



On curves with circles as their isoptics

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Abstract. In the present paper we describe the family of all closed convex plane curves of class C^1 which have circles as their isoptics. In the first part of the paper we give a certain characterization of all ellipses based on the notion of isoptic and we give a geometric characterization of curves whose orthoptics are circles. The second part of the paper contains considerations on curves which have circles as their isoptics and we show the form of support functions of all considered curves.

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1. Introduction

In the present paper we consider the family \mathcal{M} of all closed convex plane curves of class C^1 . We denote by p a support function of the curve $\mathcal{C} \in \mathcal{M}$ with respect to the origin O . Then the curve \mathcal{C} has the following parametrization

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for } t \in [0, 2\pi], \quad (1.1)$$

where p' denotes the derivative of the support function p , see [46] (Fig. 1).

Let us fix $\alpha \in (0, \pi)$. Let \mathcal{C}_α be the locus of vertices of a fixed angle $\pi - \alpha$ formed by two tangent lines of the curve \mathcal{C} . The curve \mathcal{C}_α will be called an α -isoptic of \mathcal{C} , see [1].

The curve \mathcal{C}_α is given by the formula

$$z_\alpha(t) = p(t)e^{it} + \frac{p(t + \alpha) - p(t)\cos \alpha}{\sin \alpha} ie^{it} \quad \text{for } t \in [0, 2\pi], \quad (1.2)$$

see [2, 3].

With each curve $\mathcal{C} \in \mathcal{M}$ we associate a certain family \mathcal{C}^* consisting of lines constructed in the following way.

We fix a chord of the curve \mathcal{C} such that its tangents at points $A, B \in \mathcal{C}$ intersect. Let us denote by U the intersection point of these tangents and by S

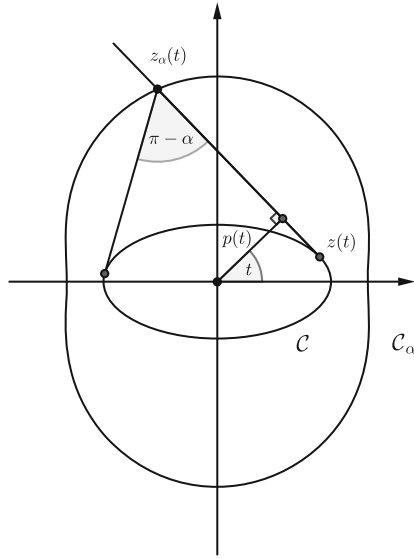


FIGURE 1. α -isoptic C_α of C

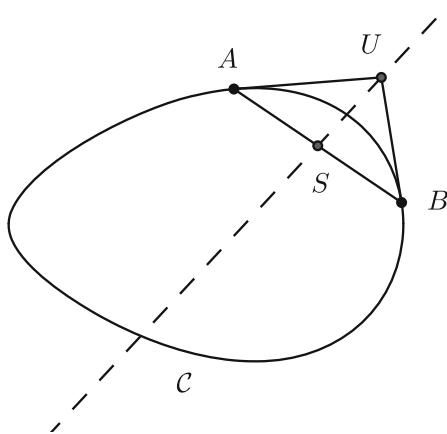


FIGURE 2. A line belonging to the family C^*

the midpoint of the segment AB . The line passing through U and S belongs to the family C^* . Moreover, given an angle $\alpha \in (0, \pi)$ denote by C_α^* the subfamily of C^* such that $\angle AUB = \pi - \alpha$ (Fig. 2).

In the first part of the paper we give the following characterization of ellipses.

Theorem 1.1. *Let $\alpha \in (0, \pi)$ be a fixed angle such that $\frac{\alpha}{\pi}$ is a rational number and $\alpha \neq \frac{\pi}{2}$. A curve $\mathcal{C} \in \mathcal{M}$ is an ellipse if and only if all lines from \mathcal{C}_α^* are concurrent.*

In the second part of the paper using some considerations from the previous sections we find curves with special isoptics called orthoptics. We recall a definition of an orthoptic curve. A $\frac{\pi}{2}$ -isoptic of the curve \mathcal{C} is said to be an orthoptic of \mathcal{C} . The ellipses give an example of curves whose orthoptics are circles. We present there a certain characterization of a class of curves with circles as their orthoptics. Moreover, we find explicitly a support function of a curve $\mathcal{C} \in \mathcal{M}$, different from a circle, which has a circle as its isoptic. These curves were considered in a very interesting paper [32] and in a paper [37] by the second author.

