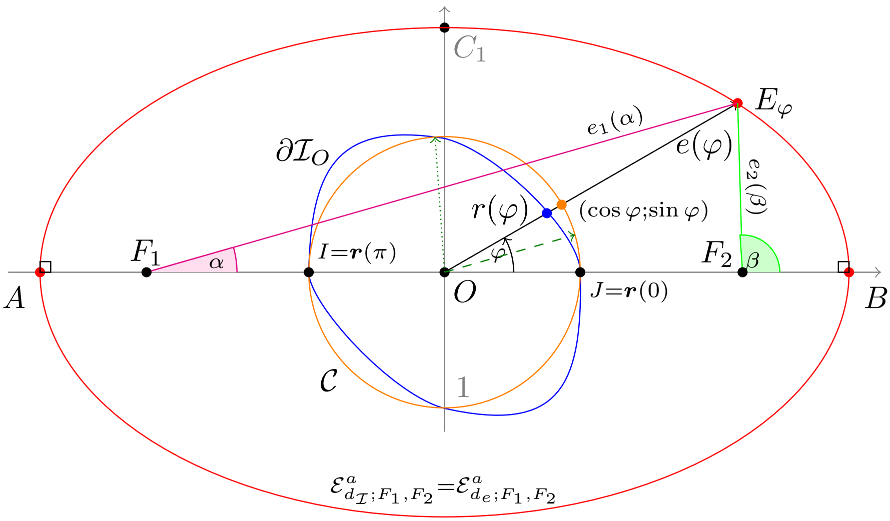

FIGURE 4. Coinciding ellipses $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \equiv \mathcal{E}^a_{d_e;F_1,F_2}$

zero by Lemma 3.2. This is a contradiction as δ vanishes in a neighborhood of J, hence $\delta \equiv 0$, so the theorem is proved. □

This kind of implication extends over to analyticity too.

Theorem 4.4. *The boundary of the indicatrix of a Minkowski plane is analytic if and only if one of the ellipses of the Minkowski plane is analytic.*

Proof. First, assume that the Minkowski plane $(\mathbb{R}^2, d_\mathcal{I})$ is analytic. Then the circles are also analytic, because they are homothetic to the boundary of the indicatrix, so we only need to prove the analyticity of ellipses $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ with different focuses.

Fix an arbitrary point $E_0 \in \mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$, and let the point $R_i \in \mathcal{I}$ $(i = 1, 2)$ be such that $O\overline{R_i} \parallel F_i\overline{E_0}$. Let the straight line t_i $(i = 1, 2)$ be tangent to $\mathcal{I}$ at R_i. Let d_e be the Euclidean metric which satisfies $t_2 \perp OR_2$, $d_e(O, R_1) = d_e(O, R_2)$, and $d_e(O, J) = 1$. Then we have

$$e_2^2(\beta) = e_1^2(\alpha) + 4c^2 - 4e_1(\alpha)c\cos\alpha, \quad \text{and} \quad \beta = \arcsin\frac{e_1(\alpha)\sin\alpha}{e_2(\beta)}.$$

Substituting this into (4.1) results in the analytic equation

$$F(\alpha, e_1(\alpha)) := \Big(2a - \frac{e_1(\alpha)}{r(\alpha)}\Big)^2 - \frac{e_1^2(\alpha) + 4c^2 - 4e_1(\alpha)c\cos\alpha}{r^2\Big(\arcsin\frac{e_1(\alpha)\sin\alpha}{\sqrt{e_1^2(\alpha)+4c^2-4e_1(\alpha)c\cos\alpha}}\Big)} = 0.$$

Since

$$\partial_2 F(\alpha, e_1(\alpha)) = 2\frac{e_2(\beta)}{r(\beta)}\frac{-1}{r(\alpha)} - \frac{2e_1(\alpha) - 4c\cos\alpha}{r^2(\beta)} + 2\frac{e_2^2(\beta)}{r^3(\beta)}\frac{\dot r(\beta)}{\cos\beta}\Big(\frac{\sin\alpha}{e_2(\beta)} - \frac{1}{2}\frac{e_1(\alpha)\sin\alpha(2e_1(\alpha) - 4c\cos\alpha)}{e_2^3(\beta)}\Big),$$

