Aequat. Math. 94 (2020), 1269–1279

0001-9054/20/061269-11
published online January 18, 2020
https://doi.org/10.1007/s00010-020-00708-3

Aequationes Mathematicae

D'Alembert's and Wilson's equations on semigroups

Radosław Łukasik

Abstract. In this paper we consider a generalization of d'Alembert's equation and Wilson's equation on commutative semigroups using only the semigroup operation, ie. we consider the functional equation

$$h(x+2y)+h(x)=2f(y)h(x+y),\ x,y\in S,$$

where $f,h\colon S\to\mathbb{K}$, $(S,+)$ is a commutative semigroup, $\mathbb{K}$ is a quadratically closed field, $\operatorname{char}\mathbb{K}\neq 2$.

Mathematics Subject Classification. Primary 39B52; Secondary 20M14.

Keywords. D'Alembert equation, Wilson equation, Semigroup.

1. Introduction

If we look at d'Alembert's functional equation

$$f(x+y)+f(x-y)=2f(x)f(y),\ x,y\in S, \tag{1.1}$$

on a group S, then we have two possible ways of generalizing it to semigroups.

The first one is the functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)f(y),\ x,y\in S, \tag{1.2}$$

where $(S,+)$ is an abelian semigroup, $\sigma\in\operatorname{Aut}(S)$ is an involution, which has been studied by many mathematicians, (see [3,4,7,12–14,16,18,19]). For non-abelian groups the solutions of d'Alembert's functional equation may be different from those of the abelian case (see [8,10,11,20]).

The above equation has a generalization of the form

$$\int_K f(x+\lambda y)d\mu(\lambda)=f(x)f(y),\ x,y\in G,$$

where $(G, +)$ is a locally compact group, K is a compact subgroup of the automorphism group of G with the normalized Haar measure μ, $f\colon G \to \mathbb{C}$. It is a generalization of the cosine equation and it is studied in the theory of group representations, being the relation defining K-spherical functions (for the terminology see [5, p. 88]). This equation has been studied by many mathematicians (for example see [6,15,17,21,22]).

The second way is the functional equation

$$f(x+2y) + f(x) = 2f(x+y)f(y),\ \ x, y \in S, \tag{1.3}$$

which we obtain by the substitution $x \mapsto x + y$. This equation is equivalent to d'Alembert's functional equation on groups and we will show that its solutions are the same as those of d'Alembert's functional equation.

2. Preliminaries

Throughout the present paper, we assume that $(S, +)$ is an abelian semigroup and the relation $\sim \subseteq S \times S$ is given by

$$\forall_{x,y\in S}\left(x \sim y \Leftrightarrow \exists_{z\in S}\ (x + z = y + z)\right), \tag{2.1}$$

$\mathbb{K}$ is a quadratically closed field, $\operatorname{char}\mathbb{K} \neq 2$.

Lemma 2.1. ([15, Lemma 2.2]) *The relation $\sim$ given by (2.1) is an equivalence relation, $S/_\sim$ with the operation $+\colon {S/_\sim}^2 \to S/_\sim$ defined by*

$$[x]_\sim + [y]_\sim := [x + y]_\sim,\ \ x, y \in S, \tag{2.2}$$

is a cancellative abelian semigroup and the function $\varkappa\colon S \to S/_\sim$ given by

$$\varkappa(x) = [x]_\sim,\ \ x \in S, \tag{2.3}$$

is a semigroup epimorphism.

Theorem 2.2. ([18, Theorem 1]) *Let $\sigma\colon S \to S$ be an involution, $f\colon S \to \mathbb{K}$. Then f satisfies Eq. (1.2) iff there exists an exponential function $m\colon S \to \mathbb{K}$ such that*

$$f(x) = \frac{m(x) + m(\sigma x)}{2},\ \ x \in S. \tag{2.4}$$

The exponential function m from the above theorem is on groups either zero everywhere or non-zero everywhere (e.g. $m\colon S \to \mathbb{K}^*$ is a homomorphism). Similarly on semigroups, m has the same zero behavior as in groups iff f satisfies

$$f(x + \sigma x) \neq 0,\ \ x \in S, \tag{2.5}$$

(see [15, Theorem 2.8]). But generally on semigroups there may exist exponential functions which have zeros on some non-trivial subset of S.