We would like to emphasize that all the papers in the bibliography, that is [1–10, 12–45, 47, 49–56], with the exception of Santaló's and Su's books, [46, 48], and the paper by Cyr, [11], present a wide spectrum of results in isoptics theory and are included here for the interested reader to have a complete overview of isoptics theory.

2. Some property of ellipses

In this section we prove the simple part of Theorem 1.1, namely:

If \mathcal{C} is an ellipse then all lines of the family \mathcal{C}^ intersect in the center of this ellipse.*

Proof. Using the properties of affine transformations the ellipse \mathcal{C} can be transformed into a circle and for the circle the mentioned property is very easy to establish. \square

3. Some characterization of ellipses

We will now deal with the second part of Theorem 1.1, namely:

If $\mathcal{C} \in \mathcal{M}$ and all lines of the family \mathcal{C}_α^ are concurrent then \mathcal{C} is an ellipse.*

Proof. The proof of this fact is divided into steps.

Step 1. Let $\mathcal{C} \in \mathcal{M}$ and all lines of the family \mathcal{C}_α^* be concurrent. This common point O we take as the origin of the coordinate system and the support function p in the Eq. (1.1) is determined with respect to this point.

Each point $z_\alpha(t)$ of a fixed α -isoptic determines a chord of the curve \mathcal{C} joining the points $z(t)$ and $z_\alpha(t)$. The midpoint of that chord we denote by $s(t)$. The formula (1.1) yields

$$\begin{aligned} 2s(t) = z(t) + z(t + \alpha) &= (p(t) + p(t + \alpha) \cos \alpha - p'(t + \alpha) \sin \alpha)e^{it} \\ &+ (p'(t) + p(t + \alpha) \sin \alpha + p'(t + \alpha) \cos \alpha)ie^{it}. \end{aligned} \quad (3.1)$$

From our assumptions the points $O, s(t), z_\alpha(t)$ lie on the same line (Fig.3), that is we have

$$\det[s(t), z_\alpha(t)] = 0. \tag{3.2}$$

Thus substituting the formulae (1.2) and (3.1) into (3.2) we get the following equation for the support function p , namely

$$(p^2(t + \alpha) - p^2(t)) \cos \alpha - (p(t + \alpha)p'(t + \alpha) + p(t)p'(t)) \sin \alpha = 0. \tag{3.3}$$

Substituting $p = \sqrt{f}$ we get a simpler condition for f than (3.3)

$$2(f(t + \alpha) - f(t)) \cos \alpha - (f'(t + \alpha) + f'(t)) \sin \alpha = 0. \tag{3.4}$$

Step 2.

Now, we develop the function f in the Fourier series. Let

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{3.5}$$

From the formula (3.5) we obtain

$$f'(t) = \sum_{n=1}^{\infty} (nb_n \cos nt - na_n \sin nt),$$

$$f(t + \alpha) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [(a_n \cos n\alpha + b_n \sin n\alpha) \cos nt + (b_n \cos n\alpha - a_n \sin n\alpha) \sin nt],$$

$$f'(t + \alpha) = \sum_{n=1}^{\infty} [n(b_n \cos n\alpha - a_n \sin n\alpha) \cos nt - n(a_n \cos n\alpha + b_n \sin n\alpha) \sin nt].$$

