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Aequationes Mathematicae

On a linearization of the recursion $U(x_0, x_1, x_2, \ldots) = \varphi(x_0, U(x_1, x_2, \ldots))$ and its application in economics

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Dedicated to Professor János Aczél on his 95th birthday.

Abstract. Let I be an interval, X be a metric space and $\succeq$ be an order relation on the infinite product X^∞. Let $U : X^\infty \to \mathbb{R}$ be a continuous mapping, representing $\succeq$, that is such that $(x_0, x_1, x_2, \ldots) \succeq (y_0, y_1, y_2, \ldots) \Leftrightarrow U(x_0, x_1, x_2, \ldots) \geq U(y_0, y_1, y_2, \ldots)$. We interpret X as a space of consumption outcomes and the relation $\succeq$ represents how an individual would rank all consumption sequences. One assumes that U, called the utility function, satisfies the recursion $U(x_0, x_1, x_2, \ldots) = \varphi(x_0, U(x_1, x_2, \ldots))$, where $\varphi : X \times I \to I$ is a continuous function strictly increasing in its second variable such that each function $\varphi(x, \cdot)$ has a unique fixed point. We consider an open problem in economics, when the relation $\succeq$ can be represented by another continuous function V satisfying the affine recursion $V(x_0, x_1, x_2, \ldots) = \alpha(x_0)V(x_1, x_2, \ldots) + \beta(x_0)$. We prove that this property holds if and only if there exists a homeomorphic solution of the system of simultaneous affine functional equations $F(\varphi(x, t)) = \alpha(x)F(t) + \beta(x), x \in X, t \in I$ for some functions $\alpha, \beta : X \to \mathbb{R}$. We give necessary and sufficient conditions for the existence of homeomorhic solutions of this system.

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1. Introduction

Let X be a topological space. Let X^∞ be the infinite Cartesian product endowed with product topology. Let $\succeq$ be a transitive and connex binary relation on X^∞.

The economic interpretation of these objects is as follows. The space X is treated as a set of consumption outcomes, $(x_0, x_1, x_2, \ldots) \in X^\infty$ as a sequence of outcomes consumed over time, where the element x_n represents the outcome consumed in period n.

The order relation "$\succeq$" describes how an individual would rank all consumption sequences.

Economists working with binary relations usually assume that they can be represented by real functions.

We will say that the continuous function $U : X^\infty \to \mathbb{R}$ represents "$\succeq$" if for all $(x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots) \in X^\infty$

$$(x_0, x_1, x_2, \ldots) \succeq (y_0, y_1, y_2, \ldots) \Leftrightarrow U(x_0, x_1, x_2, \ldots) \geq U(y_0, y_1, y_2, \ldots).$$

A function U satisfying the above condition is said to be *a utility function*. In the problem under consideration the economists assume that the utility function U satisfies the recursion

$$U(x_0, x_1, x_2, \ldots) = \varphi(x_0, U(x_1, x_2, \ldots)), \tag{1}$$

where $\varphi : X \times \mathrm{I} \to \mathrm{I}$ is a continuous function strictly increasing in its second variable and I is an interval.

Assume that X^∞ is a connected topological space and U satisfies (1). Then the set $U(X^\infty)$ is a subinterval of I as the image of a continuous function of a connected space and $\varphi(x, U(X^\infty)) \subset U(X^\infty))$ for every $x \in X$. Thus for a given U we may restrict the domain of φ to the set $X \times U(X^\infty)$ and further assume that $U(X^\infty) = I$.

The recursion (1) was introduced by Koopmans et al. in paper [3], which is considered as a classical one in economics. In this paper it is explained, why it is natural to impose this recursive structure on U (see also [1]).

The key role in this theory is played by the property of "impatience" on the part of the individual defined as follows.

Impatience For all $n \geq 1$, $a, b \in X^n$ and all $(x_0, x_1, \ldots) \in X^\infty$

$$(a, a, a, \ldots) \succeq (b, b, b, \ldots) \Leftrightarrow (a, b, x_0, x_1, \ldots) \succeq (b, a, x_0, x_1, \ldots).$$

In simple terms this means that, if the repeated consumption $a \in X^n$ is preferred over the repeated consumption $b \in X^n$, so that a is "better" then b , than the individual would sooner consume a than b.

Koopmans started with the conjecture that any binary relation that admits a utility function satisfying (1) would satisfy impatience. It turned out that this supposition is false. Koopmans left the problem how to represent the relation of preference that do satisfy impatience. Next Asen Kochov (in personal correspondence) posed a conjecture that impatience holds if and only if the recursion representing $\succeq$ can be chosen to be affine admitting non-constant coefficients.

In this paper we show that this conjecture without additional assumptions is not true. We prove even something more, we give necessary and sufficient conditions when the above conjecture holds.

2. Preliminary remarks

We make the following general assumption:

(A) X is a Hausdorff topological space satisfying the first axiom of countability and the product topology in X^∞ is such that it is connected and the convergence of the sequences is equivalent to the convergence with respect to coordinates, that is $\lim_{k\to\infty}(x_{0,k}, x_{1,k}, \ldots) = (x_0, x_1, \ldots)$ if and only if $\lim_{k\to\infty} x_{n,k} = x_n$ for every $n \in \mathbb{N}$.

