A functional equation related to generalized entropies and the modular group

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Abstract. We solve a functional equation connected to the algebraic characterization of generalized information functions. To prove the symmetry of the solution, we study a related system of functional equations, which involves two homographies. These transformations generate the modular group, and this fact plays a crucial role in solving the system. The method suggests a more general relation between conditional probabilities and arithmetic.

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1. Motivation and results

In this paper, we study the measurable solutions $u : [0, 1] \rightarrow \mathbb{R}$ of the functional equation

$$u(1-x) + (1-x)^\alpha u\left(\frac{y}{1-x}\right) = u(y) + (1-y)^\alpha u\left(\frac{1-x-y}{1-y}\right) \quad (1.1)$$

for all $x, y \in [0, 1)$ such that $x + y \in [0, 1]$. The parameter α can take any positive real value.

This equation appears in the context of algebraic characterizations of information functions. Given a random variable X whose range is a finite set E_X , a measure of its “information content” is supposed to be a function

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$f[X] : \Delta(E_X) \rightarrow \mathbb{R}$, where $\Delta(E_X)$ denotes the set of probabilities on E_X ,

$$\Delta(E_X) = \left\{ p : E_X \rightarrow [0, 1] \mid \sum_{x \in E_X} p(x) = 1 \right\}. \quad (1.2)$$

The most important example of such a function is the Shannon-Gibbs entropy

$$S_1[X](p) := - \sum_{x \in E_X} p(x) \log p(x), \quad (1.3)$$

where $0 \log 0$ equals 0 by convention.

Shannon entropy satisfies a remarkable property, called the *chain rule*, that we now describe. Let X (resp. Y) be a variable with range E_X (resp. E_Y); both E_X and E_Y are supposed to be finite sets. The couple (X, Y) takes values in a subset E_{XY} of $E_X \times E_Y$, and any probability p on E_{XY} induce by marginalization laws X_*p on E_X and Y_*p on E_Y . For instance,

$$X_*p(x) = \sum_{y: (x, y) \in E_{XY}} p(x, y). \quad (1.4)$$

The chain rule corresponds to the identities

$$S_1[(X, Y)](p) = S_1[X](X_*p) + \sum_{x \in E_X} X_*p(x) S_1[Y](Y_*(p|_{X=x})), \quad (1.5)$$

$$S_1[(X, Y)](p) = S_1[Y](Y_*p) + \sum_{y \in E_Y} Y_*p(y) S_1[X](X_*(p|_{Y=y})), \quad (1.6)$$

where $p|_{X=x}$ denotes the conditional probability $y \mapsto p(y, x)/X_*p(x)$. These identities reflect the third axiom used by Shannon to characterize an information measure H : “if a choice be broken down into two successive choices, the original H should be the weighted sum of the individual values of H ” [7].

There is a deformed version of Shannon entropy, called generalized entropy of degree α [1, Ch. 6]. For any $\alpha \in (0, \infty) \setminus \{1\}$, it is defined as

$$S_\alpha[X](p) := \frac{1}{1 - \alpha} \left(\sum_{x \in E_X} p(x)^\alpha - 1 \right). \quad (1.7)$$

This function was introduced by Havrda and Charvát [4]. Constantino Tsallis popularized its use in physics, as the fundamental quantity of non-extensive statistical mechanics [8], so S_α is also called Tsallis α -entropy. It satisfies a deformed version of the chain rule:

$$S_\alpha[(X, Y)](p) = S_\alpha[X](X_*p) + \sum_{x \in X} (X_*p(x))^\alpha S_\alpha[Y](Y_*(p|_{X=x})). \quad (1.8)$$

Suppose now that, given $\alpha > 0$, we want to find the most general functions $f[X]$ —for a given collection of finite random variables X —such that

- A. $f[X](\delta) = 0$ whenever δ is any Dirac measure—a measure concentrated on a singleton—, which means that variables with deterministic outputs do not give (new) information when measured;
- B. the generalized α -chain rule holds, i.e. for any variables X and Y with finite range¹

$$f[(X, Y)](p) = f[X](X_*p) + \sum_{x \in E_X} (X_*p(x))^\alpha f[Y](Y_*(p|_{X=x})), \quad (1.9)$$

$$f[(X, Y)](p) = f[Y](Y_*p) + \sum_{y \in E_Y} (Y_*p(y))^\alpha f[X](X_*(p|_{Y=y})). \quad (1.10)$$

