Continuous solutions to two iterative functional equations

KAROL BARON

Dedicated to Professor Ludwig Reich on his 80th birthday.

Abstract. Based on iteration of random-valued functions we study the problem of solvability in the class of continuous and Hölder continuous functions φ of the equations

$$\begin{aligned}\varphi(x) &= F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega), \\ \varphi(x) &= F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),\end{aligned}$$

where P is a probability measure on a σ -algebra of subsets of Ω .

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space (X, ρ) with the σ -algebra $\mathcal{B}$ of all its Borel subsets, and a $\mathcal{B} \otimes \mathcal{A}$ -measurable function $f : X \times \Omega \rightarrow X$.

We continue the research of continuous solutions $\varphi : X \rightarrow \mathbb{R}$ of the equations

$$\varphi(x) = F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega), \quad (1)$$

$$\varphi(x) = F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega). \quad (2)$$

We refer mainly to [2, 5]. Like in these papers we focus on the iteration of random-valued functions:

$$f^0(x, \omega_1, \omega_2, \dots) = x, \quad f^n(x, \omega_1, \omega_2, \dots) = f(f^{n-1}(x, \omega_1, \omega_2, \dots), \omega_n)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^\mathbb{N}$. Note that for $n \in \mathbb{N}$ the n th iterate f^n mapping $X \times \Omega^\infty$ into X is $\mathcal{B} \otimes \mathcal{A}_n$ -measurable, where $\mathcal{A}_n$ denotes the σ -algebra of all sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ -algebra $\mathcal{A}^n$. (See [13, section 1.4], [10].)

Let $\pi_n^f(x, \cdot)$ denote the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n^f(x, B) = P^\infty(f^n(x, \cdot) \in B) \quad \text{for } n \in \mathbb{N} \cup \{0\}, \quad x \in X \text{ and } B \in \mathcal{B}.$$

If

$$\int_\Omega \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \rho(x, z) \quad \text{for } x, z \in X \quad (3)$$

with a $\lambda \in (0, 1)$, and

$$\int_\Omega \rho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X, \quad (4)$$

then (see [2, Theorem 3.1]) there exists a probability Borel measure π^f on X such that for every $x \in X$ the sequence $(\pi_n^f(x, \cdot))_{n \in \mathbb{N}}$ converges weakly to π^f , and (see [11, Corollary 5.6 and Lemma 3.1], also [3, Lemma 2.2]) for every non-expansive $u : X \rightarrow \mathbb{R}$ the inequality

$$\left| \int_X u(z) \pi_n^f(x, dz) - \int_X u(z) \pi^f(dz) \right| \leq \frac{\lambda^n}{1 - \lambda} \int_\Omega \rho(f(x, \omega), x) P(d\omega) \quad (5)$$

holds for $x \in X$ and $n \in \mathbb{N}$.

This limit distribution π^f plays an important role in solving (1) and (2), see [5, Theorem 3.1], [2, Corollary 4.1], [3, Theorem 2.1]. In particular:

(I) If $F : X \rightarrow \mathbb{R}$ is continuous and bounded, then any continuous and bounded solution $\varphi : X \rightarrow \mathbb{R}$ of (1) has the form

$$\begin{aligned} \varphi(x) = & F(x) - \frac{1}{2} \int_X F(z) \pi^f(dz) \\ & + \sum_{n=1}^{\infty} (-1)^n \left(\int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right) \quad \text{for } x \in X; \end{aligned} \quad (6)$$

if additionally F is Lipschitz, then (6) defines a Lipschitz solution $\varphi : X \rightarrow \mathbb{R}$ of (1).

(II) If $F : X \rightarrow \mathbb{R}$ is continuous and bounded and (2) has a continuous and bounded solution $\varphi : X \rightarrow \mathbb{R}$, then

$$\int_X F(x) \pi^f(dx) = 0, \quad (7)$$

and any such solution has the form

$$\varphi(x) = c + F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X$$

with a real constant c .

(III) If $F : X \rightarrow \mathbb{R}$ is Lipschitz, then it is integrable for π^f and (2) has a Lipschitz solution $\varphi : X \rightarrow \mathbb{R}$ if and only if (7) holds.

The limit distribution π^f and facts cited above will be used in the main part of the paper. A characterization of this limit for some special random-valued functions in Hilbert spaces have been given by [3, Theorem 3.1] and, in Banach spaces, by [4, Theorem 2.1].

