On the geometric concomitant equation

ZENON MOSZNER

Dedicated to Professor János Aczél on 95th anniversary of his birth.

Abstract. We solve in some cases the equation in the definition of the concomitant of a geometric object for each of the four functions in this equation. Remarks and examples are given, too.

Mathematics Subject Classification. 39B52.

Keywords. Equation of geometric concomitant, Associated equation, Translation equation.

1. Introduction

For $i = 1, 2$ let S_i be a set and let $(G_i, +)$ be a groupoid. Let the function $F_i : S_i \times G_i \rightarrow S_i$ be the solution of the translation equation, i.e., such that

$$F_i(F_i(x, t_1), t_2) = F_i(x, t_1 + t_2) \text{ for } x \in S_i, t_1, t_2 \in G_i \quad \text{for } i = 1, 2.$$

The equation

$$f[F_1(x, t)] = F_2[f(x), h(t)] \quad \text{for } (x, t) \in S_1 \times G_1, \quad (1)$$

where $f : S_1 \rightarrow S_2$ and h is a homomorphism from G_1 to G_2 , is said to be the geometric concomitant equation.

This equation means that the diagram

$$\begin{array}{ccc} x & \xrightarrow{t} & F_1(x, t) \\ f \downarrow & & \downarrow f \\ f(x) & \xrightarrow{h(t)} & F_2(f(x), h(t)) \end{array} \quad \begin{array}{c} \\ \\ \parallel \end{array}$$

is commutative.

If $G_1 = G_2$, $h(t) = t$ for $t \in G_1$, G_1 has the neutral element 0 and $F_i(x, 0) = x$ for $x \in S_i$, $i = 1, 2$, then in the theory of geometric objects [1] the function F_1 is the transformation law of the object x , $f(x)$ is the concomitant of this object x [4] and F_2 is the transformation law of the object $f(x)$.

If f is a bijection, then F_1 and F_2 are said to be equivalent (e.g., in the theory of abstract automata [11]).

If G is a group, then G/G^* means the family of the right cosets of the group G for the subgroup G^* throughout the paper.

We have four functions in Eq. (1), thus there are four possibilities for the unknown function.

2. Unknown function f

Let G_i be a group and let $F_i(x, 0) = x$ for $x \in S_i$ and $i = 1, 2$. The solution F_i of the translation equation has the form

$$F_i(x, t) = b_i^{-1}[b_i(x) + t] \quad \text{for } (x, t) \in S_i \times G_i, \quad (2)$$

where $b_i = \cup_{k \in K_i} : b_{ik}$, $S_i = \cup_{k \in K_i} : S_{ik}$, S_{ik} for $k \in K_i$ are non-empty disjoint sets such that for every $k \in K_i$ there exists a subgroup G_{ik} of the group G_i for which $\text{card} S_{ik} = \text{card} G_i / G_{ik}$ and $b_{ik} : S_{ik} \rightarrow G_i / G_{ik}$ is a bijection ([5, 7]).

From here the above function F_i and the solution of the translation equation $F_i(u, t) = u + t : (\cup_{k \in K_i} G_i / G_{ik}) \times G_i \rightarrow \cup_{k \in K_i} G_i / G_{ik}$ are equivalent since $b_i(F_i(x, t)) = b_i(x) + t = F_i(b_i(x), t)$. This implies that the solution $F : S \times G \rightarrow S$ such that G is a group and $F(s, 0) = s$ for $s \in S$ may be identified with the family of the right cosets of the group G .

Theorem 1. *If F_i are of the form (2), then a function $f : S_1 \rightarrow S_2$ is a solution of (1) if and only if there exist functions $\alpha : K_1 \rightarrow G_2$ and $\beta : K_1 \rightarrow K_2$ such that*

$$h(G_{1k}) \subset [-\alpha(k) + G_{2\beta(k)} + \alpha(k)] \text{ for } k \in K_1 \quad (3)$$

and f is of the form $f = b_2^{-1} \phi b_1$, where

$$\phi(C) = \phi(G_{1k} + t) = G_{2\beta(k)} + \alpha(k) + h(t) \text{ for } C = G_{1k} + t \in G_1 / G_{1k} \text{ and } k \in K_1. \quad (4)$$

Proof. Note that the function ϕ defined in (4) is well-defined. In fact, if $G_{1k} + t_1 = G_{1k} + t_2$, then $t_2 - t_1 \in G_{1k}$, thus the function

$$h(t_2 - t_1) \in [-\alpha(k) + G_{2\beta(k)} + \alpha(k)] =: A.$$

Since A is a group, $A + h(t_2 - t_1) = A$. This yields that $A + h(t_2) = A + h(t_1)$.

