On a derivation of the Goldstein–Einhorn probability weighting functions

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Dedicated to Professor Janos Aczél on the occasion of his 95-th birthday

Abstract. The well known Goldstein–Einhorn probability weighting functions have been characterized by Gonzalez and Wu using the linear in log odds preference condition. We provide an alternative way of deriving this class of functions. It is based on the concept of indifference prices.

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1. Introduction

Many behavioral models of decision making under risk (e.g. rank-dependent utility, cumulative prospect theory) lead to the well known subjective utility representations involving *a utility* (or *a value*) function and *a probability weighting function*. The latter one is also called *a probability distortion function*. Possible forms of probability weighting functions have been investigated by several authors. Such functions are usually assumed to be non-decreasing and to satisfy the boundary conditions $w(0) = 0$ and $w(1) = 1$. The following classes of probability weighting functions are frequently used in a literature (cf. e.g. [3] and [11]):

- the Prelec function ([14])

$$w(p) = \exp(-\alpha(-\ln p)^\gamma) \quad \text{for } p \in (0, 1], \quad \alpha, \gamma > 0; \quad (1)$$

- the Wu–Gonzalez function ([19])

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^\eta} \quad \text{for } p \in [0, 1], \quad \eta, \gamma > 0;$$

- the Tversky–Kahneman function ([18])

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \quad \text{for } p \in [0, 1], \quad \gamma > 0;$$

- the Goldstein–Einhorn function ([5])

$$w(p) = \frac{\alpha p^\gamma}{\alpha p^\gamma + (1-p)^\gamma} \quad \text{for } p \in [0, 1], \quad \alpha, \gamma > 0. \quad (2)$$

It is worth noting that Prelec [14] was the first to offer an axiomatic approach for estimating probability weighting functions. He derived the functions of the form (1) from a condition called *compound invariance*. Simpler axiomatizations than used in Prelec [14] was provided by Luce [11] and al-Nowaihi and Dhami [12]. Some problems concerning characterization of Prelec functions were investigated in Aczél and Luce [1]. It is remarkable that in Luce [11] some further methods for deriving probability weighting functions, namely *ϕ -reduction invariance* and *ϕ -compound invariance*, were suggested. It was shown in Luce [11] that, applying them, one can characterize several classes of probability weighting functions.

Probability weighting functions of the form (2) were introduced by Goldstein and Einhorn [5]. Some of their variants were applied e.g. by Birnbaum and McIntosh [2], Lattimore, Baker and Witte [10], Tversky and Fox [17]. The special case $\alpha = 1$ has been used by Karmarkar [7, 8]. Gonzalez and Wu [6] established a preference condition, called *the linear-in-log-odds preference condition*, that in the context of rank-dependent utility theory is necessary and sufficient for (2).

The aim of this paper is to give an alternative way of deriving probability weighting functions of the Goldstein–Einhorn type. It is based on the concept of indifference prices. Let us point out that the idea of using utility functions to price random outcomes has a long history. In modern economics literature utility based pricing dates back at least to Pratt [13]. For more details concerning utility based pricing we refer e.g. to Föllmer and Schied [4] and Raiffa [16].

We prove that, under the rank-dependent utility model, the positive homogeneity of the indifference bid price on the family of all binary lotteries characterizes not only the value functions but also the probability weighting functions. Furthermore, we show that this, rather surprising, property of bid prices can be applied to characterization of the Goldstein–Einhorn weighting function. The extension to three or more outcome gambles is investigated as well.

2. Indifference bid prices under rank-dependent utility

Assume that $\mathcal{X}^{(fin)}$ is a family of all *gambles*, that is finitely supported probability distributions on $\mathbb{R}$. For every $n \in \mathbb{N}$, with $n \geq 2$, let $\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle$ be a gamble in which consequences $x_1 < \dots < x_n$ occur with probabilities $p_1, \dots, p_n \in (0, 1)$, respectively. Denote by $\mathcal{X}^{(n)}$ a family of all such gambles. Furthermore, for every $x \in \mathbb{R}$, let $\langle x; 1 \rangle$ be a gamble offering a single payoff x for sure. In the sequel, for $x \in \mathbb{R}$, we identify $\langle x; 1 \rangle$ with x . Assume that $\preceq$ is a preference relation on the family $\mathcal{X}^{(fin)}$ and $\sim$ is the indifference relation on $\mathcal{X}^{(fin)}$, induced by $\preceq$, that is, for every $X, Y \in \mathcal{X}^{(fin)}$,

$$X \sim Y \quad \text{if and only if} \quad X \preceq Y \quad \text{and} \quad Y \preceq X.$$