Actually we do not have a sufficiently satisfactory theorem to guarantee the existence of continuous solutions to the equations considered. An explanation of this situation is given in the paper [9] by Witold Jarczyk (see also [13, Note 3.8.4]). Namely, in the case where Ω is a singleton and X is a compact real interval, for the appropriate f the set of continuous $F : X \rightarrow \mathbb{R}$ such that the equation has a continuous solution is small in the sense of Baire category. It is also small from the measure point of view (see [1]). We will go also in this direction but, above all, we are looking for conditions under which Eqs. (1) and (2) have continuous and Hölder continuous solutions $\varphi : X \rightarrow \mathbb{R}$. In the case where Ω is a singleton, see [12, Chapter II, §7].

2. Results

We will consider Eqs. (1) and (2) assuming the following hypothesis (H).

(H) $(\Omega, \mathcal{A}, P)$ is a probability space, (X, ρ) is a complete and separable metric space, $f : X \times \Omega \rightarrow X$ is $\mathcal{B} \otimes \mathcal{A}$ -measurable, (3) holds with a $\lambda \in (0, 1)$ and (4) is satisfied.

We regard λ as fixed in $(0, 1)$, and for any metric space X we define $\mathcal{F}(X)$ as the set of all continuous functions $F : X \rightarrow \mathbb{R}$ such that there are a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ such that

$$|F(x) - F_n(x)| \leq \alpha \vartheta^n \quad \text{for } x \in X, \quad n \in \mathbb{N},$$

and

$$|F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X, \quad n \in \mathbb{N}.$$

Clearly any real Lipschitz function defined on X belongs to $\mathcal{F}(X)$.

Theorem 2.1. Assume (H). If $F \in \mathcal{F}(X)$, then formula (6) defines a continuous solution $\varphi : X \rightarrow \mathbb{R}$ of (1), and if additionally (7) holds, then the formula

$$\varphi_0(x) = F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X \quad (8)$$

defines a continuous solution $\varphi_0 : X \rightarrow \mathbb{R}$ of (2).

The proof will be based on three lemmas. In each of them we assume (H).

Lemma 2.2. *If $F \in \mathcal{F}(X)$, then the integrals*

$$\int_{\Omega} |F(f(x, \omega))| P(d\omega) \quad \text{for } x \in X, \quad \int_X |F(z)| \pi^f(dz)$$

are finite, and the function

$$x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X, \quad (9)$$

is continuous.

Proof. Corresponding to F choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$. Then

$$\int_{\Omega} |F(f(x, \omega))| P(d\omega) \leq \alpha \vartheta + \beta L \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) + |F_1(x)|$$

for $x \in X$, and

$$\int_X |F(z)| \pi^f(dz) \leq \alpha \vartheta + \int_X |F_1(z)| \pi^f(dz),$$

see also (III). Moreover, for every $n \in \mathbb{N}$ the function

$$x \mapsto \int_{\Omega} F_n(f(x, \omega)) P(d\omega), \quad x \in X,$$

is Lipschitz:

$$\left| \int_{\Omega} F_n(f(x, \omega)) P(d\omega) - \int_{\Omega} F_n(f(z, \omega)) P(d\omega) \right| \leq \beta L^n \lambda \rho(x, z)$$

for $x, z \in X$, and therefore function (9), as their uniform limit, is continuous. $\square$

Lemma 2.3. *If $F \in \mathcal{F}(X)$, then*

$$\int_X |F(z)| \pi_n^f(x, dz) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N},$$

and for every $n \in \mathbb{N}$ the function

$$x \mapsto \int_X F(z) \pi_n^f(x, dz), \quad x \in X,$$

is continuous.

Proof. By induction, (3) and (4),

$$\int_{\Omega^\infty} \rho(f^n(x, \omega), f^n(z, \omega)) P^\infty(d\omega) \leq \lambda^n \rho(x, z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N} \quad (10)$$

and

$$\int_{\Omega^\infty} \rho(f^n(x, \omega), x) P^\infty(d\omega) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$

Since

$$\int_X F(z) \pi_n^f(x, dz) = \int_{\Omega^\infty} F(f^n(x, \omega)) P^\infty(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}, \quad (11)$$

an application of Lemma 2.2 with f replaced by f^n , $n \in \mathbb{N}$, finishes the proof. $\square$

Lemma 2.4. *If $F \in \mathcal{F}(X)$, then there are constants $\theta \in (0, 1)$ and $M \in (0, \infty)$ such that*

$$\begin{aligned} & \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right| \\ & \leq M \theta^n (1 + \rho(x, x_0) + \int_\Omega \rho(f(x_0, \omega), x_0) P(d\omega)) \end{aligned}$$

for $x, x_0 \in X$ and $n \in \mathbb{N}$.