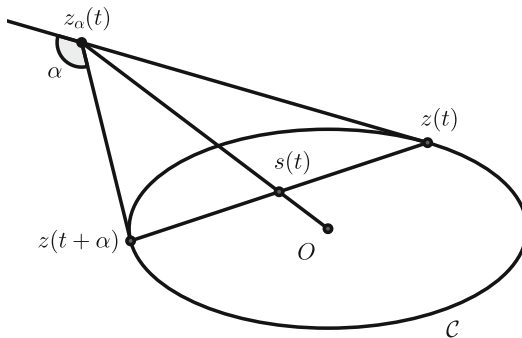


FIGURE 3. Points $z(t), z(t + \alpha), z_\alpha(t), s(t)$

Hence we get

$$\begin{aligned}
 & f(t + \alpha) - f(t) \\
 &= \sum_{n=1}^{\infty} [(a_n(\cos n\alpha - 1) + b_n \sin n\alpha) \cos nt + (-a_n \sin n\alpha + b_n(\cos n\alpha - 1)) \sin nt], \\
 & f'(t + \alpha) + f'(t) \\
 &= \sum_{n=1}^{\infty} [n(-a_n \sin n\alpha + b_n(1 + \cos n\alpha)) \cos nt - n(a_n(1 + \cos n\alpha) + b_n \sin n\alpha) \sin nt].
 \end{aligned}$$

The above relations substituted into (3.4) yield the following system of equations

$$\begin{cases} [2(\cos n\alpha - 1) \cos \alpha + n \sin n\alpha \sin \alpha]a_n \\ \quad + [2 \sin n\alpha \cos \alpha - n(1 + \cos n\alpha) \sin \alpha]b_n = 0 \\ [-2 \cos \alpha \sin n\alpha + n(1 + \cos n\alpha) \sin \alpha]a_n \\ \quad + [2(\cos n\alpha - 1) \cos \alpha + n \sin n\alpha \sin \alpha]b_n = 0. \end{cases} \tag{3.6}$$

Since the determinant of this system is equal to

$$[2(\cos n\alpha - 1) \cos \alpha + n \sin n\alpha \sin \alpha]^2 + [2 \sin n\alpha \cos \alpha - n(1 + \cos n\alpha) \sin \alpha]^2,$$

for the existence of a non-zero solution, the following system of equations must be satisfied

$$\begin{cases} 2(\cos n\alpha - 1) \cos \alpha + n \sin n\alpha \sin \alpha = 0, \\ 2 \sin n\alpha \cos \alpha - n(1 + \cos n\alpha) \sin \alpha = 0, \end{cases} \tag{3.7}$$

hence

$$\begin{cases} \cos n\alpha = \frac{4 \cos^2 \alpha - n^2 \sin^2 \alpha}{4 \cos^2 \alpha + n^2 \sin^2 \alpha}, \\ \sin n\alpha = \frac{4n \sin \alpha \cos \alpha}{4 \cos^2 \alpha + n^2 \sin^2 \alpha}. \end{cases} \tag{3.8}$$

Now, arguing as Jerónimo-Castro, Rojas-Tapia, Velasco-García and Yee-Romero in [22], we prove that f has only the coefficients a_0, a_2 and b_2 .

From the first equation of (3.7) we obtain that

$$\begin{aligned}
 n \sin n\alpha \sin \alpha &= 2(1 - \cos n\alpha) \cos \alpha, \\
 2n \sin \frac{n\alpha}{2} \cos \frac{n\alpha}{2} \sin \alpha &= 4 \left(\sin \frac{n\alpha}{2} \right)^2 \cos \alpha,
 \end{aligned}$$

which gives

$$\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}. \tag{3.9}$$

Now, we will prove that there is no integer number $n > 2$ such that the Eq. (3.9) is fulfilled. In order to do this, we shall prove two lemmas. The first lemma below is inspired by Lemma 3 in [22].

Lemma 3.1. *Suppose $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. If there exists a natural number $n > 2$ such that the Eq. (3.9) is satisfied then*

$$(n + 2) \sin \frac{(n - 2)\alpha}{2} = (n - 2) \sin \frac{(n + 2)\alpha}{2}.$$

Proof. We know that for any complex number $z \in \mathbb{C} \setminus \{(k + \frac{1}{2})\pi : k \in \mathbb{Z}\}$, it holds that

$$\tan z = i \frac{e^{-iz} - e^{iz}}{e^{iz} + e^{-iz}}.$$

By the assumptions of the lemma $\tan \alpha$ and $\tan \frac{n\alpha}{2}$ are simultaneously finite. The condition of the lemma in these terms is

$$\frac{e^{-i\frac{n}{2}\alpha} - e^{i\frac{n}{2}\alpha}}{e^{i\frac{n}{2}\alpha} + e^{-i\frac{n}{2}\alpha}} = \frac{n}{2} \cdot \frac{e^{-i\alpha} - e^{i\alpha}}{e^{i\alpha} + e^{-i\alpha}}.$$