One of the frequently used notions related to utility based pricing is the indifference bid price. In order to recall it, assume that x_0 is a decision maker's initial wealth. The indifference *bid price* for a given $X \in \mathcal{X}^{(fin)}$ is a real number $P_{x_0}(X)$ such that

$$x_0 - P_{x_0}(X) + X \sim x_0. \quad (3)$$

Condition (3) has the following interpretation. Suppose that an individual has to decide between entering into the gamble X and rejecting it. The left hand side of (3) represents the case where the decision maker decides to play the gamble. In such a case the amount $P_{x_0}(X)$ is subtracted from the decision maker's initial wealth, who now has an opportunity to get one of the outcomes of the gamble. The right hand side of (3) represents the status quo. The decision maker rejects the gamble and remains with the initial wealth x_0 . In general, the bid price need not exist and, even if it exists, it need not be unique. However, if $P_{x_0}(X)$ is uniquely determined, it expresses the maximal price at which the investor, with an initial wealth level x_0 , is willing to buy X .

In what follows we analyze the bid price under *rank-dependent* utility, one of the most frequently applied behavioral alternatives of the classical expected utility model. Assume that the decision maker has a continuous strictly increasing value function $u : \mathbb{R} \rightarrow \mathbb{R}$, with $u(0) = 0$, and a continuous non-decreasing weighting function $w : [0, 1] \rightarrow [0, 1]$, with $w(0) = 0$ and $w(1) = 1$. Then, for every $n \in \mathbb{N}$, a gamble $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$ is evaluated by the decision maker as (we adopt a convention $\sum_{i=n+1}^n p_i = 0$)

$$\text{RDU}(X) = \sum_{k=1}^n \left[w \left(\sum_{i=k}^n p_i \right) - w \left(\sum_{i=k+1}^n p_i \right) \right] u(x_k).$$

Therefore, for every $n \in \mathbb{N}$ and $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$, condition (3) is equivalent to

$$\sum_{k=1}^n \left[w \left(\sum_{i=k}^n p_i \right) - w \left(\sum_{i=k+1}^n p_i \right) \right] u(x_0 - P_{x_0}(X) + x_k) = u(x_0). \quad (4)$$

In the particular case $n = 2$, for every $X = \langle x_1, x_2; 1 - p, p \rangle \in \chi^{(2)}$, we have

$$(1 - w(p))u(x_0 - P_{x_0}(X) + x_1) + w(p)u(x_0 - P_{x_0}(X) + x_2) = u(x_0). \quad (5)$$

Since u is continuous and strictly increasing, for every $n \in \mathbb{N}$ and $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$ there exists a unique real number $P_{x_0}(X)$ such that (4) is valid. Hence, condition (4) defines in an implicit way a functional $P_{x_0} : \mathcal{X}^{(fin)} \rightarrow \mathbb{R}$, which typically does not admit an explicit formula. Nevertheless, it follows from (4) that P_{x_0} is a *mean*, i.e. for every $n \in \mathbb{N}$ and $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$ it holds that

$$x_1 \leq P_{x_0}(\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle) \leq x_n. \quad (6)$$

Furthermore, P_{x_0} is *translation invariant*, that is

$$P_{x_0}(\langle x_1 + z, \dots, x_n + z; p_1, \dots, p_n \rangle) = P_{x_0}(\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle) + z \quad (7)$$

for every $n \in \mathbb{N}$, $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$ and $z \in \mathbb{R}$.