The simplest non-trivial case corresponds to $E_X = E_Y = \{0, 1\}$ and $E_{XY} = \{(0, 0), (1, 0), (0, 1)\}$; a probability p on E_{XY} is a triple $p(0, 0) = a$, $p(1, 0) = b$, $p(0, 1) = c$, such that $X_*p = (a + c, b)$ and $Y_*p = (a + b, c)$. The equality between the right-hand sides of (1.9) and (1.10) reads

$$\begin{aligned} f[X](a + c, b) + (1 - b)^\alpha f[Y]\left(\frac{a}{1 - b}, \frac{c}{1 - b}\right) \\ = f[Y](a + b, c) + (1 - c)^\alpha f[X]\left(\frac{a}{1 - c}, \frac{b}{1 - c}\right), \end{aligned} \quad (1.11)$$

for any triple $(a, b, c) \in [0, 1]^2$ such that $a + b + c = 1$. Setting $a = 0$ and using assumption A, we conclude that $f[X](c, 1 - c) = f[Y](1 - c, c) =: u(c)$ for any $c \in [0, 1]$. Therefore, (1.11) can be written in terms of this unique unknown u ; if moreover we set $c = y$, $b = x$ and consequently $a = 1 - x - y$, we get the functional equation (1.1), with the stated boundary conditions.

The main result of this article is the following.

Theorem 1.1. *Let α be a positive real number. Suppose $u : [0, 1] \rightarrow \mathbb{R}$ is a measurable function that satisfies (1.1) for every $x, y \in [0, 1]$ such that $x + y \in [0, 1]$. Then, there exists $\lambda \in \mathbb{R}$ such that $u(x) = \lambda s_\alpha(x)$, where*

$$s_1(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$$

and

$$s_\alpha(x) = \frac{1}{1 - \alpha} (x^\alpha + (1 - x)^\alpha - 1)$$

when $\alpha \neq 1$.

By convention, $0 \log_2 0 := \lim_{x \rightarrow 0} x \log_2 x = 0$. For $\alpha = 1$, Theorem 1.1 is essentially Lemma 2 in [5]. Our proof depends on two independent results.

Theorem 1.2. (Regularity) *Any measurable solution of (1.1) is infinitely differentiable on the interval $(0, 1)$.*

¹Assumption A can be deduced from B if one identifies X with (X, X) through the diagonal map $E_X \rightarrow E_X \times E_X$, $x \mapsto (x, x)$ and then evaluates (1.9) at $Y = X$ and $p = \delta_{x_0}$, for any $x_0 \in E_X$.

Theorem 1.3. (Symmetry) *Any solution of (1.1) satisfies $u(x) = u(1 - x)$ for all $x \in \mathbb{Q} \cap [0, 1]$.*

The first is proved analytically, by means of standard techniques in the field of functional equations (cf. [1, 5, 9]), and the second by a novel geometrical argument, relating the equation to the action of the modular group on the projective line.

Theorems 1.2 and 1.3 above imply that any measurable solution u of (1.1) must be symmetric, i.e. $u(x) = u(1 - x)$ for all $x \in [0, 1]$, and therefore

$$u(x) + (1 - x)^\alpha u\left(\frac{y}{1 - x}\right) = u(y) + (1 - y)^\alpha u\left(\frac{x}{1 - y}\right) \quad (1.12)$$

whenever $x, y \in [0, 1)$ and $x + y \in [0, 1]$. When $\alpha = 1$, this equation is called “the fundamental equation of information theory”; it first appeared in the work of Tverberg [9], who deduced it from a characterization of an “information function” that not only supposed a version of the chain rule, but also the invariance of the function under permutations of its arguments. Daróczy introduced the fundamental equation for general $\alpha > 0$, and showed that it can be deduced from an axiomatic characterization analogue to that of Tverberg, that again supposed invariance under permutations along with a deformed chain rule akin to (1.8), see [3, Thm. 5].

For $\alpha = 1$, Tverberg [9] showed that, if $u : [0, 1] \rightarrow \mathbb{R}$ is symmetric, Lebesgue integrable and satisfies (1.12), then it must be a multiple of $s_1(x)$. In [5], Kannappan and Ng weakened the regularity condition, showing that all measurable solutions of (1.12) have the form $u(x) = As_1(x) + Bx$ (where A and B are arbitrary real constants), which reduces to $u(x) = As_1(x)$ when u is symmetric. In fact, they solved some generalizations of the fundamental equation, proving among other things that, when $\alpha = 1$, the only measurable solutions of (1.1) are multiples of $s_1(x)$.