Proof. Corresponding to F choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$, and put

$$\theta = \max\{\vartheta, \lambda L\}, \quad M = 2 \max \left\{ \alpha, \frac{\beta}{1 - \lambda} \right\}.$$

Then $\theta \in (0, 1)$, and by Lemmas 2.3 and 2.2, (5) with $u = \frac{F_n}{\beta L^n}$ and (3) for every $x, x_0 \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
& \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right| \\
& \leq \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi_n^f(x, dz) \right| \\
& \quad + \left| \int_X F_n(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi^f(dz) \right| \\
& \quad + \left| \int_X F_n(z) \pi^f(dz) - \int_X F(z) \pi^f(dz) \right| \\
& \leq \int_X |F(z) - F_n(z)| \pi_n^f(x, dz) + \beta L^n \frac{\lambda^n}{1-\lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \\
& \quad + \int_X |F_n(z) - F(z)| \pi^f(dz) \leq 2\alpha \vartheta^n + \beta L^n \frac{\lambda^n}{1-\lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \\
& \leq 2\alpha \theta^n + \frac{\beta}{1-\lambda} \theta^n (\lambda \rho(x, x_0) + \int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega) + \rho(x, x_0)) \\
& \leq M \theta^n (1 + \rho(x, x_0) + \int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega)).
\end{aligned}$$

□

Proof of Theorem 2.1. It follows from Lemmas 2.2–2.4 that formula (6) defines a continuous function $\varphi : X \rightarrow \mathbb{R}$ and arguing like in the proof of Theorem 3.1(ii) of [5] (see also the calculations below) we show that it solves (1).

Assume now that also (7) holds. Then it follows from Lemmas 2.3 and 2.4 that formula (8) defines a continuous function $\varphi_0 : X \rightarrow \mathbb{R}$. Applying (11), Lemma 2.4, the Lebesgue dominated convergence theorem and the Fubini theorem we observe that for every $x \in X$ the function $\varphi_0 \circ f(x, \cdot)$ is integrable for P and

$$\begin{aligned}
& \int_{\Omega} \varphi_0(f(x, \omega)) P(d\omega) = \int_{\Omega} F(f(x, \omega)) P(d\omega) \\
& \quad + \int_{\Omega} \sum_{n=1}^{\infty} \left(\int_X F(z) \pi_n^f(f(x, \omega), dz) \right) P(d\omega) = \int_X F(z) \pi_1^f(x, dz) \\
& \quad + \sum_{n=1}^{\infty} \int_{\Omega} \left(\int_{\Omega^{\infty}} F(f^n(f(x, \omega_1), \omega_2, \omega_3, \dots)) P^{\infty}(d(\omega_2, \omega_3, \dots)) \right) P(d\omega_1) \\
& = \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} F(f^{n+1}(x, \omega_1, \omega_2, \dots)) P^{\infty}(d(\omega_1, \omega_2, \dots)) \\
& = \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_X F(z) \pi_{n+1}^f(x, dz) = \varphi_0(x) - F(x).
\end{aligned}$$

□

Proposition 2.5. *If F is a real function on a metric space X and*

$$|F(x) - F(z)| \leq \beta \rho(x, z)^\alpha \quad \text{for } x, z \in X \quad (12)$$

with some constants $\alpha \in (0, 1)$, $\beta \in [0, \infty)$, then $F \in \mathcal{F}(X)$.

Proof. Fix $L \in (1, \frac{1}{\lambda})$, put

$$\vartheta = L^{-\frac{\alpha}{1-\alpha}}, \quad \theta = \vartheta^{\frac{1}{\alpha}},$$

and for every $n \in \mathbb{N}$ let A_n be a maximal for inclusion subset of X such that

$$\rho(x, z) \geq \theta^n \quad \text{for every pair of distinct points } x, z \text{ of } A_n.$$

By the maximality,

$$X = \bigcup_{z \in A_n} \{x \in X : \rho(x, z) < \theta^n\} \quad \text{for } n \in \mathbb{N}.$$