Note that if X is connected then X^∞ is connect, moreover condition (A) holds for metric spaces.

In fact, let (X, d) be a metric space. Define on X^∞ the metric

$$\varrho(\{x_n\}, \{y_n\}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

Note that (X^∞, ϱ) is a metric space satisfying condition (A).

Let us introduce the notation

$$f_a := \varphi(a, \cdot), \quad a \in X.$$

Obviously $f_a : I \to I$. It is convenient to consider X as the set of parameters. Let U satisfy (1). Put

$$U_a := U(a, a, \ldots),\ a \in X.$$

By (1) we have $U_a = U(a, a, \ldots) = \varphi(a, U(a, a, \ldots)) = f_a(U(a, a, \ldots)) = f_a(U_a)$. Thus U_a is a fixed point of f_a.

Remark 1. Every function f_a has a unique fixed point.

Proof. Suppose that $f_a(p) = p$ for a $p \in I$. By the surjectivity of U there exists a sequence $(a_1, a_2, \ldots) \in X^\infty$ such that $U(a_1, a_2, \ldots) = p$. Since $f_a(p) = p$ we have by (1)

$$p = U(a_1, a_2, \ldots) = f_a(U(a_1, a_2, \ldots)) = U(a, a_1, a_2, \ldots).$$

Hence $f_a^2(U(a_1, a_2, \ldots)) = f_a(U(a, a_1, a_2, \ldots)) = U(a, a, a_1, a_2, \ldots)$. Further, by induction, we get $p = f_a^n(p) = U(a, a, \ldots, a, a_1, a_2, \ldots)$ (a is repeated n-times).

Since $(a, a, \ldots, a, a_1, a_2, \ldots) \to (a, a, a, \ldots)$, the continuity of U implies that

$$U(a, a, \ldots, a, a_1, a_2, \ldots) \to U(a, a, \ldots),$$

so $p = U_a$. $\square$

Remark 2. If a function $V : X^\infty \to \mathbb{R}$ satisfies (1) and $V_a := V(a, a, \ldots)$ then $V_a = U_a$.

This is a simple consequence of the fact that $f_a(U_a) = U_a$, $f_a(V_a) = V_a$ and f_a has a unique fixed point.

Remark 3. If $\inf I < \inf f_a$ and $\sup f_a < \sup I$ then $U_a \in \mathrm{Int} I$ and

(H) $f_a(t) < t$ for $t > U_a$ and $f_a(t) > t$ for $t < U_a$.

Moreover, (H) holds if and only if for all sequences $(x_1, x_2, \dots), (y_1, y_2, \dots) \in X^\infty$ such that $(x_1, x_2, \dots) \succeq (a, a, \dots) \succeq (y_1, y_2, \dots)$ we have $(x_1, x_2, \dots) \succeq (a, x_1, x_2, \dots)$ and $(a, y_1, y_2, \dots) \succeq (y_1, y_2, \dots)$.

Proof. Since f_a has a unique fixed point, the inequalities $\inf I < \inf f_a$, $\sup f_a < \sup I$, imply that $U_a \in \mathrm{Int} I$. Moreover, $\inf I < \inf f_a$ implies that $f_a(t) > t$ for $t < U_a$ and $\sup f_a < \sup I$ implies that $f_a(t) < t$ for $t > U_a$.

Let now $t \in I$ and $t > U_a$. Then there exists a sequence $(x_1, x_2, \dots)$ such that $U(x_1, x_2, \dots) = t$. Since $U(x_1, x_2, \dots) > U(a, a, \dots)$ we have $(x_1, x_2, \dots) \succeq (a, a, \dots)$. Thus by the assumption $(x_1, x_2, \dots) \succeq (a, x_1, x_2, \dots)$, so

$$t = U(x_1, x_2, \dots) \geq U(a, x_1, x_2, \dots) = f_a(U(x_1, x_2, \dots)) = f_a(t).$$

Since f_a has a unique fixed point and $t \neq U_a$ we get $f_a(t) < t$. Similarly we get the second inequality in (H).

Now, let (H) hold and $(x_1, x_2, \dots) \succeq (a, a, \dots)$. Put $t = U(x_1, x_2, \dots)$. We have $t \geq U_a$. Then $f_a(t) \leq t$, so $U(a, x_1, x_2, \dots) = f_a(U(x_1, x_2, \dots)) = f_a(t) \leq t = U(x_1, x_2, \dots)$. Thus $(x_1, x_2, \dots) \succeq (a, x_1, x_2, \dots)$. Similarly we get the second inequality. □

If I is a compact interval and $\inf f_a = \inf I$ then $f_a(t) < t$ for $t \neq U_a$, however if $\sup f_a < \sup I$ then $f_a(t) > t$ for $t \neq U_a$. Thus, if I is a compact interval then (H) holds.

Further we assume that every mapping f_a satisfies (H) .