Another desirable property of P_{x_0} is positive homogeneity which, however, generally does not hold. Recall that P_{x_0} is said to be *positively homogeneous* provided, for every $n \in \mathbb{N}$, $X = \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$ and $\lambda \in [0, \infty)$, we have

$$P_{x_0}(\langle \lambda x_1, \dots, \lambda x_n; p_1, \dots, p_n \rangle) = \lambda P_{x_0}(\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle). \quad (8)$$

Positive homogeneity of P_{x_0} implies that the pricing is currency-free, that is the indifference bid price does not depend on a currency in which the outcomes are expressed.

Remark 2.1. Assume that P_{x_0} is positively homogeneous on $\mathcal{X}^{(2)}$, that is, for every $\langle x_1, x_2; 1 - p, p \rangle \in \mathcal{X}^{(2)}$ and $\lambda \in (0, \infty)$, it holds that

$$P_{x_0}(\langle \lambda x_1, \lambda x_2; 1 - p, p \rangle) = \lambda P_{x_0}(\langle x_1, x_2; 1 - p, p \rangle).$$

Then, taking $x_1 = 0$ and $x_2 = 1$, we obtain

$$P_{x_0}(\langle 0, \lambda; 1 - p, p \rangle) = \lambda P_{x_0}(\langle 0, 1; 1 - p, p \rangle) \quad \text{for } \lambda \in (0, \infty), p \in (0, 1).$$

Moreover, in view of (6), we have $P_{x_0}(\langle 0, 1; 1 - p, p \rangle) \in [0, 1]$ for $p \in (0, 1)$. Hence

$$P_{x_0}(\langle 0, x; 1 - p, p \rangle) = f(p)x \quad \text{for } x \in (0, \infty), p \in (0, 1),$$

where $f : (0, 1) \rightarrow [0, 1]$ is given by

$$f(p) = P_{x_0}(\langle 0, 1; 1 - p, p \rangle) \quad \text{for } p \in (0, 1).$$

Furthermore, taking (7) into account, for every $\langle x_1, x_2; 1 - p, p \rangle \in \mathcal{X}^{(2)}$, we get

$$P_{x_0}(\langle x_1, x_2; 1 - p, p \rangle) = P_{x_0}(\langle 0, x_2 - x_1; 1 - p, p \rangle) + x_1 = f(p)(x_2 - x_1) + x_1.$$

Thus

$$P_{x_0}(\langle x_1, x_2; 1 - p, p \rangle) = (1 - f(p))x_1 + f(p)x_2 \quad \text{for } \langle x_1, x_2; 1 - p, p \rangle \in \mathcal{X}^{(2)}. \quad (9)$$

3. Results

In the whole section we assume that a decision maker, having an initial wealth x_0 , a continuous strictly increasing value function $u : \mathbb{R} \rightarrow \mathbb{R}$, with $u(0) = 0$, and a continuous non-decreasing weighting function $w : [0, 1] \rightarrow [0, 1]$ such that $w(0) = 0$ and $w(1) = 1$, follows rank-dependent utility.

The first result characterizes those indifference bid prices whose restriction to the family $\mathcal{X}^{(2)}$ is of the form (9). As we have already mentioned, in the Introduction, this characterization involves not only the value function but also the probability weighting function.

Theorem 3.1. *Let $f : (0, 1) \rightarrow [0, 1]$. The following two statements are equivalent:*

- (i) P_{x_0} is of the form (9);
- (ii) there exist $\alpha, \beta, \gamma \in (0, \infty)$ and $\delta \in \mathbb{R}$ such that

$$u(x) = \begin{cases} -\alpha\beta(x_0 - x)^\gamma + \delta & \text{for } x \in (-\infty, x_0], \\ \beta(x - x_0)^\gamma + \delta & \text{for } x \in (x_0, \infty) \end{cases} \quad (10)$$

and

$$w(p) = \frac{\alpha f(p)^\gamma}{\alpha f(p)^\gamma + (1 - f(p))^\gamma} \quad \text{for } p \in (0, 1). \quad (11)$$

Proof. Assume that (i) holds. Then, applying (5) and (9), for every $\langle x_1, x_2; 1 - p, p \rangle \in \mathcal{X}^{(2)}$, we obtain

$$(1 - w(p))u(x_0 + f(p)(x_1 - x_2)) + w(p)u(x_0 + (1 - f(p))(x_2 - x_1)) = u(x_0).$$