$\partial_2 F(\alpha, e_1(\alpha))$ vanishes if and only if

$$\frac{e_2(\beta)}{r(\alpha)} + \frac{e_1(\alpha) - 2c\cos\alpha}{r(\beta)} = \frac{e_2(\beta)\sin\alpha}{r(\beta)\cos\beta}\frac{\dot r(\beta)}{r(\beta)}\Big(1 - \frac{e_1(\alpha)(e_1(\alpha) - 2c\cos\alpha)}{e_2^2(\beta)}\Big).$$

By (3.2), we have $\frac{\dot r(\beta)}{r(\beta)} = \cot\theta$, where θ is the angle between $F_1\overline{E}$ and the tangent vector at E of $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$. Furthermore, it can be easily seen that $e_1(\alpha) - 2c\cos\alpha = e_2(\beta)\cos(\beta-\alpha)$. Thus, the above equation is equivalent to

$$\frac{e_2(\beta)}{r(\alpha)} + \frac{e_2(\beta)\cos(\beta-\alpha)}{r(\beta)} = \frac{\cot\theta}{r(\beta)\cos\beta}(e_2(\beta)\sin\alpha - e_1(\alpha)\cos(\beta-\alpha)\sin\alpha).$$

Since $e_2(\beta)\sin\beta = e_1(\alpha)\sin\alpha$, this equation simplifies to

$$\frac{1}{r(\alpha)} + \frac{\cos(\beta-\alpha)}{r(\beta)} = \frac{\cot\theta}{r(\beta)}\frac{\sin\alpha - \sin\beta\cos(\beta-\alpha)}{\cos\beta} = -\sin(\beta-\alpha)\frac{\cot\theta}{r(\beta)}.$$

In sum, $\partial_2 F(\alpha, e_1(\alpha))$ vanishes if and only if

$$\frac{r(\beta)}{r(\alpha)} + \sin(\beta-\alpha)(\cot\theta + \cot(\beta-\alpha)) = 0. \tag{4.5}$$

At E_0 we have $\theta = \pi/2$, and $r(\beta) = r(\alpha)$, therefore (4.5) reduces to $1+\cos(\beta-\alpha)=0$, resulting $\beta = \pi + \alpha$, a contradiction. Thus $\partial_2 F(\alpha, e_1(\alpha)) \neq 0$ at E_0, hence the analytic implicit function theorem [3, Theorem 4.1] implies the analyticity of e_1 in a neighborhood of α. As the point E_0 was chosen arbitrarily on $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$, this proves that $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2}$ is analytic.

Assuming now that the ellipse is analytic, Lemma 4.2 proves the analyticity of the boundary of the indicatrix, where F_1F_2 intersects it. By (4.3) we have

$$\bar r(\beta(\alpha)) = \frac{e_1(\alpha)}{-e_2(\beta(\alpha))}\bar r(\alpha) + \frac{2a}{e_2(\beta(\alpha))}.$$

This shows that if $\bar r$ is analytic in an interval $(-\varepsilon, \varepsilon)$, then it is also analytic in the interval $(-\beta(\varepsilon), \beta(\varepsilon))$. According to Lemma 3.2, this means that the boundary of the indicatrix is analytic. □

5. Quadrics in a Minkowski geometry

Note that in the planar case Theorem 5.1 states that if one ellipse is a quadric, then the Minkowski plane is a model of the Euclidean geometry.

Theorem 5.1. *A Minkowski geometry is a model of the Euclidean geometry if and only if every central planar section of at least one quadric is an ellipse.*

Proof. As every central planar section of each elliptic quadric is an ellipse in the Euclidean geometry, we only have to prove that a Minkowski geometry is Euclidean if every central planar section of at least one quadric is an ellipse.

Let the quadric $\mathcal{Q}$ be such that its every central planar section is an ellipse. Then Theorem 4.3 implies, that every central planar section of the indicatrix is an ellipse, hence the statement of the theorem follows immediately from [2, II.16.12] which states for any integers $1 < k < n$ that the border $\partial\mathcal{K}$ of a convex body $\mathcal{K} \subset \mathbb{R}^n$ is an ellipsoid if and only if every k-plane through an inner point of $\mathcal{K}$ intersects $\partial\mathcal{K}$ in a k-dimensional ellipsoid. $\square$

For the other deduction we only have the following result that we put here without its easy proof.

Theorem 5.2. *A Minkowski geometry is a model of the Euclidean geometry if and only if there is a hyperplane and a point in that hyperplane such that every line in the hyperplane through the point is parallel to the main axis of some ellipsoid that is a quadric.*

It is worth noting that the question whether finding an ellipsoid that is a quadric would imply that a Minkowski geometry is Euclidean remains open for the non-planar cases.

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