3. Results

Let us introduce the notation

$$f_{(x_0, x_1, \dots, x_k)} := f_{x_0} \circ f_{x_1} \circ \dots f_{x_k}, \quad x_i \in X.$$

By (1) we get that for every $k \in \mathbb{N} \setminus 0$ and all $a \in X^k$ and $x \in X^\infty$

$$U(a, x) = f_a(U(x)).$$

Hence, similarly as in Remark 1, it follows that every f_a for $a \in X^k$ has a unique fixed point U_a.

Note that $G := \{f_a : a \in \bigcup_{k \geq 1} X^k\}$ is a semigroup of strictly increasing continuous functions possessing a unique fixed point (in this notation $a = (a_0, a_1, \dots, a_k)$).

Theorem 1. *The relation $\succeq$ satisfies impatience if and only if*

$$\forall_{k \geq 1} \forall_{a, b \in X^k} \; U_a \geq U_b \Leftrightarrow f_a \circ f_b \geq f_b \circ f_a.$$

Proof. This theorem is a consequence of the surjectivity of U and the following equivalences and equalities

$$(a,a,a,\dots) \succeq (b,b,b,\dots) \Leftrightarrow U(a,a,a,\dots) \geq U(b,b,b,\dots) \Leftrightarrow U_a \geq U_b,$$
$$(a,b,x_0,x_1,\dots) \succeq (b,a,x_0,x_1,\dots) \Leftrightarrow U(a,b,x_0,x_1,\dots) \geq U(b,a,x_0,x_1,\dots),$$
$$U(a,b,x_0,x_1,\dots) = f_a \circ f_b(U(x_0,x_1,\dots)),$$
$$U(b,a,x_0,x_1,\dots) = f_b \circ f_a(U(x_0,x_1,\dots)),$$
$$U(a,b,x_0,x_1..) \geq U(b,a,x_0,x_1,..) \Leftrightarrow f_a \circ f_b(U(x_0,x_1,..)) \geq f_b \circ f_a(U(x_0,x_1,..)).$$

□

Corollary 1. *If the relation $\succeq$ satisfies impatience then f_a and f_b have a joint fixed point if and only if $f_a \circ f_b = f_b \circ f_a$.*

Let us consider the particular case, where $f_a(s) = \alpha(a)s + \beta(a)$, where $\alpha : X \to (0,1)$ and $\beta : X \to \mathbb{R}$ are continuous functions. Then f_a satisfies (H) and (1) has the form

$$V(x_0,x_1,x_2,\dots) = \alpha(x_0)V(x_1,x_2,\dots) + \beta(x_0). \tag{2}$$

Theorem 2. *If V fulfills (2) then the order relation represented by V satisfies impatience.*

Proof. We have $f_x(s) = \alpha(x)s + \beta(x)$, $x \in X$ and $0 < \alpha(x) < 1$. The composition of affine functions is an affine function. Thus we may extend the domain of the functions α and β on $\bigcup_{k\geq 1} X^k$ as follows

$$f_{(x_1,x_2,\dots x_k)} = f_{x_1} \circ f_{x_2} \circ \dots f_{x_k} = \alpha(x_1,x_2,\dots,x_k)s + \beta(x_1,x_2,\dots,x_k).$$

Hence $f_x(s) = \alpha(x)s + \beta(x)$, $x \in \bigcup_{k\geq 1} X^k$, so for all $a,b \in \bigcup_{k\geq 1} X^k$

$$\begin{cases} f_a \circ f_b(s) = \alpha(a)\alpha(b)s + \alpha(a)\beta(b) + \beta(a) \\ f_b \circ f_a(s) = \alpha(a)\alpha(b)s + \alpha(b)\beta(a) + \beta(b). \end{cases} \tag{3}$$

Putting $V_a = V(a,a,\dots)$ and $V_b = V(b,b,\dots)$ we have $f_a(V_a) = V_a$ and $f_b(V_b) = V_b$. Hence $V_a = \frac{\beta(a)}{1-\alpha(a)}$ and $V_b = \frac{\beta(b)}{1-\alpha(b)}$ and $V_a \geq V_b$ if and only if $\alpha(a)\beta(b) + \beta(a) \geq \alpha(b)\beta(a) + \beta(b)$. Thus, by (3), the inequalities $V_a \geq V_b$ and $f_a \circ f_b \geq f_a \circ f_b$ are equivalent, so Theorem 1 implies that the relation represented by V satisfies impatience. □

Lemma 1. *Let U and V be the utility functions. They represent the same order relation if and only if there exists an increasing homeomorphism $\Phi : \mathrm{I} \to \mathrm{J}$ such that $U = \Phi \circ V$, where $I = U[X^\infty]$ and $J = V[X^\infty]$.*

Proof. For all $x,y \in X^\infty$ $x \succeq y \Leftrightarrow U(x) \geq U(y)$ and $x \succeq y \Leftrightarrow V(x) \geq V(y)$, so $U(x) \geq U(y) \Leftrightarrow V(x) \geq V(y)$ and consequently, changing the role of x and y, we have $U(x) = U(y) \Leftrightarrow V(x) = V(y)$. Now, we may define $\Phi : I \to J$ as follows

$$\Phi(U(x)) := V(x), \quad x \in X^\infty.$$

This definition is correct since it does not depend on the choice of x. Obviously Φ is a non-decreasing surjection. Changing the role of U and V we infer that there exists a non-decreasing surjection $\Psi : J \to I$ such that $\Psi \circ V(x) = U(x)$. Hence $\Phi \circ \Psi = id$, so Φ is injective and consequently, as an increasing bijection, is a homeomorphism. $\square$

