Sandwich results for periodicity and conjugacy

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Abstract. Let P be a nonconstant selfmap of a set $\mathcal{M}$. A sandwich-type theorem for generalized sub- P -periodic functions defined on $\mathcal{M}$ with values in a reflexive Banach space is proved. In particular, given functions $f, g: \mathcal{M} \rightarrow \mathbb{R}$, we obtain necessary and sufficient conditions for the existence of a generalized P -periodic function $F: \mathcal{M} \rightarrow \mathbb{R}$ such that $f \leq F \leq g$. The formula for F is given and its Lipschitz constant is discussed. Moreover the solvability of the functional equation $f \circ p = r \circ f$ with the help of a new sandwich method, is considered.

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1. Introduction

In a recent paper Izumi et al. [5] proved the following sandwich (separation) type result: *functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$ satisfy the inequalities*

$$f(x + np) \leq g(x), \quad n \in \mathbb{Z}, x \in \mathbb{R}, \quad (1)$$

for some $p \in \mathbb{R} \setminus \{0\}$ if, and only if, there is a p -periodic function $F: \mathbb{R} \rightarrow \mathbb{R}$ separating f and g , that is such that $F(x + p) = F(x)$ for all $x \in \mathbb{R}$ and $f \leq F \leq g$.

In Sect. 2 of this paper, applying Gajda's theorem on the existence of a translatable invariant mean in the Banach space of bounded vector valued functions [4], we prove far reaching generalizations of this result. Let $\mathcal{M}$ be a nonempty set, $P: \mathcal{M} \rightarrow \mathcal{M}$ a nonconstant mapping, and $\mathfrak{X}$ a reflexive Banach space. Assuming that a function $f: \mathcal{M} \rightarrow \mathfrak{X}$ and a set-valued function $\Gamma: \mathcal{M} \rightarrow 2^{\mathfrak{X}}$ with bounded closed and convex values are such that

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$$f(P^n(x)) \in \Gamma(x), \quad n \in \mathbb{N}_0, x \in \mathbb{R}$$

(where $\mathbb{N}_0 := \{0, 1, \dots\}$), we prove that there is a P -periodic selection F of Γ (i.e. such that $F \circ P = F$) (Theorem 2). Assume that $h : \mathcal{M} \rightarrow \mathbb{R}$ is P -super-periodic (i.e. $h \circ P \geq h$) and $g : \mathcal{M} \rightarrow \mathbb{R}$ is P -sub-periodic (i.e. $h \circ P \leq h$). Applying Theorem 2 with $\mathfrak{X} = \mathbb{R}$ we prove the inequality $h \leq g$ holds if and only if there is a P -periodic function separating h and g (Theorem 3). A strengthening of Theorem 3 is given. At the end of this section, assuming that $\mathcal{M}$ is a metric space, we give conditions under which the Lipschitz continuity of a $(P\text{-}\Gamma)$ -periodic function f (see Sect. 2), implies the existence of the Lipschitz continuous P -periodic function F .

In the context of the periodic equation $f \circ P = f$, with the notation $p = P$, it becomes quite natural to consider a more general functional equation $f \circ p = g \circ f$ linked both to conjugacy and commutativity. A description of solutions of this equations, as well as some related topics, can be found in the papers [2, 3, 7, 8, 10]. They focus on the existence and uniqueness of continuous and homeomorphic solutions in the case when p and g are given self-maps of a real interval or the unit circle S_1 . In [9], Zdun investigated bounded and continuous solutions $f : [0, 1] \rightarrow Y$, where Y is a complete metric space, of the system of equations $f \circ p_k = g_k \circ f$, $k = 0, 1, \dots, n - 1$. Ciepliński and Leśniak dealt with these problems in [1]. Moreover, Kopeček [6] investigated the equation $f \circ p = g \circ f$ for given functions p and g defined on subsets of all real numbers. He obtained a nice characterization of solvability of the above equation in the case when p, g are strictly increasing and continuous. Motivated by results of Kopeček, we investigate similar topics in Sect. 3. Namely, we prove sandwich type results for the functional equation $f(p(x)) = r(f(x))$. We conclude this paper with results concerning the functional equation

$$\alpha f(p(x)) = g(f(x)), \quad x \in \mathcal{M},$$

and the functional inequality

$$\alpha f(p^n(x)) \geq g^n(f(x)), \quad x \in \mathcal{M},$$

where the constant α and the functions $p : \mathcal{M} \rightarrow \mathcal{M}$ and $g : \mathfrak{X} \rightarrow \mathfrak{X}$ are given.