Example. Let $c \in \mathbb{K} \setminus \{0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $S = \mathbb{N}_0 \times \mathbb{N}_0$, $\sigma\colon S \to S$ be a function given by $\sigma(n,k) = (k,n)$ for $n,k \in \mathbb{N}_0$. We define functions $f, m\colon S \to \mathbb{K}$ by the formulas

$$f(n,k) = \begin{cases} 1, & n = k = 0 \\ 0 & n,k \geq 1 \\ 2^{n-1}c^n, & k = 0,\ n \geq 1 \\ 2^{k-1}c^k, & n = 0,\ k \geq 1 \end{cases},$$

$$m(n,k) = \begin{cases} 0 & k \geq 1 \\ (2c)^n, & k = 0 \end{cases}.$$

Let further $n,k,p,q \in \mathbb{N}_0$. We observe that:

- if $k \geq 1$ or $q \geq 1$, then

$$m(n+p, k+q) = 0 = m(n,k) \cdot m(p,q).$$

- if $k = q = 0$, then

$$\begin{aligned} m(n+p, k+q) &= m(n+p, 0) = (2c)^{n+p} = (2c)^n \cdot (2c)^p \\ &= m(n,0) \cdot m(p,0) = m(n,k) \cdot m(p,q). \end{aligned}$$

Hence m is an exponential function. We observe also that

- if $k = n = 0$, then

$$m(n,k) + m(k,n) = 2 = 2f(n,k).$$

- if $k, n \geq 1$, then

$$m(n,k) + m(k,n) = 0 = f(n,k).$$

- if $k \geq 1$ and $n = 0$, then

$$m(n,k) + m(k,n) = 0 + (2c)^k = 2 \cdot 2^{k-1}c^k = f(n,k).$$

- if $k = 0$ and $n \geq 1$, then

$$m(n,k) + m(k,n) = (2c)^n + 0 = 2 \cdot 2^{n-1}c^n = f(n,k).$$

Hence f satisfies Eq. (2.4) and in view of Theorem 2.2 it satisfies Eq. (1.2). We have also

$$f((n,m) + (m,n)) = f(n+m, m+n) = 0,\ n \neq 0 \vee m \neq 0.$$

3. Main results

Lemma 3.1. *Let $f\colon S \to \mathbb{K}$ satisfy Eq. (1.3). Then the function $\widetilde{f}\colon S/_{\sim} \to \mathbb{K}$ given by the formula $\widetilde{f}(\varkappa(x)) = f(x)$ for $x \in S$ is well-defined and $\widetilde{f}$ satisfies Eq. (1.3).*

Proof. Let $x, y, z \in S$ be such that $x + z = y + z$. Then

$$f(x) = 2f(x+z)f(z) - f(x+2z) = 2f(y+z)f(z) - f(y+2z) = f(y),$$

so $\widetilde{f}$ is well-defined. We have also

$$\widetilde{f}(\varkappa(x)) + \widetilde{f}(\varkappa(x) + 2\varkappa(y)) = f(x) + f(x+2y)$$
$$= 2f(x+y)f(y) = 2\widetilde{f}(\varkappa(x) + \varkappa(y))\widetilde{f}(\varkappa(y)),\ x, y \in S.$$

□

Lemma 3.2. *Assume that S is abelian and cancellative, G is an abelian group such that $G = S - S$. Let $f\colon S \to \mathbb{K}$ satisfy Eq. (1.3). Then the function $F\colon G \to \mathbb{K}$ given by*