Lemma 2. *Let $0 < \alpha(x) < 1$ for $x \in X$. Then recursion (2) has at most one continuous solution.*

Proof. Let $V_1, V_2 : X^\infty \to \mathbb{R}$ be continuous solutions of (2). Put $W := V_1 - V_2$. Then for every $x_i \in X$ $i = 0, 1, 2, \dots$

$$W(x_0, x_1, x_2, \dots) = \alpha(x_0) W(x_1, x_2, x_3, \dots).$$

By induction we get that for every $k \geq 1$

$$W(x_0, \dots, x_{k-1}, \dots) = \prod_{i=0}^{k-1} \alpha(x_i) W(x_k, x_{k+1}, x_{k+2}, \dots).$$

Suppose that $x_n = x_{n+k}$ for $n \geq 0$. Then for this periodic sequence the last equality has the form

$$W((x_0, \dots, x_{k-1}), (x_k, \dots x_{2k-1}) \dots) = \prod_{i=0}^{k-1} \alpha(x_i) W(x_0, x_1, x_2, \dots).$$

Hence, by induction, we get that for every $n \geq 1$

$$W(x_0, x_1, \dots, x_{k-1}, \dots) = \left[\prod_{i=0}^{k-1} \alpha(x_i)\right]^n W(x_0, x_1, x_2, \dots).$$

Since $0 < \alpha(x_i) < 1$ we have $\lim_{n\to\infty} \left[\prod_{i=0}^{k-1} \alpha(x_i)\right]^n = 0$ and, consequently, $W(x_0, x_1, x_2, \dots) = 0$ for every periodic sequence $\{x_n\}$.

Let $\mathcal{X} := (x_0, x_1, \dots) \in X^\infty$ be a given sequence. Define the following k-periodic sequences

$$\mathcal{X}_k := ((x_0, \dots x_{k-1}), (x_0, \dots x_{k-1}), \dots), \quad k \geq 1.$$

Note that assumption (A) implies that $\lim_{k\to\infty} \mathcal{X}_k = \mathcal{X}$, since the sequence of n-coordinates of the sequence $\{\mathcal{X}_k\}$ is constant up to the index n and is equal to x_n. Since $W(\mathcal{X}) = \lim_{k\to\infty} W(\mathcal{X}_k)$ and $W(\mathcal{X}_k) = 0$ for $k \geq 1$ we get $W(\mathcal{X}) = 0$, which gives that $V_1 = V_2$. $\square$

Let $\alpha : X \to (0, 1)$ and $\beta : X \to \mathbb{R}$. It is easy to verify that, if the series defining the function

$$S(x_0, x_1, x_2, \dots) := \beta(x_0) + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \alpha(x_i) \beta(x_{k+1}) \tag{4}$$

converges for every sequence $(x_0, x_1, x_2, \ldots) \in X^\infty$, then S satisfies recursion (2).

The function S given by formula (4) for continuous α and β is known, in economic literature, as the Uzawa-Epstein utility function (see [1]). It follows, by Lemma 2, that if recursion (2) has a continuous solution, then it is given by formula (4).

If we assume that function α is constant we give the following

Corollary 2. *If $\alpha \in (0,1)$ and $\beta : X \to \mathbb{R}$ is continuous and bounded then the function*

$$S(x_0, x_1, x_2, \ldots) = \beta(x_0) + \sum_{k=0}^{\infty} \alpha^k \beta(x_k)$$

is a unique continuous solution of the recursion

$$S(x_0, x_1, x_2, \ldots) = \alpha S(x_1, x_2, \ldots) + \beta(x_0).$$

Theorem 3. *Let the utility functions U and V represent the same order relation. If U satisfies (1) and V satisfies*

$$V(x_0, x_1, \ldots) = \psi(x_0, V(x_1, x_2, \ldots)), \tag{5}$$

then there exists an increasing homeomorphism $\Phi : I \to J$ such that

$$\Phi \circ \varphi(x, t) = \psi(x, \Phi(t)), \ t \in I. \tag{6}$$

Conversely, assume additionally that all continuous solutions of (5) represent the same order relation. If there exists an increasing homeomorphism Φ satisfying (6), then U and V represent the same order. Moreover, $V = \Phi^{-1} \circ U$.

Proof. Let U and V represent the same order relation. Then, by Lemma 1, there exists an increasing homeomorphism Φ such that $V = \Phi \circ U$. Hence

$$V(x_0, x_1, \ldots) = \Phi \circ U(x_0, x_1, \ldots) = \Phi \circ \varphi(x_0, U(x_1, x_2, \ldots)),$$

so, by (5),

$$V(x_0, x_1, \ldots) = \psi(x_0, V(x_1, x_2, \ldots)) = \psi(x_0, \Phi \circ U(x_1, x_2, \ldots)),$$

hence

$$\Phi \circ \varphi(x_0, U(x_1, x_2, \ldots)) = \psi(x_0, \Phi \circ U(x_1, x_2, \ldots)).$$

By the surjectivity of $U : X^\infty \to I$ we get (6).