2. Generalized periodicity

Let S be a semigroup and $\mathfrak{X}$ be a Banach space. By $\mathcal{B}(S, \mathfrak{X})$ we denote the space of bounded $\mathfrak{X}$ -valued functions with the sup-norm $\|\cdot\|_\infty$. In our consideration the following result due to Gajda [4, Th.2.1, p. 20] is important.

Theorem 1. [4] *Let $(S, +)$ be a commutative semigroup. Suppose that $\mathfrak{X}$ is reflexive. There is a continuous linear operator $M : \mathcal{B}(S, \mathfrak{X}) \rightarrow \mathfrak{X}$ such that*

- (i) $M(\varphi_a) = M(\varphi)$ for all $a \in S$ and $\varphi \in \mathcal{B}(S, \mathfrak{X})$, where φ_a denotes the translation of φ defined by $\varphi_a(x) := \varphi(x + a)$, $x \in S$;

- (ii) for every $\varphi \in \mathcal{B}(S, \mathfrak{X})$ and closed convex bounded subset V of $\mathfrak{X}$, if $\varphi(S) \subseteq V$, then $M(\varphi) \in V$.

The linear operator M is called an *invariant mean*.

In the sequel $\mathcal{M}$ is a nonempty set, $P: \mathcal{M} \rightarrow \mathcal{M}$ is a nonconstant function, $\mathfrak{X}$ is a reflexive Banach space, and $\Gamma: \mathcal{M} \rightarrow 2^{\mathfrak{X}}$ a set-valued function such that, for every x , the value $\Gamma(x) \subsetneq \mathfrak{X}$ is nonempty and convex.

We say that the function $F: \mathcal{M} \rightarrow \mathfrak{X}$ is *P -periodic* [resp. *P - Γ -periodic*] if

$$\forall_{x \in \mathcal{M}} \quad F(P(x)) = F(x) \quad [\text{resp. } F(P(x)) \in \Gamma(x)]. \quad (2)$$

To summarize, the paper [5] and conditions (1), (2) motivate this paper.

Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Now we prove the first main result of this paper.

Theorem 2. *Let $\mathcal{M}$ be a nonempty set and let $\mathfrak{X}$ be a reflexive Banach space. Assume that $P: \mathcal{M} \rightarrow \mathcal{M}$ is not a constant mapping. Suppose that $\Gamma: \mathcal{M} \rightarrow 2^{\mathfrak{X}}$ is a set-valued function such that for all $x \in \mathcal{M}$ the set $\Gamma(x)$ is nonempty, bounded, closed and convex. Suppose that a function $f: \mathcal{M} \rightarrow \mathfrak{X}$ satisfies the condition*

$$f(P^n(x)) \in \Gamma(x), \quad x \in \mathcal{M}, \quad n \in \mathbb{N}_0. \quad (3)$$

Then there is a P -periodic function $F: \mathcal{M} \rightarrow \mathfrak{X}$ such that

$$F(x) \in \bigcap_{m \in \mathbb{N}} \overline{\text{conv}}\{f(P^n(x)) : n > m\}, \quad x \in \mathcal{M}; \quad (4)$$

in particular, $F(x) \in \Gamma(x)$ for all $x \in \mathcal{M}$.

Proof. Fix $a \in \mathfrak{X}$, $\varepsilon \in (0, 1)$, $x \in \mathcal{M}$. By the definition of $\text{dist}(a, A)$, there is a $d(x) \in \mathfrak{X}$ such that

$$d(x) \in \overline{\text{conv}}\{f(P^n(x)) : n \in \mathbb{N}_0\} \quad \text{and} \\ \|d(x) - a\| - \varepsilon < \text{dist}(a, \text{conv}\{f(P^n(x)) : n \in \mathbb{N}_0\}).$$

Given $y \in \mathcal{M}$ we define a function $\varphi^y: \mathbb{N}_0 \rightarrow \mathfrak{X}$ by $\varphi^y(k) := d(P^k(y))$. Put $\varphi_1^y(k) := \varphi^y(k+1)$; see (i). It follows that $\varphi^{P(y)}(k) = d(P^k(P(y))) = d(P^{k+1}(y)) = \varphi^y(k+1) = \varphi_1^y(k)$, which means

$$\varphi^{P(y)} = \varphi_1^y \quad \text{for all } y \in \mathcal{M}. \quad (5)$$