For $\alpha \neq 1$, Daróczy [3] established that any $u : [0, 1] \rightarrow \mathbb{R}$ that satisfies (1.12) and $u(0) = u(1)$ has the form²

$$u(x) = \frac{u(1/2)}{2^{1-\alpha} - 1} (x^\alpha + (1 - x)^\alpha - 1), \quad (1.13)$$

without any hypotheses on the regularity of u . The proof starts by proving that any solution of (1.12) must satisfy $u(0) = 0$ (setting $x = 0$), and hence be symmetric (setting $y = 1 - x$). Since we are able to prove symmetry of the solutions of (1.1) restricted to rational arguments without any regularity hypothesis, we also get the following result.

Corollary 1.4. *For any $\alpha \in (0, \infty) \setminus \{1\}$, the only functions $u : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}$ that satisfy equation (1.1) are multiples of s_α .*

²In fact, he supposes $u(1/2) = 1$, but the argument works in general.

Proof. Set $x = 0$ in (1.1) to conclude that $u(1) = 0$, and $y = 0$ to obtain $u(0) = 0$. Moreover, u must be symmetric (Theorem 1.3), hence it must fulfill (1.12) when the arguments are rational. Given these facts, Daróczy's proof in [3, p. 39] applies with no modifications when restricted to $p, q \in \mathbb{Q}$. $\square$

More details on the characterization of information functions by means of functional equations can be found in the classical reference [1], which gives a detailed historical introduction. Reference [2] summarizes more recent developments in connection with homological algebra.

It is quite remarkable that Theorem 1.1 serves as a fundamental result to prove that, up to a multiplicative constant, $\{S_\alpha[X]\}_{X \in \mathcal{S}}$ is the only collection of measurable functionals (not necessarily invariant under permutations) that satisfy the corresponding α -chain rule, for any generic set of random variables $\mathcal{S}$. In order to do this, one introduces an adapted cohomology theory, called information cohomology [2], where the chain rule corresponds to the 1-cocycle condition and thus has an algebro-topological meaning. The details can be found in the dissertation [10].

2. The modular group

The group $G = SL_2(\mathbb{Z})/\{\pm I\}$ is called the **modular group**; it is the image of $SL_2(\mathbb{Z})$ in $PGL_2(\mathbb{R})$. We keep using the matrix notation for the images in this quotient. We make G act on $P^1(\mathbb{R})$ as follows: an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

acting on $[x : y] \in P^1(\mathbb{R})$ (homogeneous coordinates) gives

$$g[x : y] = [ax + by : cx + dy].$$

Let S and T be the elements of G defined by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.1)$$

The group G is generated by S and T [6, Ch. VII, Th. 2]; in fact, one can prove that $\langle S, T; S^2, (ST)^3 \rangle$ is a presentation of G .

3. Regularity: proof of Theorem 1.2

Lemma 3 in [5] implies that u is locally bounded on $(0, 1)$ and hence locally integrable. Their proof is for $\alpha = 1$, but the argument applies to the general case with almost no modification, just replacing

$$|u(y)| = \left| u(1-x) + (1-x)u\left(\frac{y}{1-x}\right) - (1-y)u\left(\frac{1-x-y}{1-y}\right) \right| \leq 3N,$$

where x, y are such that $u(1-x) \leq N$, $u\left(\frac{y}{1-x}\right) \leq N$ and $u\left(\frac{1-x-y}{1-y}\right) \leq N$, by

$$|u(y)| = \left| u(1-x) + (1-x)^\alpha u\left(\frac{y}{1-x}\right) - (1-y)^\alpha u\left(\frac{1-x-y}{1-y}\right) \right| \leq 3N,$$

which is evidently valid too whenever $x, y \in (0, 1)$.

To prove the differentiability, we also follow the method in [5]—already present in [9]. Let us fix an arbitrary $y_0 \in (0, 1)$; then, it is possible to choose $s, t \in (0, 1)$, $s < t$, such that

$$\frac{1-y-s}{1-y}, \frac{1-y-t}{1-y} \in (0, 1),$$

for all y in certain neighborhood of y_0 . We integrate (1.1) with respect to x , between s and t , to obtain

$$(s-t)u(y) = \int_{1-t}^{1-s} u(x)dx + y^{1+\alpha} \int_{\frac{y}{1-s}}^{\frac{y}{1-t}} \frac{u(z)}{z^3} dz + (1-y)^{1+\alpha} \int_{\frac{1-y-s}{1-y}}^{\frac{1-y-t}{1-y}} u(z)dz. \quad (3.1)$$

The continuity of the right-hand side of (3.1) as a function of y at y_0 , implies that u is continuous at y_0 and therefore on $(0, 1)$. The continuity of u in the right-hand side of (3.1) implies that u is differentiable at y_0 . An iterated application of this argument shows that u is infinitely differentiable on $(0, 1)$.