Let f be a solution of (1). Then the function $\phi = b_2 f b_1^{-1}$ is a solution of the equation

$$\phi(C + t) = \phi(C) + h(t), \quad (5)$$

where $\varphi : \cup_{k \in K_1} : G_1/G_{1k} \rightarrow \cup_{k \in K_2} : G_2/G_{2k}$. Indeed, for $C \in \cup_{k \in K_1} : G_1/G_{1k}$ there exists $x \in S_1$ such that $b_1(x) = C$ thus $x = b_1^{-1}(C)$ and

$$\begin{aligned} b_2 f b_1^{-1}(C + t) &= b_2 f b_1^{-1}(b_1(x) + t) = b_2 f(F_1(x, t)) = b_2 F_2(f(x), h(t)) \\ &= b_2 b_2^{-1}[b_2(f(x)) + h(t)] = b_2 f b_1^{-1}(C) + h(t). \end{aligned}$$

Since $\varphi(G_{1k}) \in \cup_{k \in K_2} : G_2/G_{2k}$ there exist functions $\alpha : K_1 \rightarrow G_2$ and $\beta : K_1 \rightarrow K_2$ such that $\varphi(G_{1k}) = G_{2\beta(k)} + \alpha(k)$. We have thus for $t \in G_{1k}$ that $\varphi(G_{1k} + t) = \varphi(G_{1k}) = \varphi(G_{1k}) + h(t)$. This yields that $G_{2\beta(k)} + \alpha(k) = G_{2\beta(k)} + \alpha(k) + h(t)$. From here $h(t) \in [-\alpha(k) + G_{2\beta(k)} + \alpha(k)]$ if $t \in G_{1k}$. This implies $h(G_{1k}) \subset [-\alpha(k) + G_{2\beta(k)} + \alpha(k)]$. Moreover for $C = G_{1k} + t$ we obtain

$$\varphi(C) = \varphi(G_{1k} + t) = \varphi(G_{1k}) + h(t) = G_{2\beta(k)} + \alpha(k) + h(t).$$

Assume now that $f = b_2^{-1} \varphi b_1$ with φ as in (4). The function φ is a solution of the Eq. (5) since for $C = G_{1k} + t_1$

$$\begin{aligned} \varphi(C + t) &= \varphi(G_{1k} + t_1 + t) = G_{2\beta(k)} + \alpha(k) + h(t_1 + t) \\ &= G_{2\beta(k)} + \alpha(k) + h(t_1) + h(t) = \varphi(G_{1k} + t_1) + h(t) = \varphi(C) + h(t). \end{aligned}$$

From here we have

$$\begin{aligned} f[F_1(x, t)] &= b_2^{-1} \varphi b_1[b_1^{-1}(b_1(x) + t)] = b_2^{-1} \varphi(b_1(x) + t) = b_2^{-1}[\varphi(b_1(x)) + h(t)] \\ &= b_2^{-1}[b_2 b_2^{-1} \varphi(b_1(x)) + h(t)] = b_2^{-1}[b_2(f(x)) + h(t)] = F_2(f(x), h(t)). \end{aligned}$$

□

Remark. The solution f of (1) is a function from S_1 to S_2 but only the stability groups G_{ik} and the homomorphism h decide about the existence of this solution.

Examples. 1. Put

$$\begin{aligned} F_1(x, t) &= t n^{-1}(t n x + t) \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), t \in \mathbb{R}, \\ F_2(x, t) &= \begin{cases} \exp(\ln x + t) & \text{for } x > 0, t \in \mathbb{R}, \\ 0 & \text{for } x = 0, t \in \mathbb{R}, \\ -\exp(\ln(-x) + t) & \text{for } x < 0, t \in \mathbb{R} \end{cases} \end{aligned}$$

and $h(t) = t$.

We have $G_1 = G_2 = \mathbb{R}$, $S_1 = (-\frac{\pi}{2}, \frac{\pi}{2})$, $K_1 = \{1\}$, $S_{11} = S_1$, $b_{11} = t n$, $G_{11} = \{0\}$, $S_2 = \mathbb{R}$, $K_2 = \{1, 2, 3\}$, $S_{21} = (0, +\infty)$, $S_{22} = \{0\}$, $S_{23} = (-\infty, 0)$, $b_{21} = \ln$, $b_{22} = 0$, $b_{23}(x) = \ln(-x)$, $G_{21} = \{0\}$, $G_{22} = \mathbb{R}$, $G_{23} = \{0\}$.