$$F(x-y) = 2f(x)f(y) - f(x+y),\ x, y \in S, \tag{3.1}$$

is well-defined, $F|_S = f$ and F satisfies the equation

$$F(x+y) + F(x-y) = 2F(x)F(y),\ x, y \in G.$$

Proof. Let $x, y, u, v \in S$, $x - y = u - v$. Then $x + v = y + u$ and

$$\begin{aligned}
f(x+y) &+ 2f(u)f(v) = 2f(x+y+v)f(v) - f(x+y+2v) + 2f(u)f(v)\\
&= 2f(u+2y)f(v) - f(u+v+2y) + 2f(u)f(v)\\
&= 2\big(f(u+2y) + f(u)\big)f(v) - f(u+v+2y)\\
&= 4f(u+y)f(y)f(v) - f(u+v+2y) = 4f(x+v)f(v)f(y) - f(u+v+2y)\\
&= 2f(x+2v)f(y) + 2f(x)f(y) - f(u+v+2y)\\
&= 2f(u+v+y)f(y) + 2f(x)f(y) - f(u+v+2y)\\
&= f(u+v) + 2f(x)f(y),
\end{aligned}$$

which means that

$$2f(x)f(y) - f(x+y) = 2f(u)f(v) - f(u+v),$$

so F is well-defined. We observe also that

$$F(x) = F(2x - x) = 2f(2x)f(x) - f(3x) = f(x),\ x \in S.$$

Now we show that

$$F(x-y-z) + F(x-y+z) = 2F(x-y)f(z),\ x, y, z \in S. \tag{3.2}$$

Indeed, for $x, y, z \in S$ we have

$$\begin{aligned}
2F(x-y)f(z) &= 2F(x+z-z-y)f(z)\\
&= 4f(x+z)f(y+z)f(z) - 2f(x+z+y+z)f(z)\\
&= 2\Big(f(x) + f(x+2z)\Big)f(y+z) - f(x+z+y) - f(x+z+y+2z)
\end{aligned}$$

$$= 2f(x)f(y+z) - f(x+y+z) + 2f(x+2z)f(y+z) - f(x+2z+y+z)$$
$$= F(x-y-z) + F(x+2z-y-z) = F(x-y-z) + F(x-y+z).$$

Hence, for $x, y, u, v \in S$ we get

$$\begin{aligned} 2F(x-y)F(u-v) &= 4F(x-y)f(u)f(v) - 2F(x-y)f(u+v) \\ &= 2F(x-y+u)f(v) + 2F(x-y-u)f(v) - F(x-y+u+v) \\ &\quad - F(x-y-u-v) = F(x-y+u+v) + F(x-y+u-v) \\ &\quad + F(x-y-u-v) + F(x-y-u+v) - F(x-y+u+v) \\ &\quad - F(x-y-u-v) = F(x-y+u-v) + F(x-y-u+v), \end{aligned}$$

which ends the proof. □

Using Lemmas 2.1, 3.1, 3.2 and the fact that $\varkappa$ is a homomorphism we easily obtain the following result.

Corollary 3.3. *Let G be an abelian group such that $G = S/_\sim - S/_\sim$.*

1. *Let $f: S \to \mathbb{K}$ satisfy Eq. (1.3). Then the function $F: G \to \mathbb{K}$ given by*
$$F(\varkappa(x) - \varkappa(y)) = 2f(x)f(y) - f(x+y), \; x, y \in S, \tag{3.3}$$
is well-defined, $F \circ \varkappa = f$ and F satisfies d'Alembert's functional equation.
2. *Let $F: G \to \mathbb{K}$ satisfy d'Alembert's functional equation. Then $f = F \circ \varkappa: S \to \mathbb{K}$ satisfies Eq. (1.3).*

Theorem 3.4. *Let $f: S \to \mathbb{K}$ be a non-zero function. Then f satisfies Eq. (1.3) iff there exists a homomorphism $m: S \to \mathbb{K}^*$ such that*

$$f(x) = \frac{m(x) + m(x)^{-1}}{2}, \; x \in S. \tag{3.4}$$

Proof. It is easy to check that the function given by (3.4) satisfies Eq. (1.3).