If $n \in \mathbb{N}$ and x, z are distinct points of A_n , then by (12),

$$|F(x) - F(z)| \leq \beta \rho(x, z)^{\alpha-1} \rho(x, z) \leq \beta \theta^{(\alpha-1)n} \rho(x, z) = \beta L^n \rho(x, z).$$

It follows from this, using Kirszbraun–McShane extension theorem [7, Theorem 6.1.1], that for every $n \in \mathbb{N}$ there exists an $F_n : X \rightarrow \mathbb{R}$ such that

$$F_n|_{A_n} = F|_{A_n} \quad \text{and} \quad |F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X.$$

If $n \in \mathbb{N}$ and $x \in X$, then there is a $z \in A_n$ such that $\rho(x, z) < \theta^n$, and

$$\begin{aligned} |F(x) - F_n(x)| &\leq |F(x) - F(z)| + |F_n(z) - F_n(x)| \\ &\leq \beta \rho(x, z)^\alpha + \beta L^n \rho(x, z) \\ &\leq \beta \theta^{\alpha n} + \beta L^n \theta^n = 2\beta \vartheta^n. \end{aligned}$$

□

Corollary 2.6. *Assume (H). If $F : X \rightarrow \mathbb{R}$ satisfies (12) with some constants $\alpha \in (0, 1)$, $\beta \in [0, \infty)$, then formula (6) defines a solution $\varphi : X \rightarrow \mathbb{R}$ of (1) such that*

$$|\varphi(x) - \varphi(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^\alpha \quad \text{for } x, z \in X,$$

and if additionally (7) holds, then formula (8) defines a solution $\varphi_0 : X \rightarrow \mathbb{R}$ of (2) such that

$$|\varphi_0(x) - \varphi_0(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^\alpha \quad \text{for } x, z \in X.$$

Proof. By Proposition 2.5 and Theorem 2.1 formula (6) defines a solution $\varphi : X \rightarrow \mathbb{R}$ of (1). Using (6), (11), (12), Jensen's inequality and (10) for every $x, z \in X$ we have

$$\begin{aligned} |\varphi(x) - \varphi(z)| &\leq |F(x) - F(z)| \\ &\quad + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} |F(f^n(x, \omega)) - F(f^n(z, \omega))| P^{\infty}(d\omega) \\ &\leq \beta \rho(x, z)^{\alpha} + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} \beta \rho(f^n(x, \omega), f^n(z, \omega))^{\alpha} P^{\infty}(d\omega) \\ &\leq \beta \rho(x, z)^{\alpha} + \beta \sum_{n=1}^{\infty} \left(\int_{\Omega^{\infty}} \rho(f^n(x, \omega), f^n(z, \omega)) P^{\infty}(d\omega) \right)^{\alpha} \\ &\leq \beta \rho(x, z)^{\alpha} + \beta \sum_{n=1}^{\infty} (\lambda^n \rho(x, z))^{\alpha} = \frac{\beta}{1 - \lambda^{\alpha}} \rho(x, z)^{\alpha}. \end{aligned}$$

For the second part we argue similarly. $\square$

Regarding the uniqueness of solutions, we have the following theorem.

Theorem 2.7. Assume (H) and let $F : X \rightarrow \mathbb{R}$.

(i) If $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ are solutions of (1), then $\varphi_1 = \varphi_2$.

(ii) If $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ are solutions of (2), then $\varphi_1 - \varphi_2$ is a constant function.

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ and put $\varphi = \varphi_1 - \varphi_2$. Then $\varphi \in \mathcal{F}(X)$. Corresponding to φ choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$.

If φ_1, φ_2 are solutions of (1), then φ solves (1) with $F = 0$, and, by induction,

$$\varphi(x) = (-1)^n \int_{\Omega^{\infty}} \varphi(f^n(x, \omega)) P^{\infty}(d\omega) \quad \text{for } x \in X, \quad n \in \mathbb{N}.$$

If φ_1, φ_2 are solutions of (2), then φ solves (2) with $F = 0$, and

$$\varphi(x) = \int_{\Omega^{\infty}} \varphi(f^n(x, \omega)) P^{\infty}(d\omega) \quad \text{for } x \in X, \quad n \in \mathbb{N}.$$