The condition (3) has the form $G_{11} \subset G_{2b(1)}$ and this is true for $\beta(1) = 1, 2, 3$ and for every $\alpha(1) = a \in \mathbb{R}$.

If $\beta(1) = 1$, then $\varphi(G_{11} + t) = \varphi(\{0\} + t) = \{0\} + a + t$. For $C(t) := \{0\} + t$ we have $\varphi(C(t)) = C(a + t)$. From here $\varphi(b_{11}(x)) = \varphi(C(t n x)) = C(a + t n x)$. This yields that

$$f(x) = b_{21}^{-1}(C(a + t n x)) = \exp(a + t n x).$$

Analogously if $\beta(1) = 2$, then $f(x) = 0$ and if $\beta(1) = 3$, then $f(x) = -\exp(a + tnx)$.

From here the functions $f(x) = \varepsilon \exp(a + tnx)$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a \in \mathbb{R}$, $\varepsilon \in \{-1, 0, 1\}$ are the only solutions of (1) in the case of the above functions F_1 and F_2 .

2. The equation

$$f[F_2(x, t)] = F_1(f(x), h(t)),$$

where F_1, F_2, h are as in Example 1, does not have a solution. In fact, there does not exist a function $\beta : K_2 \rightarrow K_1$ for which $\mathbb{R} = G_{22} \subset G_{1\beta(k)} = G_{11} = \{0\}$.

We prove this fact directly, too. We have $f[F_2(0, t)] = f(0) \neq F_1(f(0), t)$ since $F_1(f(0), \mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

3. Let the function F be of the form

$$F(x, t) = \begin{cases} b_n^{-1}[b_n(x) + t] & \text{for } x \in S_n, \quad t \in G, \\ x & \text{for } x \in S \setminus \cup S_n =: N_2, t \in G, \end{cases} \quad (6)$$

where G is an abelian group, S_n are disjoint subsets of S such that $\text{card} S_n = \text{card} G$ for n in a set N_1 of indices which is disjoint with N_2 and $h_n : S_n \rightarrow G$ are bijections.

Consider the equation

$$f[F(x, t)] = F(f(x), t). \quad (7)$$

Put $N = N_1 \cup N_2$. The formula

$$f(x) = \begin{cases} \gamma(x) & \text{for } x \in N_2, \\ \gamma(n) & \text{for } x \in S_n, \gamma(n) \in N_2, \\ b_{\gamma(n)}^{-1}[b_n(x) + \delta(n)] & \text{for } x \in S_n, \gamma(n) \in N_1, \end{cases} \quad (8)$$

where the function $\gamma : N \rightarrow N$ is such that $\gamma(N_2) \subset N_2$ and δ is an arbitrary function from N_1 to G , gives all the solutions of (7).

For the proof assume that the function $f : S \rightarrow S$ is a solution of (7). If $x \in S \setminus \cup S_n = N_2$, then $f(x) = f[F(x, t)] = F(f(x), t)$, thus $f(x) \in N_2$ and we put $\gamma(x) = f(x)$.

If $x \in S_n$ and $f(x) \in N_2$, then we have $f[F(x, t)] = F(f(x), t) = f(x)$. This yields that f is a constant function on S_n and this constant, denoted $\gamma(n)$, is in N_2 . From here $f(x) = \gamma(n)$ for $x \in S_n$.

If $x \in S_n$ and $f(x) \in \cup I_n$, then $f(x) \in S_m$ for some $m \in N_1$. Since $f[F(x, t)] = F(f(x), t) \in S_m$, thus the values of the function f on S_n are in S_m . From here m is determined by n , thus m is a function of $n : m = \gamma(n) \in N_1$. Let x_0 be a fixed point of S_n , then $f(x_0) \in S_{\gamma(n)}$. Since $x \in S_n$, for $\tau = b_n(x) - b_n(x_0)$ we have $F(x_0, \tau) = b_n^{-1}(b_n(x_0) + \tau) = x$. This implies that

$$\begin{aligned} f(x) &= f[F(x_0, \tau)] = F(f(x_0), \tau) = b_{\gamma(n)}^{-1}[b_{\gamma(n)}(f(x_0)) + b_n(x) - b_n(x_0)] \\ &= b_{\gamma(n)}^{-1}(b_n(x) + \delta(n)), \end{aligned}$$

where $\delta(n) = b_{\gamma(n)}(f(x_0)) - b_n(x_0)$ is a function from N_1 to G .