Assume that f satisfies Eq. (1.3). In view of Corollary 3.3 there exists a function $F: G \to \mathbb{K}$ such that $F \circ \varkappa = f$, F satisfies d'Alembert's functional equation, where G is an abelian group such that $G = S/_\sim - S/_\sim$. Hence there exists a homomorphism $M: G \to \mathbb{K}^*$ such that

$$F(x) = \frac{M(x) + M(-x)}{2} = \frac{M(x) + M(x)^{-1}}{2}, \; x \in G.$$

We define $m: S \to \mathbb{K}^*$ by the formula

$$m(x) = M(\varkappa(x)), \; x \in S.$$

Since M and $\varkappa$ are homomorphisms, m is a homomorphism. We have also

$$f(x) = F(\varkappa(x)) = \frac{M(\varkappa(x)) + M(\varkappa(x))^{-1}}{2} = \frac{m(x) + m(x)^{-1}}{2},$$

which ends the proof. □

A well-known generalization of d'Alembert's functional equation is Wilson's functional equation (see e.g. [1] for references).

Solutions of this equation can be found as a special case of some more general functional equation in [2], but we use a more readable result from the paper [9, Theorem 8].

Theorem 3.5. *Let G be an abelian group, $F, H\colon G \to \mathbb{K}$. The ordered pair (F, H) satisfies Wilson's functional equation*

$$H(x+y) + H(x-y) = 2F(y)H(x),\ \ x, y \in G,$$

iff F, H have one of the following forms:

1. *$H = 0$ and F is arbitrary;*
2. *$F(x) = \frac{M(x)+M(x)^{-1}}{2}$, $H(x) = c\frac{M(x)+M(x)^{-1}}{2} + d\frac{M(x)-M(x)^{-1}}{2}$ for $x \in G$;*
3. *$F(x) = M(x)$, $H(x) = M(x)\Big(A(x) + c\Big)$, $M(x) \in \{1, -1\}$ for $x \in G$;*

where $M\colon G \to \mathbb{K}^*$ is a homomorphism, $A\colon G \to \mathbb{K}$ is additive, $c, d \in \mathbb{K}$.

We can equivalently write Wilson's functional equation in the form

$$h(x+2y) + h(x) = 2f(y)h(x+y),\ \ x, y \in S, \tag{3.5}$$

and now we can consider it on semigroups.

Lemma 3.6. *Let $f, h\colon S \to \mathbb{K}$, $h \neq 0$, (f, h) satisfies (3.5) and. Then f is a non-zero function which satisfies (1.3).*

Proof. We observe that

$$\begin{aligned}
&2\Big(f(y+2z) + f(y) - 2f(y+z)f(z)\Big)h(x+y+2z)\\
&\quad = 2f(y+2z)h(x+y+2z) + 2f(y)h(x+y+2z)\\
&\qquad - 4f(y+z)f(z)h(z+y+2z) = h(x+2y+4z) + h(x) + h(x+2y+2z)\\
&\qquad + h(x+2z) - 2f(y+z)h(x+y+3z) - 2f(y+z)h(x+y+z)\\
&\quad = h(x+2y+4z) + h(x) + h(x+2y+2z) + h(x+2z) - h(x+2y+2z)\\
&\qquad - h(x) - h(x+2y+4z) - h(x+2z) = 0,\ \ x, y, z \in S.
\end{aligned}$$

Suppose that there exist $y, z \in S$ such that $h(x+y+2z) = 0$ for all $x \in S$. Then

$$\begin{aligned}
h(x+y+z) &= h(x+y+z) + h(x+y+3z)\\
&= 2f(z)h(x+y+2z) = 0,\ \ x \in S,
\end{aligned}$$

so

$$h(x+y) = h(x+y) + h(x+y+2z) = 2f(z)h(x+y+z) = 0,\ \ x \in S,$$

and

$$h(x) = h(x) + h(x+2y) = 2f(y)h(x+y) = 0,\ \ x \in S,$$

which gives us a contradiction. Hence f satisfies (1.3).

Suppose that $f=0$. Then

$$h(2x)+h(2x+2y)=0=h(2y)+h(2y+2x),\ x,y\in S,$$

which means that $h(2x)=0$ for all $x\in S$. We have also

$$h(x+2y)=-h(x),\ x,y\in S,$$

so

$$2h(x)=h(x)+h(x)=-h(x+2y)-h(x+4y)=0,\ x,y\in S,$$

which gives us a contradiction. □

Lemma 3.7. *Let $f,h\colon S\to\mathbb{K}$, $h\neq 0$, (f,h) satisfies Eq. (3.5). Then functions $\widetilde{f},\widetilde{h}\colon S/_\sim\to\mathbb{K}$ given by the formulas $\widetilde{f}(\varkappa(x))=f(x)$, $\widetilde{h}(\varkappa(x))=h(x)$ for $x\in S$ are well-defined and $(\widetilde{f},\widetilde{h})$ satisfies Eq. (3.5).*

Proof. In view of Lemmas 3.1 and 3.6 the map $\widetilde{f}$ is well-defined.