4. Symmetry: proof of Theorem 1.3

Define the function $h : [0, 1] \rightarrow \mathbb{R}$ through

$$\forall x \in [0, 1], \quad h(x) = u(x) - u(1-x). \quad (4.1)$$

Observe that h is anti-symmetric around $1/2$, that is, we have

$$\forall x \in [0, 1], \quad h(x) = -h(1-x). \quad (4.2)$$

Let now $z \in [\frac{1}{2}, 1]$ be arbitrary and use the substitutions $x = 1-z$ and $y = 1-z$ in (1.1) to derive the identity

$$\forall z \in \left[\frac{1}{2}, 1\right], \quad h(z) = z^\alpha h(2-z^{-1}). \quad (4.3)$$

Using the anti-symmetry of h to modify the right-hand side of the previous equation, we also deduce that

$$\forall z \in \left[\frac{1}{2}, 1\right], \quad h(z) = -z^\alpha h(z^{-1} - 1). \quad (4.4)$$

Setting $x = 0$ (respectively $y = 0$) in (1.1), we conclude that $u(1) = 0$ (resp. $u(0) = 0$). Hence, the function h is subject to the boundary conditions

$h(0) = h(1) = 0$. From (4.3), it follows that $h(1/2) = h(0)/2^\alpha = 0$. If the domain of h is extended to the whole real line imposing 1-periodicity:

$$\forall x \in]-\infty, \infty[, \quad h(x+1) = h(x), \quad (4.5)$$

a similar argument can be used to determine the value of h at any rational argument. To that end, it is important to establish first that (4.3) and (4.4) hold for the extended function.

Theorem 4.1. *The function h , extended periodically to $\mathbb{R}$, satisfies the equations*

$$\forall x \in \mathbb{R}, \quad h(x) = |x|^\alpha h\left(\frac{2x-1}{x}\right), \quad (4.6)$$

$$\forall x \in \mathbb{R}, \quad h(x) = -|x|^\alpha h\left(\frac{1-x}{x}\right). \quad (4.7)$$

We establish first the anti-symmetry around $1/2$ of the extended h (Lemma 4.2), which implies that (4.7) follows from (4.6); the latter is a consequence of Lemmas 4.3–4.7.

Lemma 4.2.

$$\forall x \in \mathbb{R}, \quad h(x) = -h(1-x).$$

Proof. We write $x = [x] + \{x\}$, where $\{x\} := x - [x]$. Then,

$$h(x) \stackrel{(4.5)}{=} h(\{x\}) \stackrel{(4.2)}{=} -h(1-\{x\}) \stackrel{(4.5)}{=} -h(1-\{x\}-[x]) = -h(1-x).$$

□

Lemma 4.3.

$$\forall x \in [1, 2], \quad h(x) = x^\alpha h(2-x^{-1}). \quad (4.8)$$

Proof. For h is periodic, (4.8) is equivalent to

$$\forall x \in [1, 2], \quad h(x-1) = x^\alpha h(1-x^{-1}), \quad (4.9)$$

and the change of variables $u = x-1$ gives

$$\forall u \in [0, 1], \quad h(u) = (u+1)^\alpha h\left(\frac{u}{u+1}\right). \quad (4.10)$$

Note that $1 - \frac{u}{u+1} = \frac{1}{u+1} \in [1/2, 1]$ whenever $u \in [0, 1]$. Therefore,

$$h\left(\frac{u}{u+1}\right) \stackrel{(\text{Lemma } 4.2)}{=} -h\left(\frac{1}{u+1}\right) \stackrel{(4.4)}{=} \left(\frac{1}{u+1}\right)^\alpha h(u).$$

This establishes (4.10). □

Lemma 4.4.

$$\forall x \in [2, \infty[, \quad h(x) = x^\alpha h(2-x^{-1}). \quad (4.11)$$

Proof. If $x \in [2, \infty[$, then $1 - \frac{1}{x} \in [\frac{1}{2}, 1]$ and we can apply Eq. (4.3) to obtain

$$h\left(1 - \frac{1}{x}\right) \stackrel{(4.3)}{=} \left(1 - \frac{1}{x}\right)^\alpha h\left(2 - \left(1 - \frac{1}{x}\right)^{-1}\right) = \left(\frac{x-1}{x}\right)^\alpha h\left(1 - \frac{1}{x-1}\right). \quad (4.12)$$