Conversely, let (6) hold. Then putting in (6) $x = x_0$ and $t = U(x_1, x_2 \ldots)$ we have

$$\Phi \circ \varphi(x_0, U(x_1, x_2 \ldots)) = \psi(x_0, \Phi(U(x_1, x_2, \ldots)),$$

so, by (1),

$$\Phi \circ U(x_0, x_1, \ldots)) = \psi(x_0, \Phi \circ U(x_1, x_2, \ldots)).$$

Putting $W := \Phi \circ U$ we get

$$W(x_0, x_1, \ldots)) = \psi(x_0, W(x_1, x_2, \ldots)).$$

Thus W is a continuous solution of (5), so by the assumption W and V represent the same order. By Lemma 1 there exists an increasing homeomorphism Λ such that $W = \Lambda \circ V$. Hence $U = \Phi^{-1} \circ W = \Phi^{-1} \circ \Lambda \circ V$. Thus U and V represent the same relation. □

Putting in (5) $\psi(x,t) = \alpha(x)t + \beta(x)$ we get by Theorem 3 and Lemma 2 the following.

Theorem 4. *Let U satisfy (1) and V satisfy (2). Then U and V represent the same relation if and only if there exists an increasing homeomorphism $\Phi : \mathrm{I} \to \mathbb{R}$ such that*

$$\Phi(\varphi(x,t)) = \alpha(x)\Phi(t) + \beta(x), \; x \in X, \; t \in \mathrm{I}.$$

Let us write the last system in more convenient form

$$\Phi(f_x(t)) = \alpha_x \Phi(t) + \beta_x, \; t \in \mathrm{I}, \; x \in X, \tag{7}$$

for some $0 < \alpha_x < 1$ and $\beta_x \in \mathbb{R}$.

Theorem 5. *If U satisfies (1) and there exists an increasing homeomorphic solution of (7) with some coefficients $\alpha_x \in (0,1)$ and $\beta_x \in \mathbb{R}$, then the relation represented by U satisfies impatience.*

Proof. Let U represent $\succeq$ and satisfy (1). Put $V := \Phi \circ U$, where Φ is an increasing homeomorphic solution of (7). Note that V satisfies (2). In fact,

$$\begin{aligned} V(x_0, x_1, \ldots) &= \Phi(U(x_0, x_1, \ldots)) = \Phi(f_x(U(x_1, x_2, \ldots))) \\ &= \alpha_{x_0}\Phi(U(x_1, x_2, \ldots)) + \beta_{x_0} = \alpha_{x_0} V(x_1, x_2, \ldots) + \beta_{x_0}. \end{aligned}$$

Hence, by Theorem 4, the relation $\succeq$ is also represented by V. Thus by Theorem 2 $\succeq$ satisfies impatience. □

Remark 4. If a given Φ satisfies (7) with coefficients α_x and β_x, then α_x and β_x are uniquely determined. Moreover $0 < \alpha_x < 1$.

Proof. Let $t_1, t_2 \in I$ be such that $\Phi(t_1) \neq \Phi(t_2)$. Putting in (7) $t = t_1$ and $t = t_2$ we get a system of two linear equations

$$\begin{aligned} \Phi(f_x(t_1)) &= \alpha_x \Phi(t_1) + \beta_x, \\ \Phi(f_x(t_2)) &= \alpha_x \Phi(t_2) + \beta_x, \end{aligned}$$

which determine α_x and β_x uniquely.

Let $f_x(p) = p$. Then we have $\Phi(p)(1 - \alpha_x) = \beta_x$. Let $t > p$. Then, by (H), $f_x(t) < t$ and $\Phi(f_x(t)) < \Phi(t)$ so, by (6), $\alpha_x \Phi(t) + \beta_x < \Phi(t)$. Hence $\alpha_x \Phi(t) + \Phi(p)(1 - \alpha_x) < \Phi(t)$, so $(1 - \alpha_x)(\Phi(t) - \Phi(p)) > 0$. Since $\Phi(t) > \Phi(p)$ we get $\alpha_x < 1$. □

We consider the inverse problem. When does impatience generated by a utility function U satisfying (1) implies the existence of another utility function V satisfying the affine recursion (2) generating the same relation?

Assume that relation $\succeq$ satisfies impatience. We give some necessary and sufficient conditions for the existence of a homeomorphic solution of the system (7).

Further we make the general assumption that I is an open interval and $f_x(I) = I$ for $x \in X$.

First consider a special trivial case where the family of functions $\{f_x, x \in X\}$ is a subset of a cyclic group.

Theorem 6. *If the family of functions $\{f_x, x \in X\}$ is a subset of a cyclic group then there exists V satisfying (2) which determines the same order relation as U.*

Proof. By the assumption there exists an increasing homeomorphism h satisfying (H) such that $\{f_x, x \in X\} \subset \{h^n, n \in \mathbb{N}\}$ and, as a consequence, there exists a function $n : X \to \mathbb{N}$ such that $f_x = h^{n(x)}$.