By Theorem 1 (with $S = \mathbb{N}_0$) there is an invariant mean $M: \mathcal{B}(\mathbb{N}_0, \mathfrak{X}) \rightarrow \mathfrak{X}$. Now we define $F: \mathcal{M} \rightarrow \mathfrak{X}$ by

$$F(x) := M(\varphi^x) \quad \text{for } x \in \mathcal{M}. \quad (6)$$

Applying in turn (5) and (i), we have $F(P(x)) = M(\varphi^{P(x)}) = M(\varphi_1^x) = M(\varphi^x) = F(x)$, so the function F is P -periodic. It is easy to see that

$$d(P^k(x)) \in \text{conv}\{f(P^{n+k}(x)) : n \in \mathbb{N}_0\} \subseteq \text{conv}\{f(P^n(x)) : n \in \mathbb{N}_0\}.$$

Thus we get $d(P^k(x)) \in \overline{\text{conv}}\{f(P^n(x)) : n \in \mathbb{N}_0\} \subseteq \Gamma(x)$ for all k in $\mathbb{N}_0$. Therefore, $\varphi^x(\mathbb{N}_0) \subseteq \overline{\text{conv}}\{f(P^n(x)) : n \in \mathbb{N}_0\} \subseteq \Gamma(x)$, and so by (ii) we obtain $F(x) = M(\varphi^x) \in \overline{\text{conv}}\{f(P^n(x)) : n \in \mathbb{N}_0\}$.

Putting $P^k(x)$ in place of x in the above equality we get $F(P^k(x)) \in \overline{\text{conv}}\{f(P^n(P^k(x))) : n \in \mathbb{N}_0\}$. On the other hand, we also have $F(P^k(x)) = F(x)$, hence $F(x) \in \overline{\text{conv}}\{f(P^{n+k}(x)) : n \in \mathbb{N}_0\}$. Since $k \in \mathbb{N}_0$ is arbitrarily chosen, we conclude that $F(x) \in \bigcap_{m \in \mathbb{N}} \overline{\text{conv}}\{f(P^n(x)) : n > m\}$. $\square$

Theorem 3. *Let $\mathcal{M}$ be a nonempty set and assume that $P: \mathcal{M} \rightarrow \mathcal{M}$ is not a constant mapping. Suppose that $h: \mathcal{M} \rightarrow \mathbb{R}$ is a P -super-periodic function, i.e. $h(P(\cdot)) \geq h(\cdot)$, and $g: \mathcal{M} \rightarrow \mathbb{R}$ is a P -sub-periodic, i.e. $g(P(\cdot)) \leq g(\cdot)$. Then the functions $h, g: \mathcal{M} \rightarrow \mathbb{R}$ satisfy the condition*

$$\forall x \in \mathcal{M} \quad h(x) \leq g(x), \quad (7)$$

if and only if there is a P -periodic function $F: \mathcal{M} \rightarrow \mathbb{R}$ such that $h \leq F \leq g$.

Proof. Assume (7). Define $f: \mathcal{M} \rightarrow \mathbb{R}$ by $f(x) := \frac{1}{2}h(x) + \frac{1}{2}g(x)$. By the assumptions we have $h(P^n(x)) \geq h(x)$ and $g(P^n(x)) \leq g(x)$. It follows that $h(x) \leq f(P^n(x)) \leq g(x)$ for all x in $\mathcal{M}$. For the set-valued function $\Gamma: \mathcal{M} \rightarrow 2^{\mathbb{R}}$ given by $\Gamma(x) := [h(x), g(x)]$ we have $f(P^n(x)) \in \Gamma(x)$. Now applying Theorem 2 we arrive at the desired assertion. The converse implication has a trivial verification. $\square$

The above result can be strengthened (in some circumstances) as follows.