We prove (4.11) by recurrence. The case $x \in [1, 2]$ corresponds to Lemma 4.3. Suppose it is valid on $[n-1, n]$, for certain $n \geq 2$; for $x \in [n, n+1]$,

$$\begin{aligned} h(x) &\stackrel{(4.5)}{=} h(x-1) \\ &\stackrel{(\text{rec.})}{=} (x-1)^\alpha h(2 - (x-1)^{-1}) \\ &\stackrel{(4.5)}{=} (x-1)^\alpha h(1 - (x-1)^{-1}) \\ &\stackrel{(4.12)}{=} x^\alpha h(1 - x^{-1}) \\ &\stackrel{(4.5)}{=} x^\alpha h(1 - x^{-1}). \end{aligned}$$

□

Lemma 4.5.

$$\forall x \in \left[0, \frac{1}{2}\right], \quad h(x) = -x^\alpha h(x^{-1} - 1). \quad (4.13)$$

Proof. The previous lemma and periodicity imply that $h(x-1) = x^\alpha h(1-x^{-1})$ for all $x \geq 2$, i.e.

$$\forall u \geq 1, \quad h(u) = (u+1)^\alpha h\left(1 - \frac{1}{u+1}\right). \quad (4.14)$$

Then, for $u \geq 1$,

$$h\left(\frac{1}{u+1}\right) \stackrel{(\text{Lem. 4.2})}{=} -h\left(1 - \frac{1}{u+1}\right) \stackrel{(4.14)}{=} -\left(\frac{1}{u+1}\right)^\alpha h(u). \quad (4.15)$$

We set $y = (u+1)^{-1} \in (0, \frac{1}{2}]$. Equation (4.15) reads

$$\forall y \in \left(0, \frac{1}{2}\right], \quad h(y) = -y^\alpha h(y^{-1} - 1). \quad (4.16)$$

Since $h(0) = 0$, the lemma is proved. □

Lemma 4.6.

$$\forall x \in \left[0, \frac{1}{2}\right], \quad h(x) = x^\alpha h(2 - x^{-1}). \quad (4.17)$$

Proof. Immediately deduced from the previous lemma using the anti-symmetric property in Lemma 4.2. □

Lemma 4.7.

$$\forall x \in]-\infty, 0], \quad h(x) = -x^\alpha h(2 - x^{-1}).$$

Proof. On the one hand, periodicity implies that $h(x) = h(x+1) \stackrel{(\text{Lem. 4.2})}{=} -h(1-(x+1)) = -h(-x)$. On the other, for $x \leq 0$, the preceding results imply that $h(-x) = (-x)^\alpha h(2 - (-x)^{-1}) = |x|^\alpha h(2 - (-x)^{-1})$. Therefore,

$$\begin{aligned} h(x) &= -h(-x) = -|x|^\alpha h\left(2 + \frac{1}{x}\right) \\ &\stackrel{(\text{Lem. 4.2})}{=} |x|^\alpha h\left(1 - \left(2 + \frac{1}{x}\right)\right) \stackrel{(4.5)}{=} |x|^\alpha h\left(2 - \frac{1}{x}\right). \end{aligned}$$

□

The transformations $x \mapsto \frac{2x-1}{x}$ and $x \mapsto \frac{1-x}{x}$ in Eqs. (4.6) and (4.7) are homographies of the real projective line $P^1(\mathbb{R})$, that we denote respectively by α and β . They correspond to elements

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.18)$$

in G , that satisfy

$$B^2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad BA^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (4.19)$$

This last matrix corresponds to $x \mapsto 1 - x$.

Lemma 4.8. *The matrices A and B^2 generate G .*

Proof. Let

$$P = S^{-1}T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

One has

$$PAP^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (4.20)$$

and

$$PB^2P^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.21)$$

Therefore, $PAP^{-1} = T^{-1}$ and $S = T^{-3}PB^{-2}P^{-1}$. Inverting these relations, we obtain

$$T = PA^{-1}P^{-1}; \quad S = PA^3B^{-2}P^{-1}. \quad (4.22)$$

Let X be an arbitrary element of G . Since $Y = PXP^{-1} \in G$ and G is generated by S and T , the element Y is a word in S and T . In consequence, X is a word in $P^{-1}SP$ and $P^{-1}TP$, which in turn are words A and B^2 . □