It is well-known that there exists a homeomorphic solution Φ of the equation

$$\Phi(h(t)) = \alpha\Phi(t) + \beta$$

(see [6]). It depends on an arbitrary function. It is obvious that every solution of this equation satisfies the system

$$\Phi(h^n(t)) = \alpha^n\Phi(t) + \beta_n,$$

where $\beta_n = \beta\frac{\alpha^n-1}{\alpha^n-1}$. Hence Φ satisfies (7), where $\alpha_x = \alpha^{n(x)}$ and $\beta_x = \beta\frac{\alpha^{n(x)}-1}{\alpha^{n(x)}-1}$. $\square$

Further we concentrate on the case where, for some a and b, $f_a^n \neq f_b^m$ for $n, m \geq 1$.

Remark 5. If (7) has a homeomorphic solution, $x, y \in X$ and $f_x \neq f_y$, then their graphs are either disjoint or intersect in one point.

Proof. Let homeomorphism Φ satisfy (7). Suppose that

$$\text{card } (\mathrm{graph}\, \mathrm{f_x} \cap \mathrm{graph}\, \mathrm{f_y}) \geq 2.$$

Then there exists $t_1, t_2 \in I$, $t_1 \neq t_2$ such that $f_x(t_1) = f_y(t_1)$ and $f_x(t_2) = f_y(t_2)$. Hence, by (7),

$$\alpha_x\Phi(t_1) + \beta_x = \alpha_y\Phi(t_1) + \beta_y$$

and

$$\alpha_x\Phi(t_2) + \beta_x = \alpha_y\Phi(t_2) + \beta_y,$$

which gives $\alpha_x(\Phi(t_1) - \Phi(t_2)) = \alpha_y(\Phi(t_1) - \Phi(t_2))$, so $\alpha_x = \alpha_y$ and $\beta_x = \beta_y$.

Now, again by (6), we get $\Phi(f_x(t)) = \Phi(f_y(t))$ for $t \in I$, thus $f_x = f_y$, which contradicts our assumption. $\square$

Note that if $\Phi(f_a(t)) = \alpha_a\Phi(t) + \beta_a$, then for every $n \geq 1$, $\Phi(f_a^n(t)) = \alpha_a^n\Phi(t) + \beta_{a,n}$ for a $\beta_{a,n} \in \mathbb{R}$. Hence, by Remark 5, we get

Remark 6. If system (7) has a homeomorphic solution then, for every $a, b \in I$ if $f_a^n \neq f_b^m$ then $f_a^n(t) \neq f_b^m(t)$ for all $t \in I$, except for only one point.

Recall that the functions f_a and f_b are said to be iteratively incommensurable if $f_a^n(t) \neq f_b^m(t)$ for all $t \in I$ and for all $n, m \in \mathbb{N}$ (see [8]).

In view of Corollary 1 we know that f_a and f_b commute if and only if they have a joint fixed point.

To solve system (7) we consider two cases.

(I) There exist $a, b \in X$, $a \neq b$ such that f_a and f_b have a joint fixed point.
(II) For every $a, b \in X$, $a \neq b$ f_a and f_b has no joint fixed point.

Case (I)

Let $f_a(p) = f_b(p) = p$ and f_a and f_b be iteratively incommensurable except one point. First we deal with the system of two functional equations

$$\begin{cases} \Psi(f_a(t)) = \alpha_a\Psi(t) + \beta_a, \\ \Psi(f_b(t)) = \alpha_b\Psi(t) + \beta_b. \end{cases} \tag{8}$$

By (8) we get $\Psi(p) = \frac{\beta_a}{1-\alpha_a} = \frac{\beta_b}{1-\alpha_a}$. Putting

$$G(t) := \Psi(t) - \frac{\beta_a}{1-\alpha_a}, \quad t \in I$$

we have $G(p) = 0$ and

$$\begin{cases} G(f_a(t)) = \alpha_a G(t), \ t \in I \\ G(f_b(t)) = \alpha_b G(t), \ t \in I. \end{cases}$$

In fact,

$$\begin{aligned} G \circ f_a &= \Psi \circ f_a - \frac{\beta_a}{1-\alpha_a} = \alpha_a\Psi + \beta_a - \frac{\beta_a}{1-\alpha_a} = \alpha_a\Psi - \beta_a\frac{\alpha_a}{1-\alpha_a} \\ &= \alpha_a\left(\Psi - \frac{\beta_a}{1-\alpha_a}\right) = \alpha_a G. \end{aligned}$$

Similarly we obtain that $G \circ f_b = \alpha_b G$. Thus the last system is equivalent to system (8).

Introduce the notation $I^- := I \cap (-\infty, p)$, $I^+ := I \cap (p, \infty)$ and

$$f_a^- := f_a|I^-, \ f_b^- := f_b|I^-, \ f_a^+ := f_a|I^+, \ f_b^+ := f_b|I^+.$$

Let Ψ be an increasing solution of (8). Then $G(t) < 0$ for $t < p$ and $G(t) > 0$ for $t > 0$.