Theorem 4. *Let $\mathcal{M}$ be a nonempty set assume that $P: \mathcal{M} \rightarrow \mathcal{M}$ is not a constant mapping. Suppose that $h: \mathcal{M} \rightarrow \mathbb{R}$ is a P -super-periodic function. Then the functions $h, g: \mathcal{M} \rightarrow \mathbb{R}$ satisfy the condition*

$$\forall x \in \mathcal{M} \quad \forall n \in \mathbb{N}_0 \quad h(P^n(x)) \leq g(x), \quad (8)$$

if and only if there is a P -periodic function $F: \mathcal{M} \rightarrow \mathbb{R}$ such that $h \leq F \leq g$.

Proof. Assume (8). It is clear that $h(x) \leq h(P^n(x)) \leq g(x)$. Define $\hat{g}: \mathcal{M} \rightarrow \mathbb{R}$ by $\hat{g}(x) := \sup\{h(P^n(x)) : n \in \mathbb{N}_0\}$. It is easy to check that $h(x) \leq \hat{g}(x) \leq g(x)$ and $\{h(P^{n+1}(x)) : n \in \mathbb{N}_0\} \subseteq \{h(P^n(x)) : n \in \mathbb{N}_0\}$. Therefore we obtain $\hat{g}(P(x)) = \sup\{h(P^{n+1}(x)) : n \in \mathbb{N}_0\} \leq \sup\{h(P^n(x)) : n \in \mathbb{N}_0\} = \hat{g}(x)$, which means that $\hat{g}$ is P -sub-periodic. Moreover $h(x) \leq \hat{g}(x)$. Applying Theorem 3, we complete the proof. The converse result is immediate. $\square$

The next result shows that the Lipschitz property of a P - Γ -periodic function can be inherited by some P -periodic function.

Theorem 5. *Let $(\mathcal{M}, \varrho)$ be a metric space and $\mathfrak{X}$ a reflexive Banach space. Suppose that $P: \mathcal{M} \rightarrow \mathcal{M}$ is a uniformly Lipschitz function, i.e. there is $\gamma > 0$ such that*

$$\varrho(P^n(x), P^n(y)) \leq \gamma \cdot \varrho(x, y) \quad \text{for } x, y \in \mathcal{M};$$

$\Gamma: \mathcal{M} \rightarrow 2^{\mathfrak{X}}$ is a set-valued function such that for all $x \in \mathcal{M}$ the set $\Gamma(x)$ is bounded, closed, convex; and $f: \mathcal{M} \rightarrow \mathfrak{X}$ is a P - Γ -periodic function. If f is a Lipschitz function with Lipschitz constant β , then there is a P -periodic function $F: \mathcal{M} \rightarrow \mathfrak{X}$ such that F is a function with Lipschitz constant $\beta\gamma$.

Proof. Given $y \in \mathcal{M}$ we define a function $\varphi^y: \mathbb{N}_0 \rightarrow \mathfrak{X}$ by $\varphi^y(k) := f(P^k(y))$. Put $\varphi_1^y(k) := \varphi^y(k+1)$. It follows that $\varphi^{P(y)}(k) = f(P^k(P(y))) = f(P^{k+1}(y)) = \varphi^y(k+1) = \varphi_1^y(k)$, which means

$$\varphi^{P(y)} = \varphi_1^y \quad \text{for all } y \in \mathcal{M}. \quad (9)$$

Now Theorem 1 implies that there is an invariant mean $M: \mathcal{B}(\mathbb{N}_0, \mathfrak{X}) \rightarrow \mathfrak{X}$. Define a function $F: \mathcal{M} \rightarrow \mathfrak{X}$ by the formula $F(x) := M(\varphi^x)$ for $x \in \mathcal{M}$. Hence, by (9) and (i) we get $F(P(x)) = M(\varphi^{P(x)}) = M(\varphi_1^x) = M(\varphi^x) = F(x)$, so F is P -periodic.