It is possible to find explicit formulas for S and T in terms of A and B^2 . Since $P = S^{-1}T^{-1}$, we deduce that $PSP^{-1} = S^{-1}T^{-1}STS$ and $PTP^{-1} = S^{-1}T^{-1}TTS = S^{-1}TS$. Hence, in virtue of (4.22),

$$S = P^{-1}S^{-1}T^{-1}STSP$$

$$\begin{aligned}
 &= (P^{-1}S^{-1}P)(P^{-1}T^{-1}P)(P^{-1}SP)(P^{-1}TP)(P^{-1}SP) \\
 &= B^2AB^{-2}A^2B^{-2}
 \end{aligned}$$

and

$$\begin{aligned}
 T &= P^{-1}S^{-1}TSP \\
 &= (P^{-1}S^{-1}P)(P^{-1}TP)(P^{-1}SP) \\
 &= B^2A^{-1}B^{-2}.
 \end{aligned}$$

To finish our proof of Proposition 1.3, we remark that the orbit of 0 by the action of G on $P^1(\mathbb{R})$ is $\mathbb{Q} \cup \{\infty\}$, where $\mathbb{Q} \cup \{\infty\}$ has been identified with $\{[p : q] \in P^1(\mathbb{R}) \mid p, q \in \mathbb{Z}\} \subset P^1(\mathbb{R})$. This is a consequence of Bezout's identity: for every point $[p : q] \in P^1(\mathbb{R})$ representing a reduced fraction $\frac{p}{q} \neq 0$ ($p, q \in \mathbb{Z} \setminus \{0\}$ and coprime), there are two integers x, y such that $xq - yp = 1$. Therefore

$$g' = \begin{pmatrix} x & p \\ y & q \end{pmatrix}$$

is an element of G and $g'[0 : 1] = [p : q]$. The case $q = 0$ is covered by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} [0 : 1] = [1 : 0].$$

The extended Eqs. (4.6) and (4.7) are such that

1. For all $x \in \mathbb{R}$, if $h(x) = 0$ then $h(\alpha^{-1}x) = 0$ and $h(\beta^{-1}x) = 0$;
2. For all $x \in \mathbb{R} \setminus \{0\}$, if $h(x) = 0$ then $h(\alpha x) = 0$ and $h(\beta x) = 0$.

Since $h(1/2) = 0$, the following lemma is the missing piece to establish that the extended h vanishes on $\mathbb{Q}$ (and hence the original h necessarily vanishes on $[0, 1] \cap \mathbb{Q}$).

Lemma 4.9. *For any $r \in \mathbb{Q} \setminus \{0\}$, there exists a finite sequence*

$$w = (w_i)_{i=1}^n \in \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}^n$$

such that $r = w_n \circ \cdots \circ w_1(1/2)$ and, for all $i \in \{1, \dots, n\}$, the iterate $x_i := w_i \circ \cdots \circ w_1(1/2)$ does not equal 0 or ∞ .

Proof. Since the orbit in $P^1(\mathbb{R})$ of $1/2$ by the group of homographies generated by A and B^2 (i.e. G itself) contains the whole set of rational numbers $\mathbb{Q}$, there exists a w such that $r = w_n \circ \cdots \circ w_1(1/2)$, where each w_i equals α, β or one of their inverses.

If some iterate equals 0 or ∞ , the sequence w can be modified to avoid this. Let $i \in \{0, \dots, n\}$ be the largest index such that $x_i \in \{0, \infty\}$; in fact, $i < n$ because $r \neq 0, \infty$.

- If $x_i = 0$, then $x_{i+1} \in \{1/2, 1\}$ (the possibility $x_{i+1} = \infty$ is ruled out by the choice of i). In the case $x_{i+1} = 1/2$, the equality $r = w_n \circ \cdots \circ w_{i+2}(1/2)$ holds, and when $x_{i+1} = 1$, we have $r = w_n \circ \cdots \circ w_{i+2} \circ \beta(1/2)$.

- If $x_i = \infty$, then $x_{i+1} \in \{2, -1\}$ (again, $x_{i+1} = 0$ is ruled out). When $x_{i+1} = 2$, we have $r = w_n \circ \cdots \circ w_{i+2} \circ \beta \circ \alpha \circ \beta^{-1} \circ \beta^{-1}(1/2)$, and when $x_{i+1} = -1$, it also holds that $r = w_n \circ \cdots \circ w_{i+2} \circ \alpha \circ \alpha \circ \beta^{-1} \circ \beta^{-1}(1/2)$.

□

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