Let $x,y,z\in S$ be such that $x+z=y+z$. Then

$$h(x)=2h(x+z)f(z)-h(x+2z)=2h(y+z)f(z)-h(y+2z)=h(y),$$

so $\widetilde{h}$ is well-defined. We have also

$$\begin{aligned}\widetilde{h}(\varkappa(x))+\widetilde{h}(\varkappa(x)+2\varkappa(y))&=h(x)+h(x+2y)\\&=2h(x+y)f(y)=2\widetilde{h}(\varkappa(x)+\varkappa(y))\widetilde{f}(\varkappa(y)),\ x,y\in S.\end{aligned}$$

□

Lemma 3.8. *Assume that S is abelian and cancellative, G is an abelian group such that $G=S-S$. Let $f,h\colon S\to\mathbb{K}$, $h\neq 0$, (f,h) satisfies Eq. (3.5). Then functions $F,H\colon G\to\mathbb{K}$ given by*

$$F(x-y)=2f(x)f(y)-f(x+y),\ x,y\in S, \tag{3.6}$$

$$H(x-y)=2h(x)f(y)-h(x+y),\ x,y\in S, \tag{3.7}$$

are well-defined, $F|_S=f$, $H|_S=h$ and (F,H) satisfies the equation

$$H(x+y)+H(x-y)=2H(x)F(y),\ x,y\in G.$$

Proof. In view of Lemmas 3.2 and 3.6 the map F is well-defined.

Let $x,y,u,v\in S$, $x-y=u-v$. Then $x+v=y+u$ and

$$\begin{aligned}h(x+y)+2h(u)f(v)&=2h(x+y+v)f(v)-h(x+y+2v)+2h(u)f(v)\\&=2h(u+2y)f(v)-h(u+v+2y)+2h(u)f(v)\\&=2\big(h(u+2y)+h(u)\big)f(v)-h(u+v+2y)\\&=4h(u+y)f(y)f(v)-h(u+v+2y)=4h(x+v)f(v)f(y)-h(u+v+2y)\\&=2h(x+2v)f(y)+2h(x)f(y)-h(u+v+2y)\end{aligned}$$

$$= 2h(u+v+y)f(y) + 2h(x)f(y) - h(u+v+2y)$$
$$= h(u+v) + 2h(x)f(y),$$

which means that

$$2h(x)f(y) - h(x+y) = 2h(u)f(v) - h(u+v),$$

so H is well-defined. We observe also that

$$H(x) = H(2x-x) = 2h(2x)f(x) - h(3x) = h(x), \ x \in S.$$

Now we show that

$$H(x-y-z) + H(x-y+z) = 2f(z)H(x-y), \ x, y, z \in S. \tag{3.8}$$

Indeed, we have

$$\begin{aligned}
2H(x-y)f(z) &= 2H(x+z-z-y)f(z)\\
&= 4h(x+z)f(y+z)f(z) - 2h(x+z+y+z)f(z)\\
&= 2\Big(h(x) + h(x+2z)\Big)f(y+z) - h(x+z+y) - h(x+z+y+2z)\\
&= 2h(x)f(y+z) - h(x+y+z) + 2h(x+2z)f(y+z) - h(x+2z+y+z)\\
&= H(x-y-z) + H(x+2z-y-z)\\
&= H(x-y-z) + H(x-y+z), \ x, y, z \in S.
\end{aligned}$$

Hence, for $x, y, u, v \in S$ we get

$$\begin{aligned}
2F(u-v)H(x-y) &= 4H(x-y)f(u)f(v) - 2H(x-y)f(u+v)\\
&= 2H(x-y+u)f(v) + 2H(x-y-u)f(v) - H(x-y+u+v)\\
&\quad - H(x-y-u-v) = H(x-y+u+v) + H(x-y+u-v)\\
&\quad + H(x-y-u-v) + H(x-y-u+v) - H(x-y+u+v)\\
&\quad - H(x-y-u-v) = H(x-y+u-v) + H(x-y-u+v),
\end{aligned}$$

which ends the proof. □

Using Lemmas 2.1, 3.7, 3.8 and the fact that $\varkappa$ is a homomorphism we easily obtain the following result.