It is easily to verify that a function of the form (8) is the solution of the Eq. (7).

This result is proven in [9] if $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the continuous solution of the translation equation. Note that the formula (8) gives all the functions f which are commutative with the functions of the family $\{F(., t)\}_{t \in G}$ of functions from S to S .

If a solution $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ of the translation equation, such that $F(x, 0) = x$ for $x \in \mathbb{R}$, is continuous with respect to the second variable for every $x \in \mathbb{R}$ and for which at least one of the functions $F(., t): \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then the function F has the form (6). In this case S_n are open intervals and b_n are homeomorphisms and the function F is continuous [10]. This solution F of the translation equation is continuous, too, if F is Carathéodory, i.e., the function $F(x, .): \mathbb{R} \rightarrow \mathbb{R}$ is measurable for every $x \in \mathbb{R}$ and function $F(., t): \mathbb{R} \rightarrow \mathbb{R}$ is continuous for every $t \in \mathbb{R}$ [2].

Recall that x_0 is said to be a fixed point of the function $F: S \times G \rightarrow G$ if $F(x_0, t) = x_0$ for $t \in G$.

Theorem 2. *Let $F: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous solution of the translation equation such that $F(x, 0) = x$ for $x \in \mathbb{R}$. Every solution f of (7) is continuous if and only if F does not have any fixed points.*

Proof. Assume that f is a solution of (7). If F does not have any fixed points, then the set N_1 in the form (8) of F is a singleton $N_1 = \{m\}$ and $S_m = \mathbb{R}$. From here $\gamma(m) = m$ and $f(x) = b_m^{-1}(b_m(x) + \delta(m))$ for $x \in \mathbb{R}$, where $b_m: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism and $\delta(m) \in \mathbb{R}$. The function f is thus continuous.

If a is the only fixed point of F , i.e., $N_2 = \{a\}$ in (6), then N_1 has two elements, which we denote as 1, 2, so that $S_1 = (-\infty, a)$ and $S_2 = (a, +\infty)$. If $\gamma(1) = 2$, $\gamma(2) = 1$, $\gamma(a) = a$, $\delta: \{1, 2\} \rightarrow \mathbb{R}$ is an arbitrary function and $b_1: S_1 \rightarrow \mathbb{R}$ and $b_2: S_2 \rightarrow \mathbb{R}$ are arbitrary increasing bijections, then the function f defined as in (8) is a solution of (7). For $x \in S_1$ we obtain $f(x) = b_2^{-1}(b_1(x) + \delta(1))$. We have for $x \rightarrow a$ that $b_1(x) + \delta(1) \rightarrow +\infty$, thus $b_2^{-1}(b_1(x) + \delta(1)) \rightarrow +\infty$ and from here f is not continuous.

Let a and b be two different fixed point of F . We consider two cases.

- a/ Every point of the interval (a, b) is a fixed point of F .
- b/ The opposite case to a/.

In case a/ let the function γ in (8) be such that $\gamma(a) = b$ and $\gamma(x) = x$ for $x \in (a, b)$. From here we have for the function f defined as in (8) that

$$\lim_{x \rightarrow a+} f(x) = a \neq b = \gamma(a) = f(a),$$

thus f is discontinuous.

In case b/ there exists at least one $n \in N_1$ in the form (6) of the function F such that $S_n \subset (a, b)$. Let $c = \inf S_n$. Then $c \in N_2$. Let γ in (8) be such that

$\gamma(c) = b$, $\gamma(n) = n$ and let $\delta(n) = 0$ in (8). From here $f(x) = x$ for $x \in S_n$. This yields that

$$\lim_{x \rightarrow c+} f(x) = c \neq b = \gamma(c) = f(c),$$

thus f is discontinuous. $\square$

If in Eq. (7) we take the function F_1 of Example 1 as the function F , then the solution f has the form $f(x) = tn^{-1}(tnx + \delta)$ for $\delta \in \mathbb{R}$. If in this equation we have F_2 , then

$$f(x) = \begin{cases} 0 & \text{for } x = 0, \\ 0 & \text{for } x \in S_n, \gamma(n) = 0, \\ b_{\gamma(n)}^{-1}(b_n(x) + \delta(n)) & \text{for } x \in S_n, \gamma(n) \neq 0, \end{cases}$$

where $S_1 = (0, +\infty)$, $S_2 = (-\infty, 0)$, $\gamma : \{0, 1, 2\} \rightarrow \{0, 1, 2\}$, $\gamma(0) = 0$, $\delta : \{1, 2\} \rightarrow \mathbb{R}$, $b_1(x) = \ln x$, $b_2(x) = \ln(-x)$.