Finally, we show that f is a Lipschitz function. Recall that the function M is a continuous operator with $\|M\| = 1$. From the definition of F , we have

$$\begin{aligned} \|F(x) - F(y)\|_{\mathfrak{X}} &= \|M(\varphi^x) - M(\varphi^y)\|_{\mathfrak{X}} \leq \|M\| \cdot \|\varphi^x - \varphi^y\|_{\infty} \\ &= \|M\| \cdot \sup \{\|\varphi^x(k) - \varphi^y(k)\|_{\mathfrak{X}} : k \in \mathbb{N}_0\} \\ &= \sup \{\|f(P^k(x)) - f(P^k(y))\|_{\mathfrak{X}} : k \in \mathbb{N}_0\} \\ &\leq \sup \{\beta \varrho(P^k(x), P^k(y)) : k \in \mathbb{N}_0\} \leq \beta \gamma \varrho(x, y). \end{aligned}$$

The Proof of Theorem 5 is complete. $\square$

3. Sandwich with functional equation $f \circ p = r \circ f$

In the previous section we saw that the problem of determining periodic (resp. p -super-periodic) functions leads to the functional equation $f(p(x)) = f(x)$ (resp. the functional inequality $f(p(x)) \geq f(x)$). So, it becomes quite natural to consider the functional equation $f(p(x)) = r(f(x))$ (resp. the functional inequality $f(p(x)) \geq r(f(x))$).

The results presented in this section are of the type as in the previous section, but focused on the functional equation $f(p(x)) = r(f(x))$ instead of periodic functions. First we prove a sandwich type result for the functional equation $f(p(x)) = g(f(x))$.

Theorem 6. Let $I \subseteq \mathbb{R}$ be a compact interval. Assume that functions $p: I \rightarrow I$ and $r: I \rightarrow I$ satisfy the following conditions: p is invertible and

$$|p^{-1}(x) - p^{-1}(y)| \leq \alpha(|x - y|), \quad x, y \in I, \quad (10)$$

where $\alpha: [0, \infty) \rightarrow [0, \infty)$ is right-continuous at 0 and $\alpha(0) = 0$; r is increasing, continuous and

$$|r(x) - r(y)| \leq \beta(|x - y|), \quad x, y \in I; \quad (11)$$

where $\beta: [0, \infty) \rightarrow [0, \infty)$ is increasing, right continuous at 0 and $\beta(0) = 0$. Assume that there is a right-continuous at 0 function $\gamma: [0, \infty) \rightarrow [0, \infty)$, with $\gamma(0) = 0$ such that

$$\beta \circ \gamma \circ \alpha \leq \gamma. \quad (12)$$

If there exist continuous functions $f, g: I \rightarrow I$ such that

$$f \leq g, \quad (13)$$

$$f \circ p \leq r \circ f, \quad (14)$$

$$r \circ g \leq g \circ p, \quad (15)$$

and for some $h_0: I \rightarrow I$, we have $f \leq h_0 \leq g$ and $|h_0(x) - h_0(y)| \leq \gamma(|x - y|)$ for all $x, y \in I$, then there is a continuous function $h: I \rightarrow I$ such that

$$f \leq h \leq g \quad (16)$$

and satisfying the equality

$$h \circ p = r \circ h.$$

Proof. Let $\mathcal{C}(I)$ be the Banach space of all continuous functions $\varphi: I \rightarrow \mathbb{R}$ with the supremum norm $\|\varphi\|_\infty := \sup \{|\varphi(x)| : x \in I\}$, and denote by $\mathcal{F}$ the family of all continuous functions $\varphi: I \rightarrow I$ such that

$$f \leq \varphi \leq g, \quad (17)$$

and

$$|\varphi(x) - \varphi(y)| \leq \gamma(|x - y|), \quad x, y \in I. \quad (18)$$

By the assumption, $\mathcal{F}$ is nonempty. Define the mapping $T: \mathcal{F} \rightarrow \mathcal{C}(I)$ by

$$T(\varphi) := r \circ \varphi \circ p^{-1}.$$

Then, for every φ and $x \in I$, applying in turn: inequality (14) together with the invertibility of p ; the monotonicity of r and the inequality in (17); again the monotonicity of r and the second inequality in (17); the inequality in (15), we obtain

$$f(x) \leq r(f(p^{-1}(x))) \leq r(\varphi(p^{-1}(x))) \leq r(g(p^{-1}(x))) \leq g(x),$$

which shows that

$$f \leq T(\varphi) \leq g.$$

Moreover, using in turn: the definition of T ; inequality (11); inequality (10); and (12), we get for arbitrary $\varphi \in \mathcal{F}$ and for all $x, y \in I$,

$$\begin{aligned} |T(\varphi)(x) - T(\varphi)(y)| &= |r(\varphi(p^{-1}(x))) - r(\varphi(p^{-1}(y)))| \\ &\leq \beta(|\varphi(p^{-1}(x)) - \varphi(p^{-1}(y))|) \\ &\leq \beta(\gamma(|p^{-1}(x) - p^{-1}(y)|)) \\ &\leq \beta(\gamma(\alpha(|x - y|))) \\ &\leq \gamma(|x - y|), \end{aligned}$$