Hence, taking $x_2 = x_1 + x$, where $x \in (0, \infty)$, we conclude that

$$(1 - w(p))v(-f(p)x) + w(p)v((1 - f(p))x) = 0 \quad \text{for } x \in (0, \infty), p \in (0, 1) \quad (12)$$

where $v : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$v(x) := u(x_0 + x) - u(x_0) \quad \text{for } x \in \mathbb{R}. \quad (13)$$

Obviously $v(0) = 0$ and, as u is continuous and strictly increasing, so is v . Therefore, since w is continuous, with $w(0) = 0$ and $w(1) = 1$, from (12) we derive that

$$\lim_{p \rightarrow p_w^+} f(p) = 0, \quad \text{and} \quad \lim_{p \rightarrow 1^-} f(p) = 1 \quad (14)$$

where $p_w := \inf\{p \in (0, 1] : w(p) > 0\}$. Furthermore, if $f(p_0) = 0$ for some $p_0 \in (0, 1)$ then, in view of (12), we get $w(p_0)v(x) = 0$ for $x \in \mathbb{R}$. Thus $w(p_0) = 0$, that is $p_0 \leq p_w$. In this way we have proved that

$$f(p) > 0 \quad \text{for } p \in (p_w, 1]. \quad (15)$$

So, replacing x by $x/f(p)$ in (12), we get

$$v\left(\frac{1-f(p)}{f(p)}x\right) = -\frac{1-w(p)}{w(p)}v(-x) \quad \text{for } x \in (0, \infty), p \in (p_w, 1). \quad (16)$$

Applying (16) with $x = 1$ and using the fact that $v(-1) \neq 0$, we obtain

$$\frac{1-w(p)}{w(p)} = -\frac{1}{v(-1)}v\left(\frac{1-f(p)}{f(p)}\right) \quad \text{for } p \in (p_w, 1).$$

Hence, (16) becomes

$$v\left(\frac{1-f(p)}{f(p)}x\right) = \frac{1}{v(-1)}v\left(\frac{1-f(p)}{f(p)}\right)v(-x) \quad \text{for } x \in (0, \infty), p \in (p_w, 1). \quad (17)$$

Since v and w are continuous and v is strictly increasing, it follows from (16) that $f|_{(p_w, 1)}$ is continuous. Thus, taking (14)–(15) into account, we get

$$(0, \infty) \subseteq \left\{ \frac{1-f(p)}{f(p)} : p \in (p_w, 1) \right\}.$$

Therefore, as $v(0) = 0$, making use of (17), we obtain

$$\phi(xy) = \psi(x)\phi(y) \quad \text{for } x, y \in [0, \infty),$$

where $\phi, \psi : [0, \infty) \rightarrow \mathbb{R}$ are given by

$$\phi(x) := \frac{1}{v(-1)}v(x) \quad \text{for } x \in [0, \infty) \quad (18)$$

and

$$\psi(x) := \frac{1}{v(-1)}v(-x) \quad \text{for } x \in [0, \infty). \quad (19)$$

Note that ϕ and ψ are continuous and, as $v(-1) < 0$, ϕ is strictly decreasing, while ψ is strictly increasing. Thus, applying Theorem 13.3.8 in Kuczma [9], we conclude that there exist $a \in (-\infty, 0)$ and $\gamma \in (0, \infty)$ such that

$$\phi(x) = ax^\gamma \quad \text{for } x \in [0, \infty)$$

and

$$\psi(x) = x^\gamma \quad \text{for } x \in [0, \infty).$$

Hence, from (18)–(19), we derive that

$$v(x) = \begin{cases} v(-1)(-x)^\gamma & \text{for } x \in (-\infty, 0], \\ av(-1)x^\gamma & \text{for } x \in (0, \infty). \end{cases} \quad (20)$$