From this equality, after some simplifications, we obtain

$$(n + 2) \left(e^{i(\frac{n-2}{2}\alpha)} - e^{-i(\frac{n-2}{2}\alpha)} \right) = (n - 2) \left(e^{i(\frac{n+2}{2}\alpha)} - e^{-i(\frac{n+2}{2}\alpha)} \right).$$

Since $\sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$, we have

$$(n + 2) \sin \frac{(n - 2)\alpha}{2} = (n - 2) \sin \frac{(n + 2)\alpha}{2}.$$

□

The following lemma is due to V. Cyr.

Lemma 3.2. ([11], Lemma 3) *If $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ is such that $\frac{\alpha}{\pi}$ is a rational number, and k and m are integer numbers such that $\sin m\alpha \neq 0$ then*

$$\frac{\sin k\alpha}{\sin m\alpha}$$

is either $-1, 0, 1$ or irrational.

Using Lemmas 3.1 and 3.2 we prove the following lemma inspired by Lemma 5 in [22].

Lemma 3.3. *If $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ is such that $\frac{\alpha}{\pi}$ is a rational number then there is no integer number $n > 2$ such that*

$$\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}.$$

Proof. Suppose $\frac{\alpha}{\pi}$ is a rational number and there is an integer number $n > 2$ such that $\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}$. This condition implies, by Lemma 3.1, that

$$\frac{\sin \frac{(n-2)\alpha}{2}}{\sin \frac{(n+2)\alpha}{2}} = \frac{n-2}{n+2}.$$

From this condition we have that $\sin \frac{(n+2)\alpha}{2} \neq 0$. Moreover, since $n > 2$, we have that the fraction $\frac{n-2}{n+2}$ is different from $-1, 0, 1$, hence by Lemma 3.2 we have that

$$\frac{\sin \frac{(n-2)\alpha}{2}}{\sin \frac{(n+2)\alpha}{2}}$$

must be an irrational number, which is a contradiction since $\frac{n-2}{n+2}$ cannot be an irrational number. We conclude that there is no integer number $n > 2$ such that $\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}$ if $\frac{\alpha}{\pi}$ is a rational number. \square

Now, if $\frac{\alpha}{\pi}$ is a rational number (different from $\frac{1}{2}$), from Lemma 3.3 we have that there is no integer number $n > 2$ such that $\frac{n}{2} \tan \alpha = \tan \frac{n\alpha}{2}$. It only remains to analyze the cases $n = 1$ and $n = 2$. When $n = 1$, the only solution of the Eq. (3.8) is $\alpha = 0$, which is not a permitted value. For $n = 2$, the Eq. (3.8) become identities and so a_2 and b_2 can be chosen arbitrarily.

Finally, the function f must be of the following form

$$f(t) = \frac{a_0}{2} + a_2 \cos 2t + b_2 \sin 2t,$$

where $a_0 > 2\sqrt{a_2^2 + b_2^2}$, since it has to have a positive value.

The case $\alpha = \frac{\pi}{2}$ will be analyzed in Theorem 4.1 below.

Step 3.

We consider the function

$$p(t) = \sqrt{\frac{a_0}{2} + a_2 \cos 2t + b_2 \sin 2t}, \tag{3.10}$$

such that

$$a_0 > 2\sqrt{a_2^2 + b_2^2}. \tag{3.11}$$

First we note that the condition (3.11) guarantees that $\frac{a_0}{2} + a_2 \cos 2t + b_2 \sin 2t > 0$. On the other hand we have $4p^3(p + p'') = a_0^2 - 4a_2^2 - 4b_2^2 > 0$. The condition $p + p'' > 0$ guarantees that p is a support function. In this part of the proof we show that the function p is a support function of an ellipse. To this aim we consider a conic given by the equation

$$Ax^2 + 2Bxy + Cy^2 = -F. \tag{3.12}$$

Putting

$$a = -\frac{A}{F}, \quad b = -\frac{B}{F}, \quad c = -\frac{C}{F}$$

we rewrite the Eq. (3.12) in the form

$$ax^2 + 2bxy + cy^2 = 1, \tag{3.13}$$

which is an ellipse if and only if $ac - b^2 > 0$. This ellipse has its center at $(0, 0)$ so it is always a rotated ellipse in the canonical position. After some

straightforward calculations one gets a support function of this ellipse with respect to the origin O in the form

$$\tilde{p}(t) = \sqrt{\frac{c}{ac - b^2} \cos^2 t - \frac{2b}{ac - b^2} \sin t \cos t + \frac{a}{ac - b^2} \sin^2 t}. \quad (3.14)$$