which shows that, $T(\varphi)$ satisfies inequality (18) for every $\varphi \in \mathcal{F}$. This proves that T maps $\mathcal{F}$ into itself. Clearly, $\mathcal{F}$ is a closed subset of $C(I)$. It follows from the continuity of γ at 0 and $\gamma(0) = 0$, and the Arzela Theorem that $\mathcal{F}$ is a compact subset of $C(I)$. Moreover, one should be able to verify with little effort that $\mathcal{F}$ is convex.

Now we show that T is continuous. Take $\varphi_n \in \mathcal{F}, n \in \mathbb{N}; \varphi \in \mathcal{F}$ such $\lim_{n \rightarrow \infty} \varphi_n = \varphi$. In turns making use the definition of the norm $\|\cdot\|_\infty$; the bijectivity of p ; inequality (11); the monotonicity of β and the continuity of the involved functions on the compact interval, we have

$$\begin{aligned} \|T(\varphi_n) - T(\varphi)\|_\infty &= \|r \circ \varphi_n \circ p^{-1} - r \circ \varphi \circ p^{-1}\|_\infty \\ &= \sup \{|r(\varphi_n(p^{-1}(x))) - r(\varphi(p^{-1}(x)))| : x \in I\} \\ &= \sup \{|r(\varphi_n(x)) - r(\varphi(x))| : x \in I\} \\ &\leq \sup \{\beta(|\varphi_n(x) - \varphi(x)|) : x \in I\} \\ &\leq \beta(\max \{|\varphi_n(x) - \varphi(x)| : x \in I\}) \\ &= \beta(\|\varphi_n - \varphi\|_\infty), \end{aligned}$$

whence $\lim_{n \rightarrow \infty} T(\varphi_n) = T(\varphi)$, so T is continuous. The Schauder Fixed-Point Theorem implies that there is a functional $h \in \mathcal{F}$ such that $T(h) = h$, i.e. $r \circ h \circ p^{-1} = h$. Thus $h \circ p = r \circ h$. $\square$

Next we would like to present an application of the above theorem to Hölder functions. For a number $q \in (0, 1]$, by $\mathcal{H}_q(I)$ denote the space of Hölder continuous functions $\varphi : I \rightarrow \mathbb{R}$, that is such that

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^q, \quad x, y \in I,$$

for some $C \geq 0$. As an immediate consequence of the above result we obtain:

Corollary 7. *Let $I \subseteq \mathbb{R}$, p, r, f, g, h_0 satisfy the conditions of Theorem 6 with $\alpha(t) = \alpha(1)t, \beta(t) = \beta(1)t$ and $\gamma(t) = \gamma(1)t^q$, for some $q \in (0, 1]$. If*

$$\beta(1) \leq [\alpha(1)]^{-q},$$

then there exists a function $h \in \mathcal{H}_q(I)$ such that

$$f \leq h \leq g,$$

and satisfying the equality

$$h \circ p = r \circ h.$$

We conclude this work by proving two results concerning the functional equation $\alpha f(p(x)) = g(f(x))$ and the functional inequality $\alpha f(p^n(x)) \geq g^n(f(x))$.

Theorem 8. *Let $\mathcal{M} \subseteq (\mathbb{R}^n, \|\cdot\|)$ be a compact subset with $\text{int}\mathcal{M} \neq \emptyset$, and suppose that $\mathfrak{X}, \mathcal{W}, \mathcal{Z}$ are nontrivial Banach spaces such that $\mathfrak{X} = \mathcal{W} + \mathcal{Z}$ and $\mathcal{W} \cap \mathcal{Z} = \{0\}$, i.e. $\mathfrak{X}$ is the direct sum of $\mathcal{W}$ and $\mathcal{Z}$. Assume that $p: \mathcal{M} \rightarrow \mathcal{M}$*

is continuous. Suppose that $g: \mathfrak{X} \rightarrow \mathfrak{X}$ is a continuous surjection such that the restriction $g|_{\mathcal{W}}: \mathcal{W} \rightarrow \mathfrak{X}$ is a surjective isometry (not necessarily linear), and $\mathcal{Z} \subseteq \{x \in \mathfrak{X} : g(x) = 0\}$. Then for every $\alpha \in (0, 1)$ there exists a continuous function $f: \mathcal{M} \rightarrow \mathfrak{X}$ such that

$$\forall_{x \in \mathcal{M}} \quad \alpha f(p(x)) = g(f(x))$$

and f is not a constant function. Moreover, if $\dim \mathcal{Z} \geq n$, then the solution can be chosen in such a way that f is injective.