4. The constant function $f(x) = c$ for $x \in S_1$ is a solution of (1) if and only if $F_2(c, h(t)) = c$ for $t \in G_1$. We verify condition (3) directly in this case. If $f(x) = c \in S_2$ is a solution of (1), then there exists a $l \in K_2$ such that $c \in S_{2l}$. From here $F_2(c, s) = b_{2l}^{-1}[b_{2l}(c) + s]$ thus $c = F_2(c, h(t)) = b_{2l}^{-1}[b_{2l}(c) + h(t)]$ for $t \in G_1$. There exists $s_0 \in G_2$ for which $b_{2l}(c) = G_{2l} + s_0$. This yields that $G_{2l} + s_0 = b_{2l}(c) = G_{2l} + s_0 + h(t)$. From here $-s_0 + G_{2l} + s_0 = (-s_0 + G_{2l} + s_0) + h(t)$. Since $-s_0 + G_{2l} + s_0$ is a group, $h(G_1) \subset [-s_0 + G_{2l} + s_0]$. We have $h(G_{1k}) \subset h(G_1)$ for $k \in K_1$, thus the condition (3) is satisfied with $\beta(k) = s_0$ for $k \in K_1$.

5. Consider the equation

$$f(g(x)) = g(f(x)), \quad (9)$$

where $g : S \rightarrow S$ is a given idempotent function and $f : S \rightarrow S$ is the unknown function. This is an equation of the form (1) for $F_1(x, t) = F_2(x, t) = g(x)$.

If g is an injective function on the set $S \setminus g(S)$, then every function f of the form

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in g(S), \\ g_1^{-1}[f_1(g(x))] & \text{for } x \in S \setminus g(S), \end{cases} \quad (10)$$

where f_1 is an arbitrary function from $g(S)$ to $g(S)$ and $g_1 = g|_{S \setminus g(S)}$, is a solution of (9).

Indeed, if $x \in g(S)$, then there exists $y \in S$ such that $x = g(y)$. From here $f(g(x)) = f[g(g(y))] = f(g(y)) = f(x) = f_1(x) = g(f_1(x)) = g(f(x))$ since g is the identity on $g(S)$ as the idempotent. If $x \in S \setminus g(S)$, then $g(f(x)) = f_1(g(x)) = f(g(x))$ since $g(x) \in g(S)$.

If $S = g(S)$, i.e., if $g(x) = x$ for $x \in S$, then every function $f : S \rightarrow S$ is a solution of (9).

If $S \neq g(S)$, then a solution f of (9) is of the form (10) if and only if $f(S \setminus g(S)) \subset S \setminus g(S)$. In fact, if f is a solution of (9) and $f(S \setminus g(S)) \subset S \setminus g(S)$, then

a/ for $x \in g(S)$ we have $f(x) = f(g(x)) = g(f(x)) \in g(S)$, thus $f_1 = f|_{g(S)}$ is a function from $g(S)$ to $g(S)$ and $f = f_1$ on $g(S)$,

b/ for $x \in S \setminus g(S)$ we have $f(x) \in S \setminus g(S)$ and $f(x) = g_1^{-1}[f_1(g(x))]$ since

$$g_1(f(x)) = g(f(x)) = f(g(x)) = f_1(g(x)).$$

If f is of the form (10), then for $x \in S \setminus g(S)$ we obtain $f(x) = g_1^{-1}[f_1(g(x))] \in S \setminus g(S)$.

Note in the end that if $S \neq g(S)$, then the function $f(x) = c \in g(S)$ for $x \in S$ is a solution of (9) and f is not of the form (10) by the above.