Corollary 3.9. *Let G be an abelian group such that $G = S/_\sim - S/_\sim$.*

1. *Let $f, h \colon S \to \mathbb{K}$, $h \neq 0$, (f, h) satisfies Eq. (3.5). Then functions $F, H \colon G \to \mathbb{K}$ given by*

$$F(\varkappa(x) - \varkappa(y)) = 2f(x)f(y) - f(x+y), \ x, y \in S, \tag{3.9}$$
$$H(\varkappa(x) - \varkappa(y)) = 2h(x)f(y) - h(x+y), \ x, y \in S, \tag{3.10}$$

are well-defined, $F \circ \varkappa = f$, $H \circ \varkappa = h$ and (F, H) satisfies Wilson's functional equation.

2. *Let $F, H \colon G \to \mathbb{K}$, $f = F \circ \varkappa, h = H \circ \varkappa \colon S \to \mathbb{K}$, (F, H) satisfies Wilson's functional equation. Then (f, h) satisfies Eq. (3.5).*

Theorem 3.10. *Let $f, h\colon S \to \mathbb{K}$. Then (f,h) satisfies (3.5) iff f, h have one of the following forms:*

1. $h = 0$ *and* f *is arbitrary;*
2. $f(x) = \frac{m(x)+m(x)^{-1}}{2}$, $h(x) = c\frac{m(x)+m(x)^{-1}}{2} + d\frac{m(x)-m(x)^{-1}}{2}$ *for* $x \in S$;
3. $f(x) = m(x)$, $h(x) = m(x)\Big(a(x)+c\Big)$, $m(x) \in \{1,-1\}$ *for* $x \in S$;

where $m\colon S \to \mathbb{K}^*$ is a homomorphism, $a\colon S \to \mathbb{K}$ is additive, $c, d \in \mathbb{K}$.

Proof. It is easy to check that for functions f, h given by the forms 1–3 the pair (f,h) satisfies Eq. (3.5).

Assume that (f,h) satisfies Eq. (3.5). In view of Corollary 3.9 there exist functions $F, H\colon G \to \mathbb{K}$ such that $F \circ \varkappa = f$, $H \circ \varkappa = h$, (F,H) satisfies Wilson's functional equation, where G is an abelian group such that $G = S/_{\sim} - S/_{\sim}$. Hence we get that F, H have one of the following forms:

1. $H = 0$ and F is arbitrary;
2. $F(x) = \frac{M(x)+M(x)^{-1}}{2}$, $H(x) = c\frac{M(x)+M(x)^{-1}}{2} + d\frac{M(x)-M(x)^{-1}}{2}$ for $x \in G$;
3. $F(x) = M(x)$, $H(x) = M(x)\Big(A(x)+c\Big)$, $M(x) \in \{1,-1\}$ for $x \in G$;

where $M\colon G \to \mathbb{K}^*$ is a homomorphism, $A\colon G \to \mathbb{K}$ is additive, $c, d \in \mathbb{K}$.
We define $m\colon S \to \mathbb{K}^*$, $a\colon S \to \mathbb{K}$ by the formulas

$$\begin{aligned} m(x) &= M(\varkappa(x)),\ x \in S,\\ a(x) &= A(\varkappa(x)),\ x \in S. \end{aligned}$$

Since M and $\varkappa$ are homomorphisms, m is a homomorphism. Since A and $\varkappa$ are additive, a is additive. We have also:

- In case 2

$$\begin{aligned} f(x) &= F(\varkappa(x)) = \frac{M(\varkappa(x)) + M(\varkappa(x))^{-1}}{2} = \frac{m(x)+m(x)^{-1}}{2},\ x \in S,\\ h(x) &= H(\varkappa(x)) = c\frac{M(\varkappa(x)) + M(\varkappa(x))^{-1}}{2} + d\frac{M(\varkappa(x)) - M(\varkappa(x))^{-1}}{2}\\ &= c\frac{m(x)+m(x)^{-1}}{2} + d\frac{m(x)-m(x)^{-1}}{2},\ x \in S, \end{aligned}$$

- In case 3

$$\begin{aligned} f(x) &= F(\varkappa(x)) = M(\varkappa(x)) = m(x),\ x \in S,\\ h(x) &= H(\varkappa(x)) = M(\varkappa(x))\Big(A(\varkappa(x)) + c\Big) = m(x)\Big(a(x)+c\Big),\ x \in S, \end{aligned}$$

which ends the proof. □

References

[1] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
[2] Aczél, J., Chung, J.K., Ng, C.T.: Symmetric second differences in product form on groups. Top. Math. Anal. Ser. Pure Math. **11**, 1–12 (1989)
[3] Badora, R.: On the d'Alembert type functional equation in Hilbert algebras. Funkcial. Ekvac. **43**, 405–418 (2000)
[4] Baker, J.A.: D'Alembert's functional equation in Banach algebras. Acta Sci. Math. (Szeged) **32**, 225–234 (1971)
[5] Benson, C., Jenkins, J., Ratcliff, G.: On Gelfand pairs associated with solvable Lie groups. Trans. Am. Math. Soc. **321**, 85–116 (1990)
[6] Chojnacki, W.: On some functional equation generalizing Cauchy's and d'Alembert's functional equations. Colloq. Math. **55**, 169–178 (1988)
[7] Corovei, I.: The cosine functional equation for nilpotent groups. Aequ. Math. **15**, 99–106 (1977)
[8] Corovei, I.: The d'Alembert functional equation on metabelian groups. Aequ. Math. **57**, 201–205 (1999)
[9] Corovei, I.: Wilson's functional equation on P_3-groups. Aequ. Math. **61**, 212–220 (2001)
[10] Davison, T.M.K.: D'Alembert's functional equation on topological groups. Aequ. Math. **76**(1–2), 33–53 (2008)
[11] Davison, T.M.K.: D'Alembert's functional equation on topological monoids. Publ. Math. Debr. **75**(1/2), 41–66 (2009)
[12] Förg-Rob, W., Schwaiger, J.: A generalization of the cosine equation to n summands. Grazer Math. Ber. **316**, 219–226 (1992)
[13] Gajda, Z.: A remark on the talk of W. Förg-Rob. Grazer Math. Ber. **316**, 234–237 (1992)
[14] Kannappan, P.: The functional equation $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ for groups. Proc. Am. Math. Soc. **19**, 69–74 (1968)
[15] Łukasik, R.: K-spherical functions on abelian semigroups. Bull. Aust. Math. Soc. **96**, 479–486 (2017). https://doi.org/10.1017/S0004972717000417
[16] Penney, R.C., Rukhin, A.L.: D'Alembert's functional equation on groups. Proc. Am. Math. Soc. **77**, 73–80 (1979)
[17] Shin'ya, H.: Spherical matrix functions and Banach representability for locally compact motion groups. Jpn. J. Math. **28**, 163–201 (2002)
[18] Sinopoulos, P.: Functional equations on semigroups. Aequ. Math. **59**, 255–261 (2000)
[19] Stetkær, H.: D'Alembert's equation and spherical functions. Aequ. Math. **48**, 220–227 (1994)

[20] Stetkær, H.: D'Alembert's functional equations on metabelian groups. Aequ. Math. **56**(3), 306–320 (2000)
[21] Stetkær, H.: Functional equations and matrix-valued spherical functions. Aequ. Math. **69**, 271–292 (2005)
[22] Stetkær, H.: On operator-valued spherical functions. J. Funct. Anal. **224**, 338–351 (2005)

Radosław Łukasik
Institute of Mathematics
University of Silesia
ul. Bankowa 14
40-007 Katowice
Poland
e-mail: radoslaw.lukasik@us.edu.pl

Received: October 22, 2019
Revised: January 8, 2020