Proof. Let $N: \mathbb{R}^n \rightarrow \mathcal{Z}$ be a continuous linear mapping with $\|N\| \leq 1$. The operator N can be chosen in such a way that $N|_{\mathcal{M}}$ is not constant. Since $N(\mathcal{M})$ is compact and $N(\mathcal{M}) \subseteq \mathcal{Z} \subseteq \mathfrak{X}$, without loss of generality, it may be assumed that the set $\mathcal{M}$ is contained in the closed unit ball $B_{\mathfrak{X}} := \{y \in \mathfrak{X} : \|y\| \leq 1\}$. By $\mathcal{C}(\mathcal{M}, \mathfrak{X})$ we denote the space of continuous $\mathfrak{X}$ -valued functions with the supremum norm $\|\cdot\|_{\infty}$, and set $B_{\mathcal{C}} := \{f \in \mathcal{C}(\mathcal{M}, \mathfrak{X}) : \|f\|_{\infty} \leq 1\}$.

Put $g_1 := (g|_{\mathcal{W}})^{-1}: \mathfrak{X} \rightarrow \mathcal{W}$ and define a mapping $\varphi: B_{\mathcal{C}} \rightarrow B_{\mathcal{C}}$ by the formula $\varphi(f) := (1 - \alpha)N|_{\mathcal{M}} + \alpha g_1 \circ f \circ p$. It is easy to verify that φ is a contraction. By the Banach Fixed-Point Theorem, there is a function $f_{\alpha}: \mathcal{M} \rightarrow \mathfrak{X}$ such that $(1 - \alpha)N|_{\mathcal{M}} + \alpha g_1 \circ f_{\alpha} \circ p = f_{\alpha}$.

Since $g|_{\mathcal{W}}: \mathcal{W} \rightarrow \mathfrak{X}$ is a surjective isometry, obviously, so is $g_1: \mathfrak{X} \rightarrow \mathcal{W}$. The Mazur-Ulam Theorem implies that g_1 is an affine mapping. Since $g \circ g_1 = \text{Id}_{\mathfrak{X}}$, we have

$$\begin{aligned} g(f_{\alpha}(x)) &= g((1 - \alpha)N|_{\mathcal{M}}(x) + \alpha(g_1 \circ f_{\alpha} \circ p)(x)) \\ &= (1 - \alpha)g(N|_{\mathcal{M}}(x)) + \alpha g(g_1(f_{\alpha} \circ p(x))) \\ &= (1 - \alpha) \cdot 0 + \alpha f_{\alpha}(p(x)) = \alpha f_{\alpha}(p(x)), \end{aligned}$$

whence $g \circ f_{\alpha} = \alpha f_{\alpha} \circ p$. Now, we prove that f_{α} is not constant. Assume, contrary to our claim, that $f_{\alpha}(\cdot) = c$ for some $c \in \mathfrak{X}$. It follows that $(g_1 \circ f_{\alpha} \circ p)(\cdot) = g_1(c)$, whence $(1 - \alpha)N|_{\mathcal{M}}(\cdot) = c - \alpha g_1(c)$. Thus the function $N|_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{Z}$ would be constant, which is a contradiction.