Putting $F^-(t) := \log(-G(t))$ for $t \in I^-$ and $F^+(t) := \log G(t)$ for $t \in I^+$ we get two independent systems of Abel's equations

$$\begin{cases} F^-(f_a^-(t)) = F^-(t) + \log \alpha_a, & t \in I^- \\ F^-(f_b^-(t)) = F^-(t) + \log \alpha_b, & t \in I^- \end{cases} \tag{9}$$

and

$$\begin{cases} F^+(f_a^+(t)) = F^+(t) + \log \alpha_a, & t \in I^+ \\ F^+(f_b^+(t)) = F^+(t) + \log \alpha_b, & t \in I^+. \end{cases} \tag{10}$$

Note that f_a^- and f_b^- have no fixed points and they are bijections of I^- onto itself. It works similarly with f_a^+ and f_b^+ on the interval I^+.

The above reasoning shows that system (8) with increasing Ψ is equivalent to two systems (9) and (10) with decreasing F^- and increasing F^+. We solve these systems separately.

Denote by $L(t)$ the limit set of the sequence $\{f_a^n \circ f_b^{-m}(t)\}$ that is

$$L(t) := \{f_a^n \circ f_b^{-m}(t), n, m \in \mathbb{N}\}^d, \ t \in I \setminus \{p\}.$$

In each of intervals I^- and I^+ the set $L(t)$ does not depend on t an is either an interval or a nowhere dense and perfect set (see [7,8]).

Let

$$s_-(a,b) := \sup\{n/m : n, m \in \mathbb{N}, (f_a^-)^m > (f_b^-)^n\}$$

and

$$s_+(a,b) := \inf\{n/m : n, m \in \mathbb{N}, (f_a^+)^m < (f_b^+)^n\}.$$

It is known that if system (9) has a homeomorphic solution then $s_-(a,b) = \frac{\log \alpha_b}{\log \alpha_a}$ and $L(t) = cl\, I^-$ for $t \in I^-$ (see [2]). Similarly, if system (10) has a homeomorphic solution then $s_+(a,b) = \frac{\log \alpha_b}{\log \alpha_a}$ and $L(t) = cl\, I^+$ for $t \in I^+$. Hence if system (7) has a homeomorphic solution then $s_-(a,b) = s_+(a,b)$.

In view of Theorem 2 in [8] system (9) in I^- and system (10) in I^+ have homeomorphic solutions F^- and F^+ if and only if there exist $t_1 \in I^-$ and $t_2 \in I^+$ such that $\mathrm{Int}L(t_1) \neq \emptyset$ and $\mathrm{Int}L(t_2) \neq \emptyset$. Moreover F^- and F^+ are determined uniquely up to an additive constant. Since F^- is decreasing, F^+ is increasing, $\lim_{x\to p^-} F(t) = -\infty$ and $\lim_{x\to p^+} F(t) = -\infty$ we infer that G is increasing and $G(p) = 0$ and, as a consequence, Ψ is a homeomorphic solution of (8).

Hence we have

Theorem 7. *Let f_a and f_b commute and be iteratively incommensurable except for one point. System (8) has an increasing homeomorphic solution Ψ if and only if there exist $t_1 \in I^-$ and $t_2 \in I^+$ such that $\mathrm{Int}L(t_1) \neq \emptyset$ and $\mathrm{Int}L(t_2) \neq \emptyset$ and $s^-(a,b) = s^+(a,b)$. Moreover, in each of the intervals I^- and I^+ the solution Φ is determined uniquely up to one parameter.*

If f_a and f_b satisfy the assumptions of Theorem 7 then the homeomorphic solution of system (9) is given by the formula

$$F^+(t) = c + \sup\left\{n - m\frac{\log\alpha_b}{\log\alpha_a}, n, m \in \mathbb{N}, (f_a)^n(t_0) > (f_b)^m(t)\right\},$$

for a given $t_0 \in I^+$ and a constant $c \in \mathbb{R}$ (see Theorem 2 in [4]). Then

$$\Phi(t) = \exp F^+(t) + \frac{\beta_a}{1-\alpha_a},\ t \in I^+.$$

It works similarly on the interval I^-.

Let Ψ be an increasing homeomorphic solution of (8). Define

$$\operatorname{Realm}\Psi := \{f : \Delta \to \Delta,\quad \exists\alpha \in (0,1)\ \exists\beta \in \mathbb{R}\ \Psi \circ f = \alpha\Psi + \beta\}.$$

Note that $\operatorname{Realm}\Psi$ is a semigroup of increasing homeomorphisms possessing one fixed point.

It is easy to show the following

Remark 7. If Ψ_1 and Ψ_2 are homeomorphic solutions of system (8) then $\operatorname{Realm}\Psi_1 = \operatorname{Realm}\Psi_2$.

Let Ψ be a homeomorphic solution of (8). If there exists a homeomorphic solution Φ of (7), then Φ satisfies (8) and $\operatorname{Realm}\Psi = \operatorname{Realm}\Phi$. Hence, for every $x \in X$, $f_x \in \operatorname{Realm}\Psi$ that is

$$f_x(t) = \Psi^{-1}(\alpha_x\Psi(t) + \beta_x)$$

for some $0 < \alpha_x < 1$ and $\beta \neq 0$.