We are going to show that $p = \tilde{p}$. We can find unique coefficients a, b, c with $ac - b^2 > 0$ giving the support function (3.14) of an ellipse. It is easy to see that

$$p(t) = \sqrt{\frac{1}{2}(a_0 + 2a_2) \cos^2 t + 2b_2 \sin t \cos t + \frac{1}{2}(a_0 - 2a_2) \sin^2 t}. \quad (3.15)$$

Comparing the coefficients in (3.15) and (3.14) we get

$$\begin{cases} \frac{1}{2}(a_0 + 2a_2) = \frac{c}{ac - b^2}, \\ b_2 = \frac{-b}{ac - b^2}, \\ \frac{1}{2}(a_0 - 2a_2) = \frac{a}{ac - b^2}. \end{cases} \quad (3.16)$$

Hence we have immediately

$$\begin{cases} \frac{a_0}{2} = \frac{a + c}{2(ac - b^2)}, \\ a_2 = \frac{c - a}{2(ac - b^2)}, \\ b_2 = \frac{-b}{ac - b^2}. \end{cases} \quad (3.17)$$

Note that

$$ac - b^2 = \frac{4}{a_0^2 - 4a_2^2 - 4b_2^2}.$$

Thus support functions given by the formula (3.10), where $a_0 > 2\sqrt{a_2^2 + b_2^2}$ describe only ellipses. □

4. Curves whose orthoptics are circles

In this section we will consider a subfamily $\mathcal{M}(\frac{\pi}{2})$ of the family \mathcal{M} defined as follows

$$\mathcal{M}\left(\frac{\pi}{2}\right) = \{\mathcal{C} \in \mathcal{M} : \mathcal{C}_{\frac{\pi}{2}} \text{ is a circle}\}. \quad (4.1)$$

Let a curve $\mathcal{C} \in \mathcal{M}(\frac{\pi}{2})$ be given by (1.1). We denote by $s(t)$ the midpoint of the segment with ends at $z(t)$ and $z(t + \frac{\pi}{2})$. We present here a certain geometric characterization of the family $\mathcal{M}(\frac{\pi}{2})$.

Theorem 4.1. *A curve $\mathcal{C} \in \mathcal{M}$ belongs to $\mathcal{M}(\frac{\pi}{2})$ if and only if for each fixed t the points $s(t)$, $z_{\frac{\pi}{2}}(t)$ and the origin O lie on the same line.*

Proof. Note that

$$\begin{aligned}
 2 \det[z_{\frac{\pi}{2}}(t), s(t)] &= \det\left[z_{\frac{\pi}{2}}(t), z(t) + z\left(t + \frac{\pi}{2}\right)\right] \\
 &= \det\left[p(t)e^{it} + p\left(t + \frac{\pi}{2}\right)ie^{it}, p(t)e^{it} + p'(t)ie^{it}\right. \\
 &\quad \left.+ p\left(t + \frac{\pi}{2}\right)ie^{it} - p'\left(t + \frac{\pi}{2}\right)e^{it}\right] \\
 &= p(t)p'(t) + p\left(t + \frac{\pi}{2}\right)p'\left(t + \frac{\pi}{2}\right) \\
 &= \frac{1}{2}\left(p^2(t) + p^2\left(t + \frac{\pi}{2}\right)\right)' = \frac{1}{2}\left(|z_{\frac{\pi}{2}}(t)|^2\right)',
 \end{aligned}$$

i.e.

$$4 \det[z_{\frac{\pi}{2}}(t), s(t)] = \left(|z_{\frac{\pi}{2}}(t)|^2\right)'. \tag{4.2}$$

From (4.2) it follows that $s(t), z_{\frac{\pi}{2}}(t), O$ lie on the same line if and only if the orthoptic is a circle. \square

Let us consider a class \mathcal{F} of all positive valued Fourier series of the form

$$\frac{a_0}{2} + \sum_{k=0}^{\infty} [a_{2+4k} \cos(2 + 4k)t + b_{2+4k} \sin(2 + 4k)t]. \tag{4.3}$$

We recall that the support function p of $\mathcal{C} \in \mathcal{M}\left(\frac{\pi}{2}\right)$ satisfies the following equation

$$p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = r^2, \tag{4.4}$$

where r is the radius of the orthoptic, see [32].