Now assume that $\dim \mathcal{Z} \geq n$. Then there is a continuous injective linear operator $N: \mathbb{R}^n \rightarrow \mathcal{Z}$. Repeating the above reasoning we obtain the equality $(1 - \alpha)N|_{\mathcal{M}} + \alpha g_1 \circ f_{\alpha} \circ p = f_{\alpha}$. We show that f_{α} is an injection. Suppose that $f_{\alpha}(x) = f_{\alpha}(y)$. Then

$$(1 - \alpha)N(x) + \alpha g_1(f_{\alpha}p(x)) = f_{\alpha}(x) = f_{\alpha}(y) = (1 - \alpha)N(y) + \alpha g_1(f_{\alpha}p(y)).$$

Note that $\mathfrak{X} = \mathcal{W} \oplus \mathcal{Z}$, $g_1(\mathfrak{X}) \subseteq \mathcal{W}$ and $N(x), N(y) \in \mathcal{Z}$. So the uniqueness of the decomposition implies that $N(x) = N(y)$. Since N is injective, $x = y$. $\square$

Bearing the Proof of Theorem 8 in mind, we can formulate and prove another result. More precisely, now we want to study the existence of solutions f of the functional inequality $f(p(x)) \geq g(f(x))$.

Theorem 9. Let $\mathcal{M} \subseteq (\mathbb{R}^n, \|\cdot\|)$ be a compact convex subset. Assume that $p: \mathcal{M} \rightarrow \mathcal{M}$ is a continuous mapping. Let $0 \leq d < b < c < a$. Suppose that $g: [0, a] \rightarrow [0, a]$ is a continuous function such that $[d, b] = \{x \in [0, a] : g(x) = 0\}$ and assume that $g|_{[0, c]}: [0, c] \rightarrow [0, a]$ is convex. Moreover, let $g|_{[b, c]}: [b, c] \rightarrow [0, a]$ be an increasing bijection and assume that $g_1 := (g|_{[b, c]})^{-1}$ is a Lipschitz function with Lipschitz constant $\text{Lip}(g_1) \leq 1$. Then there exists a continuous function $f: \mathcal{M} \rightarrow \mathbb{R}$ such that

$$\forall_{x \in \mathcal{M}} \forall_{n \in \mathbb{N}} \quad g^n(f(x)) \leq \alpha f(p^n(x));$$

in particular $g \circ f \leq f \circ p$. Moreover, the solution f can be chosen to be not constant. If $\mathcal{M} \subseteq \mathbb{R}$ is a compact interval and $p: \mathcal{M} \rightarrow \mathcal{M}$ is increasing, then f can be chosen to be nondecreasing and not constant.

Proof. Fix a number $\alpha \in \left(\frac{b}{g_1(b)}, 1\right)$ and define a set

$$W := \{f: \mathcal{M} \rightarrow [b, a] \mid f \text{ is continuous}\}.$$

Consider a continuous mapping $\mu: \mathcal{M} \rightarrow [d, b]$. Suppose that μ is not constant. Define a mapping $\varphi: W \rightarrow W$ by $\varphi(f) := (1 - \alpha)\mu + \alpha g_1 \circ f \circ p$ for all $f \in W$.

The mapping φ is well-defined. Indeed, since the range of $f \circ p$ is bounded from below by b , the monotonicity of g_1 implies that $\varphi(f)(t) = (1 - \alpha)\mu(t) + \alpha g_1(f(p(t))) \geq \alpha g_1(b) \geq b$, so $\varphi(f)(t) \in [b, a]$ for every $t \in \mathcal{M}$. By the Banach Fixed-Point Theorem we get

$$f = (1 - \alpha)\mu + \alpha g_1 \circ f \circ p \tag{19}$$

for some $f \in W$. Since g is convex and $g \circ \mu = 0$, it follows from (19) that $g \circ f \leq \alpha f \circ p$. Combining this inequality and again (19), we get

$$g \circ f \leq \alpha f \circ p \leq f \circ p = (1 - \alpha)\mu \circ p + g_1 \circ f \circ p^2.$$

Since g is increasing in $[b, c]$ and convex, we obtain $g \circ g \circ f \leq \alpha \circ f \circ p^2$, that $g^2 \circ f \leq \alpha \circ f \circ p^2$. Now, taking into account that g^n is increasing for $n \geq 2$, we can prove, by induction, that $g^n \circ f \leq \alpha \circ f \circ p^n$.

Moreover, if $\mathcal{M} \subseteq \mathbb{R}$ is a compact interval, then we may choose an increasing function $\mu: \mathcal{M} \rightarrow [d, b]$. Next we define

$$W := \{f: \mathcal{M} \rightarrow [b, a] \mid f \text{ is continuous and nondecreasing}\}.$$

and consider a similar mapping $\varphi: W \rightarrow W$. The rest runs similarly. Finally, in this case it is easy to notice that the fixed point f is a nondecreasing function. $\square$

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