Moreover, inserting v of the form (20) into (12), for every $x \in (0, \infty)$ and $p \in (0, 1)$, we get

$$v(-1)(1 - w(p))(f(p)x)^\gamma + av(-1)w(p)((1 - f(p))x)^\gamma = 0.$$

Solving this equation for $w(p)$ gives

$$w(p) = \frac{-(1/a)f(p)^\gamma}{-(1/a)f(p)^\gamma + (1 - f(p))^\gamma} \quad \text{for } p \in (0, 1). \quad (21)$$

Therefore, taking (13) and (20) into account, we obtain (10)–(11) with $\alpha := -1/a$, $\beta := av(-1)$ and $\delta := u(x_0)$. Note also that as $a \in (-\infty, 0)$ and $v(-1) < 0$, we have $\alpha, \beta \in (0, \infty)$. Thus, the implication (i) $\implies$ (ii) is proved.

If (ii) holds then, inserting into (5) u and w of the form (10) and (11), respectively, making use of (6) and solving equation obtained in this way with respect to $P_{x_0}(X)$, we get (9). This proves the implication (ii) $\implies$ (i). $\square$

The following result, showing an alternative way to derive the Goldstein–Einhorn probability weighting function under rank-dependent utility, is a direct consequence of Theorem 3.1.

Theorem 3.2. *The following statements are equivalent:*

- (i) *the indifference bid price for every binary gamble coincides with the expected value of the gamble;*
- (ii) *there exist $\alpha, \beta, \gamma \in (0, \infty)$ and $\delta \in \mathbb{R}$ such that u is of the form (10) and*

$$w(p) = \frac{\alpha p^\gamma}{\alpha p^\gamma + (1 - p)^\gamma} \quad \text{for } p \in [0, 1]. \quad (22)$$

It is a natural question, whether the result presented above has an extension to three or more outcome gambles. The following theorem answers the question.

Theorem 3.3. *Let $n \in \mathbb{N}$ be greater than 2. Then, under rank-dependent utility, the indifference bid price for every gamble in $\mathcal{X}^{(n)}$ coincides with the expected value of the gamble if and only if the weighting function is the identity on $[0, 1]$ and the value function is linear.*

Proof. If the weighting function is the identity on $[0, 1]$ and the value function is linear then, in view of (4), we get

$$P_{x_0}(\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle) = \sum_{k=1}^n p_k x_k \quad \text{for } \langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}. \quad (23)$$

Thus, the indifference bid price for every gamble in $\mathcal{X}^{(n)}$ coincides with the expected value of the gamble.

In order to prove a converse implication, assume that (23) holds. Then, for every $x, y, z \in (0, \infty)$ such that $(n-2)x < y < z$, we have

$$P_{x_0}(\langle x, 2x, \dots, (n-2)x, y, z; 1/n, \dots, 1/n \rangle) = \frac{(n-1)(n-2)}{2n}x + \frac{y+z}{n}.$$

Hence, taking (4) into account, we obtain

$$\begin{aligned} \sum_{k=1}^{n-2} \left[w\left(\frac{n+1-k}{n}\right) - w\left(\frac{n-k}{n}\right) \right] u\left(x_0 + kx - \frac{(n-1)(n-2)}{2n}x - \frac{y+z}{n}\right) \\ + \left[w\left(\frac{2}{n}\right) - w\left(\frac{1}{n}\right) \right] u\left(x_0 + y - \frac{(n-1)(n-2)}{2n}x - \frac{y+z}{n}\right) \\ + w\left(\frac{1}{n}\right) u\left(x_0 + z - \frac{(n-1)(n-2)}{2n}x - \frac{y+z}{n}\right) = u(x_0). \end{aligned}$$

Thus, as u is continuous, letting $x \rightarrow 0^+$, for every $y, z \in (0, \infty)$ with $y < z$, we get

$$\alpha v\left(-\frac{y+z}{n}\right) + \beta v\left(\left(1 - \frac{1}{n}\right)y - \frac{z}{n}\right) + (1 - \alpha - \beta)v\left(\left(1 - \frac{1}{n}\right)z - \frac{y}{n}\right) = 0, \quad (24)$$

where $\alpha := 1 - w(2/n)$, $\beta := w(2/n) - w(1/n)$ and $v : \mathbb{R} \rightarrow \mathbb{R}$ is given by (13). Furthermore, taking $(a, b) \in D$, where

$$D := \{(a, b) \in \mathbb{R}^2 : -b/(n-1) < a < b\}$$

and setting

$$y = \frac{(n-1)a + b}{n-2} \quad \text{and} \quad z = \frac{a + (n-1)b}{n-2},$$

we obtain $0 < y < z$,

$$\left(1 - \frac{1}{n}\right)y - \frac{z}{n} = a, \quad \left(1 - \frac{1}{n}\right)z - \frac{y}{n} = b \quad \text{and} \quad -\frac{y+z}{n} = \frac{a+b}{2-n}.$$