We develop the function p^2 in the Fourier series

$$p^2(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} [a_n \cos nt + b_n \sin nt]. \tag{4.5}$$

Using the calculations from Step 2 we get

$$\begin{aligned}
 &p^2(t) + p^2\left(t + \frac{\pi}{2}\right) \\
 &= a_0 + \sum_{n=0}^{\infty} \left[\left(a_n \left(\cos \frac{n\pi}{2} + 1 \right) + b_n \sin \frac{n\pi}{2} \right) \cos nt \right. \\
 &\quad \left. + \left(-a_n \sin \frac{n\pi}{2} + b_n \left(\cos \frac{n\pi}{2} + 1 \right) \right) \sin nt \right].
 \end{aligned}$$

Hence we have $r^2 = a_0$ and $\frac{n\pi}{2} = \pi + 2k\pi$, that is $n = 2 + 4k$ for $k = 0, 1, 2, \dots$. Finally the Fourier series of p^2 belongs to \mathcal{F} .

In the paper [32] there was given an example of a support function of a curve $\mathcal{C} \in \mathcal{M}$ whose orthoptic is a circle, namely

$$p(t) = \sqrt{a \sin^2 3t + b \cos^2 9t + c}.$$

Note that here we have

$$\begin{aligned} p^2(t) &= a \sin^2 3t + b \cos^2 9t + c = \frac{a}{2} (1 - \cos 6t) + \frac{b}{2} (1 + \cos 18t) + c \\ &= \frac{1}{2}(a + b + 2c) + \frac{a}{2} \cos 6t + \frac{b}{2} \cos 18t, \end{aligned}$$

which is in line with (4.3)(Fig.4).

On the other hand with respect to (4.4) we may take $p(t) = r \cosh(t)$ and $p(t + \frac{\pi}{2}) = r \sinh(t)$. These formulas imply that the Fourier series of the function h belongs to \mathcal{F} , where $a_0 = \frac{\pi}{2}$. We note here that Green in [19] introduced a curve $\mathcal{C} \in \mathcal{M}$ with the support function $p(t) = \cos(\frac{\pi}{4} + k \sin 2t)$ where k is sufficiently small. We will develop this idea in a general setting in the next section.

From the above considerations it follows that all curves of the family $\mathcal{M}(\frac{\pi}{2})$ can be constructed using the Fourier series of the class \mathcal{F} . To this aim we formulate the following theorem.

Theorem 4.2. *Let $f \in \mathcal{F}$. Each function*

- (a) $p(t) = \sqrt{f(t)}$,
- (b) $p(t) = \cos f(t)$,

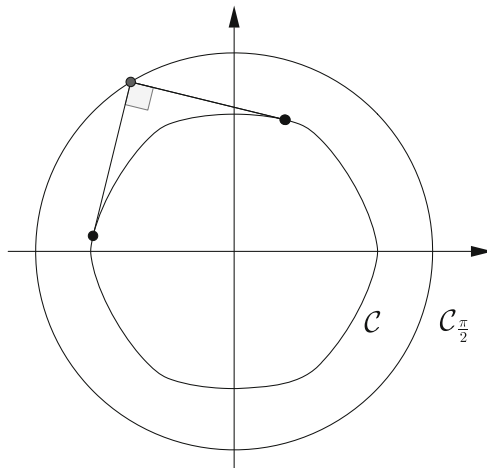


FIGURE 4. Circle of radius $\sqrt{45}$ is the orthoptic of the curve \mathcal{C} with $p(t) = \sqrt{\frac{45}{2} + \cos 6t}$

(c) $p(t) = \sin f(t)$,

such that $p(t) > 0$ and $p(t) + p''(t) > 0$ is a support function of some curve $\mathcal{C} \in \mathcal{M}(\frac{\pi}{2})$, and conversely.