In both cases

$$|\varphi(x) - \varphi(z)| \leq \int_{\Omega^{\infty}} |\varphi(f^n(x, \omega)) - \varphi(f^n(z, \omega))| P^{\infty}(d\omega)$$

for $x, z \in X$, $n \in \mathbb{N}$. Moreover,

$$\begin{aligned} |\varphi(x) - \varphi(z)| &\leq |\varphi(x) - F_n(x)| + |F_n(x) - F_n(z)| + |F_n(z) - \varphi(z)| \\ &\leq 2\alpha \vartheta^n + |F_n(x) - F_n(z)| \quad \text{for } x, z \in X, \quad n \in \mathbb{N}. \end{aligned}$$

Consequently, applying among others (10),

$$\begin{aligned} |\varphi(x) - \varphi(z)| &\leq 2\alpha\vartheta^n + \int_{\Omega^\infty} |F_n(f^n(x, \omega)) - F_n(f^n(z, \omega))| P^\infty(d\omega) \\ &\leq \beta L^n \lambda^n \rho(x, z) \quad \text{for } x, z \in X, \quad n \in \mathbb{N}, \end{aligned}$$

whence $\varphi(x) = \varphi(z)$ for $x, z \in X$, i.e., φ is a constant function. Noting that if a constant φ solves (1) with $F = 0$, then $\varphi = 0$, we end the proof. $\square$

We finish with a qualitative result.

Following [6] by Jens Peter Reus Christensen we say that a Borel subset B of an abelian Polish group G is a *Haar zero set* if there is a probability Borel measure μ on G such that $\mu(B + x) = 0$ for every $x \in G$. See also [8] where measurability in abelian Polish groups related to Christensen's Haar zero set is studied.

Assume

(H₀) $(\Omega, \mathcal{A}, P)$ is a probability space, (X, ρ) is a compact metric space, $f : X \times \Omega \rightarrow X$ is $\mathcal{B} \otimes \mathcal{A}$ -measurable and (3) holds with a $\lambda \in (0, 1)$.

Assuming (H₀) we have in particular (4):

$$\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \leq \text{diam}(X) \quad \text{for } x \in X.$$

Moreover one can consider the Banach space $C(X)$ of all continuous real functions on X with the uniform norm and its subspace C_f ,

$$C_f = \left\{ F \in C(X) : \int_X F(x) \pi^f(dx) = 0 \right\}.$$

Clearly C_f is a closed linear subspace of $C(X)$ and (see, e.g., [7, Corollary 11.2.5]) $C(X)$ is a separable Banach space. We have also the following lemma.

Lemma 2.8. *Assume (H₀). If $F : X \rightarrow \mathbb{R}$ is continuous, then so is the function*

$$x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X.$$

Proof. Fix $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that

$$|F(x) - F(z)| \leq \varepsilon \quad \text{for } x, z \in X \text{ with } \rho(x, z) \leq \delta.$$

Then, by (3), for all $x, z \in X$,

$$\begin{aligned}
 & \left| \int_{\Omega} F(f(x, \omega)) P(d\omega) - \int_{\Omega} F(f(z, \omega)) P(d\omega) \right| \\
 & \leq \int_{\Omega} |F(f(x, \omega)) - F(f(z, \omega))| P(d\omega) \\
 & \leq \varepsilon + \int_{\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}} |F(f(x, \omega)) - F(f(z, \omega))| P(d\omega) \\
 & \leq \varepsilon + 2\|F\| P(\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}) \\
 & \leq \varepsilon + 2\|F\| \frac{1}{\delta} \int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \varepsilon + \frac{2\lambda\|F\|}{\delta} \rho(x, z),
 \end{aligned}$$

and therefore the discussed function is continuous. $\square$

Let

$\mathcal{F}_1 = \{F \in C(X) : \text{equation (1) has a continuous solution } \varphi : X \rightarrow \mathbb{R}\},$

$\mathcal{F}_2 = \{F \in C_f : \text{equation (2) has a continuous solution } \varphi : X \rightarrow \mathbb{R}\}.$

Theorem 2.9. *Under the assumptions (H_0) :*

(i) $\mathcal{F}_1$ is a Borel and dense subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$.

(ii) $\mathcal{F}_2$ is a Borel and dense subset of C_f , and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in C_f and a Haar zero subset of C_f .