Therefore, in view of (24), we get

$$\beta v(a) + (1 - \alpha - \beta)v(b) = -\alpha v((a+b)/(2-n)) \quad \text{for} \quad (a, b) \in D. \quad (25)$$

Note that, by (13), v is strictly increasing and continuous, with $v(0) = 0$. Moreover, we have

$$D_1 := \{a \in \mathbb{R} : (a, b) \in D \text{ for some } b \in \mathbb{R}\} = \mathbb{R}.$$

Thus, applying Theorem 1 in Radó and Baker [15] and Theorem 13.3.5 in Kuczma [9], we obtain that there exists $c \in \mathbb{R}$ such that

$$\beta v(x) = cx \quad \text{for} \quad x \in \mathbb{R}. \quad (26)$$

Suppose that $\beta = 0$. Then, as $((2-n)z-1, 1) \in D$ for $z \in \left(\frac{2}{2-n}, \frac{1}{1-n}\right)$, setting $a = (2-n)z-1$ and $b = 1$ in (25), we get

$$(1-\alpha)v(1) + \alpha v(z) = 0 \text{ for } z \in \left(\frac{2}{2-n}, \frac{1}{1-n}\right).$$

Since v is strictly increasing, with $v(0) = 0$, this yields a contradiction. Therefore $\beta \neq 0$ and so, by (26), v is linear. Thus, as $u(0) = 0$, from (13) we deduce that u is linear. Hence, taking (4) into account, for every $\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \mathcal{X}^{(n)}$, we obtain

$$P(\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle) = \sum_{k=1}^n \left[w \left(\sum_{i=k}^n p_i \right) - w \left(\sum_{i=k+1}^n p_i \right) \right] x_k$$

which, together with (23), for every $\langle x_1, \dots, x_n; p_1, \dots, p_n \rangle \in \chi^{(n)}$ gives

$$\sum_{k=1}^n \left[w \left(\sum_{i=k}^n p_i \right) - w \left(\sum_{i=k+1}^n p_i \right) - p_k \right] x_k = 0 \quad (27)$$

Let $p \in (0, 1)$ and $q := (1-p)/(n-1)$. Then, for every $x \in (0, 1/(n-1))$, we have

$$\langle x, 2x, \dots, (n-1)x, 1; q, q, \dots, q, p \rangle \in \mathcal{X}^{(n)}$$

and so, in view of (27), we obtain

$$\sum_{k=1}^{n-1} [w((n-k)q+p) - w((n-k-1)q+p) - q]kx + w(p) - p = 0.$$

Passing in this equality to the limit with $x \rightarrow 0^+$, we get $w(p) = p$. In this way we have proved that $w(p) = p$ for $p \in (0, 1)$. Since $w(0) = 0$ and $w(1) = 1$, this means that w is the identity on $[0, 1]$. $\square$

4. Conclusion

It is known that the Goldstein–Einhorn probability weighting function can be characterized by the so called linear in log-odds-preference condition. We propose an alternative way of deriving this class of weighting functions. It is based on the concept of indifference prices. We have proved that, under the rank-dependent utility model, a necessary and sufficient condition for the indifference bid price to coincide with the expected value for binary gambles is that the weighting function is of the Goldstein–Einhorn type and the utility function is power, with coefficients strictly related to those of the weighting function. Furthermore we have shown that this phenomenon disappears in the

case of three or more outcome gambles. Namely, in such a setting the indifference bid price coincides with the expected value if and only if the weighting function is the identity on $[0, 1]$ and the value function is linear.

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