5. Curves whose isoptics are circles

In this section we extend the results from the previous section to the general case. Our goal is to describe all curves $\mathcal{C} \in \mathcal{M}$ possessing a circle as one of its isoptics. Note that such curves were called curves of generalized constant width in the paper [37].

Now, we will consider a subfamily $\mathcal{M}(\alpha, r)$ of \mathcal{M} defined as follows

$$\mathcal{M}(\alpha, r) = \{\mathcal{C} \in \mathcal{M} : \mathcal{C}_\alpha \text{ is a circle of radius } r\}. \tag{5.1}$$

We fix a curve $\mathcal{C} \in \mathcal{M}(\alpha, r)$. From Theorem 3.1 of [37] we know that the Steiner centroid O of \mathcal{C} and the center of the circle coincide. Thus we assume that the origin of the coordinate system is chosen at O , so the center of this circle is $(0, 0)$. Taking formula (1.2) into account we see that there should be

$$\begin{cases} p(t) = r \sin h(t), \\ \frac{p(t + \alpha) - p(t) \cos \alpha}{\sin \alpha} = r \cos h(t), \end{cases} \tag{5.2}$$

for some non-constant 2π -periodic function h . Thus substituting the first formula into the second one we get

$$\sin h(t + \alpha) = \sin(h(t) + \alpha).$$

Thus either $h(t + \alpha) - h(t) = \alpha$ or $h(t + \alpha) + h(t) = \pi - \alpha$. The first case is impossible since the Fourier expansion of the left hand side has no constant term and this implies $\alpha = 0$. If we substitute the Fourier expansion of $h(t) =$

$\frac{a_0}{2} + \sum_{n=0}^{\infty} [a_n \cos nt + b_n \sin nt]$ into the second formula then we obtain

$$\begin{aligned} & a_0 + \sum_{n=0}^{\infty} [(a_n \cos n\alpha + b_n \sin n\alpha + a_n) \cos nt \\ & + (b_n \cos n\alpha - a_n \sin n\alpha + b_n) \sin nt] = \pi - \alpha. \end{aligned}$$

Then we have

$$a_0 = \pi - \alpha \tag{5.3}$$

and

$$\begin{cases} (\cos n\alpha + 1)a_n + \sin n\alpha \cdot b_n = 0, \\ -\sin n\alpha \cdot a_n + (\cos n\alpha + 1)b_n = 0, \end{cases} \tag{5.4}$$

where the determinant of this system of equations is equal to $2(1 + \cos n\alpha)$. Thus in order to have the non-zero solutions $\cos n\alpha$ should be equal to -1 , so

$\alpha = \frac{1+2k}{n}\pi$, for a natural k , such that $0 < \frac{1+2k}{n} < 1$. Thus the possible angles α are rational multiples of π , and $\alpha = \frac{l}{j}\pi$, where $l, j = 1, 2, \dots, l < j$ and l is odd. Then $\frac{l}{j} = \frac{1+2k}{n}$ and $n = j\frac{1+2k}{l}$ for $k = 0, 1, 2, \dots$, so

$$h(t) = \frac{\pi - \alpha}{2} + \sum_{k=0}^{\infty} (a_{j(1+2k)} \cos j(1+2k)t + b_{j(1+2k)} \sin j(1+2k)t).$$

Moreover, the coefficients of this series should be such that $p(t) = r \sin h(t) > 0$ and $p(t) + p''(t) > 0$. Summing up our considerations we have the following theorem.

Theorem 5.1. *Let $\alpha = \frac{l}{j}\pi$ be an angle, where l is odd, l and j are relatively prime, $l < j$ and $l, j = 1, 2, \dots$. Then each function*

$$p(t) = r \sin \left(\frac{\pi - \alpha}{2} + \sum_{k=0}^{\infty} (a_{j(1+2k)} \cos j(1+2k)t + b_{j(1+2k)} \sin j(1+2k)t) \right), \quad (5.5)$$

such that $p(t) > 0$ and $p(t) + p''(t) > 0$ is a support function of some curve $\mathcal{C} \in \mathcal{M}(\alpha, r)$, and conversely.

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