Proof. By Lemma 2.8 the formulas

$$T_1(\varphi)(x) = \varphi(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),$$

$$T_2(\varphi)(x) = \varphi(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) \quad \text{for } \varphi \in C(X) \text{ and } x \in X,$$

define self-mappings T_1, T_2 of $C(X)$. Clearly, these operators are linear and continuous. Moreover,

$$T_1(C(X)) = \mathcal{F}_1.$$

Furthermore, for every $F \in T_2(C(X))$ Eq. (2) has a continuous solution $\varphi : X \rightarrow \mathbb{R}$. Hence (II) gives $T_2(C(X)) \subset C_f$, and

$$T_2(C(X)) = \mathcal{F}_2.$$

Applying now [1, Lemma] we see that $\mathcal{F}_1$ is a Borel subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$, and $\mathcal{F}_2$ is a Borel subset of C_f , and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in C_f and a Haar zero subset of C_f .

Since by (I) the set

$$\{F \in C(X) : F \text{ is Lipschitz}\}$$

is contained in $\mathcal{F}_1$ and (see [7, Theorem 11.2.4]) dense in $C(X)$, the set $\mathcal{F}_1$ is dense in $C(X)$.

To show that $\mathcal{F}_2$ is dense in C_f , fix $F \in C_f$ and $\varepsilon \in (0, \infty)$. Choose a Lipschitz $F_1 : X \rightarrow \mathbb{R}$ so that $\|F - F_1\| < \frac{\varepsilon}{2}$. According to (III), $F_1 - \int_X F_1 d\pi^f \in \mathcal{F}_2$. Moreover,

$$\left\| F - (F_1 - \int_X F_1 d\pi^f) \right\| \leq \|F - F_1\| + \left| \int_X F_1 d\pi^f - \int_X F d\pi^f \right| < \varepsilon.$$

□

Remark 2.10. It is possible that (H_0) holds and $\mathcal{F}_1 = C(X)$, $\mathcal{F}_2 = C_f$.

To see it consider an $\mathcal{A}$ -measurable $\xi : \Omega \rightarrow X$ and let $f(x, \omega) = \xi(\omega)$ for $(x, \omega) \in X \times \Omega$. Then $f^n(x, \omega) = \xi(\omega_n)$ for $(x, \omega) \in X \times \Omega^\infty$, so $\pi_n^f(x, B) = P(\xi \in B) = \pi^f(B)$ for $n \in \mathbb{N}$, $x \in X$, $B \in \mathcal{B}$, and $\int_X F d\pi^f = \int_\Omega F \circ \xi dP$ for $F \in C(X)$. Consequently for every $F \in C(X)$ the function $F - \frac{1}{2} \int_X F d\pi^f$ solves (1), and every $F \in C_f$ solves (2).

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References

- [1] Baron, K.: A remark on linear functional equations in the indeterminate case. *Glasnik Mat.* **20**(40), 373–376 (1985)
- [2] Baron, K.: On the convergence in law of iterates of random-valued functions. *Aust. J. Math. Anal. Appl.* **6**, no. 1, Art. 3 (2009)
- [3] Baron, K.: Weak limit of iterates of some random-valued functions and its application. *Aequ. Math.* **94**, 415–425; 427 (Correction) (2020)
- [4] Baron, K.: Around the weak limit of iterates of some random-valued functions. *Ann. Univ. Budapest. Sect. Comput.* **51**, 31–37 (2020)

- [5] Baron, K., Kapica, R., Morawiec, J.: On Lipschitzian solutions to an inhomogeneous linear iterative equation. *Aequ. Math.* **90**, 77–85 (2016)
- [6] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. *Isr. J. Math.* **13**, 255–260 (1972)
- [7] Dudley, R.M.: *Real Analysis and Probability*. Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge University Press, Cambridge (2002)
- [8] Fischer, P., Ślódkowski, Z.: Christensen zero sets and measurable convex functions. *Proc. Amer. Math. Soc.* **79**, 449–453 (1980)
- [9] Jarczyk, W.: On a set of functional equations having continuous solutions. *Glasnik Mat.* **17**(37), 59–64 (1982)
- [10] Kapica, R.: Sequences of iterates of random-valued vector functions and solutions of related equations. *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **213**, 113–118 (2005)
- [11] Kapica, R.: The geometric rate of convergence of random iteration in the Hutchinson distance. *Aequ. Math.* **93**, 149–160 (2019)
- [12] Kuczma, M.: *Functional equations in a single variable*. Monografie Matematyczne, vol. 48. PWN–Polish Scientific Publishers, Warszawa (1968)
- [13] Kuczma, M., Choczewski, B., Ger, R.: *Iterative Functional Equations*. Encyclopedia of Mathematics and Its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

Karol Baron
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

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