Conversely, if every $f_x \in \operatorname{Realm}\Psi$ then system (7) has an increasing homeomorphic solution.

Corollary 3. *In the case (I) system* (7) *has an increasing homeomorphic solution if and only if* $f_x \in \operatorname{Realm}\Psi$ *for all* $x \in X$.

Directly by Theorems 1, 4 and Corollary 2 we get

Theorem 8. *Suppose that the relation* $\succeq$ *satisfies impatience and is represented by a utility function fulfilling* (1) *such that for some* $a, b \in X$ f_a *and* f_b *have a joint fixed point and are iteratively incommensurable except for one point. Then* $\succeq$ *can be represented by another function satisfying the affine recursion* (2) *if and only if*

(i) *there exist* $t_1 \in I^-$ *and* $t_2 \in I^+$ *such that* $\operatorname{Int}L(t_1) \neq \emptyset$ *and* $L(t_2) \neq \emptyset$,
(ii) $s^-(a,b) = s^+(a,b)$,
(iii) $f_x \in \operatorname{Realm}\Psi$ *for* $x \in X$, *where* Ψ *is an increasing continuous solution of system* (8).

Case (II)

Now we assume that for all $x \neq y$ f_x and f_y have different fixed points. Let h_{xy} be the commutator of f_x and f_y that is

$$h_{xy} := f_x \circ f_y \circ f_x^{-1} \circ f_y^{-1}.$$

If the relation represented by the utility function U fulfilling (1) satisfies impatience then, by Theorem 1, $h_{xy} \leq id$ or $h_{xy} \geq id$ for all $x, y \in X$.

Assume that system (7) has an increasing homeomorphic solution Φ. It is easy to verify that

$$\Phi(h_{xy}(t)) = \Phi(t) + c_{xy}, \quad t \in I, \tag{11}$$

where

$$c_{xy} := \beta_x(1-\alpha_y) + \beta_y(\alpha_x - 1).$$

Let $f_x(p_x) = p_x$ and $f_y(p_y) = p_y$. By (7) $\Phi(p_x) = \frac{\beta_x}{1-\alpha_x}$ and $\Phi(p_y) = \frac{\beta_y}{1-\alpha_y}$. Hence

$$c_{xy} = (\Phi(p_x) - \Phi(p_y))(1-\alpha_x)(1-\alpha_y),$$

so $c_{xy} = 0$ if and only if $x = y$. Thus, by (11), all commutators h_{xy} are fixed point free.

By (11) we infer that all commutators h_{xy} mutually commute. Thus the derived group H', that is the group generated by all these commutators, is Abelian. This means that the group

$$H :=< f_x, x \in X >$$

generated by f_x is solvable of the derived length two.

Assume that there exist $p, q, u, w \in X$ such that h_{pq} and h_{uv} are iteratively incommensurable. Then, by Theorem 2 in [8],

$$\{h_{pq}^n \circ h_{uv}^m(t), n, m \in \mathbb{Z}\}^d = cl\, I,\ t \in I. \tag{12}$$

Note that h_{pq} and h_{uv} are iteratively incommensurable if and only if c_{pq}/c_{uv} is irrational.

Conversely, let H be a solvable group of the derived length two and there exist h_{pq}, h_{uv} iteratively incommensurable satisfying condition (12) for a $t \in I$. Then there exists a unique up to an additive constant homeomorphic solution of the system of Abel's equations (see [8])

$$\begin{cases} F(h_{pq}(t)) = F(t) + c_{pq}, \\ F(h_{uv}(t)) = F(t) + c_{uv}. \end{cases} \tag{13}$$

If system (7) has a continuous solution Φ then Φ satisfies also (13), so by the uniqueness of solutions of system (13) $\Phi = F + c$ for a constant c.

Summarising the above statements we get

Theorem 9. *Let case (II) hold and relation $\succeq$ be represented, by the utility function fulfilling* (1)*, satisfy impatience and there exist commutators h_{pq} and h_{uv} iteratively incommensurable.*

Then the relation $\succeq$ can be represented by another utility function satisfying affine recursion (2) *if and only if*

(i) *the group $< f_x, x \in X >$ is solvable of the second order,*
(ii) (12) *holds for a $t \in I$,*
(iii) *for every $x \in X$ $f_x \in \operatorname{Realm} F$, where F is a continuous solution of* (13)*.*

Note that the condition $f_x \in \operatorname{Realm} F$ means that for every $x \in X$ there exist $\alpha_x \in (0,1)$ and $\beta_x \in \mathbb{R}$ such that

$$F(f_x(t)) = \alpha_x F(t) + \beta_x, \quad t \in I.$$

This means that system (7) has a homeomorphic solution. Note that we can determine homeomorphism F (see [4]), so condition (iii) is verifiable although technically it may be difficult.

In the case (I) the assumption of the surjectivity of functions f_x can be omitted, but then the assertions in Theorems 7 and 8 have more complicated form. This generalization one can obtain by applying the method of extension of commuting, non-surjective, continuous, strictly increasing mappings to commuting homeomorphisms presented in papers [4,5].

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