6. Let G be the groupoid of $n \times n$ -matrices with the usual multiplication. We give all the solutions of Eq. (7) in which $F(x, t) = xt : \mathbb{R}^n \times G \rightarrow \mathbb{R}^n$ (the transformation law of the vector). Equation (7) is in this case of the form $f(xt) = f(x)t$. Assume that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a solution of this equation. Put in this equation $x = (1, 0, \dots, 0)$ and $t = \|t_{ik}\|_{i,k=1,\dots,n}$, where $t_{1k} = x_k$ for $k = 1, \dots, n$ and the other t_{ik} are arbitrary in $\mathbb{R}$, we have $f(x_1, \dots, x_n) = (y_1, \dots, y_n)t$, where $y := (y_1, \dots, y_n) = f(1, 0, \dots, 0)$. Assume that there exists $m \neq 1$ for which $y_m \neq 0$. Let $a_1 = \|a_{ik}\|_{i,k=1,\dots,n}$, where $a_{1k} = x_k$ for $k = 1, \dots, n$, $a_{i1} = 0$ for $i \neq 1$ and the other a_{ik} are arbitrary in $\mathbb{R}$. Let $a_2 = \|b_{ik}\|_{i,k=1,\dots,n}$, where $b_{1k} = x_k$ for $k = 1, \dots, n$, $b_{m1} = 1$, $b_{i1} = 0$ for $i \neq 1, m$ and the other b_{ik} are arbitrary in $\mathbb{R}$. Putting $t = a_1$ in our equation we have $f(x_1, \dots, x_n) = ya_1$. From here $ya_1 = ya_2$ and in particular

$$\begin{aligned} y_1 x_1 &= y_1 a_{11} + y_2 a_{21} + \dots + y_n a_{n1} = y_1 b_{11} + \dots + y_m b_{m1} + \dots \\ &+ y_n b_{n1} = y_1 x_1 + y_m, \end{aligned}$$

thus $y_m = 0$. We have a contradiction thus $y_i = 0$ for $i = 2, \dots, n$. This yields, by the equation, that $f(x_1, \dots, x_n) = y_1(x_1, \dots, x_n)$. These functions are evidently solutions of our equation.

We have the same result if G is the group of nonsingular $n \times n$ -matrices. We prove this for $n = 2$. If we put in the above

$$\begin{aligned} \text{i/ } a_1 &= \begin{vmatrix} x_1 & x_2 \\ 0 & 1 \end{vmatrix} \text{ and } a_2 = \begin{vmatrix} x_1 & x_2 \\ 1 & x_2 x_1^{-1} + 1 \end{vmatrix} \text{ for } x_1 \neq 0 \text{ and} \\ \text{ii/ } a_1 &= \begin{vmatrix} x_1 & x_2 \\ 1 & 0 \end{vmatrix} \text{ and } a_2 = \begin{vmatrix} x_1 & x_2 \\ x_1 x_2^{-1} + 1 & 1 \end{vmatrix} \text{ for } x_2 \neq 0, \end{aligned}$$

then we obtain as above $y_2 = 0$. Thus by the equation for t nonsingular (e.g., for $t = \begin{vmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{vmatrix}$) we have $f(x_1, x_2) = y_1(x_1, x_2)$ for $x_1^2 + x_2^2 \neq 0$.

For $(x_1, x_2) = (0, 0)$ put in our equation $t = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ and $f((0, 0)) = (\alpha, \beta)$. Then we have $(\beta, \alpha - \beta) = (0, 0)$, thus $\alpha = \beta = 0$ and from here $f((0, 0)) = (0, 0) = y_1(0, 0)$, too.

It is possible to obtain this result also by the method as in Theorem 1 [9]. The above result means in the theory of geometric objects that the multiplication of a vector by a scalar is the only geometric concomitant of the vector.

3. Unknown function F_2

We only need to define the function F_2 on $S_2 \times h(G_1)$ due to the form of the Eq. (1)

Theorem 3. 1/ Equation (1) has a solution for $F_2 : S_2 \times h(G_1) \rightarrow S_2$ if and only if

$$f(x) = f(y) \text{ and } h(t) = h(s) \Rightarrow f[F_1(x, t)] = f[F_1(y, s)] \text{ for } x, y \in S_1, t, s \in G_1. \quad (11)$$

2/ Let functions f, h, F_1 satisfy (11). Then a function of the form

$$F_2(u, v) = \begin{cases} f[F_1(x, t)] & \text{for } u = f(x), v = h(t) \\ H(u, v) & \text{for } u \in S_2 \setminus f(S_1), v \in h(G_1), \end{cases} \quad (12)$$

where $H(u, v) : (S_2 \setminus f(S_1)) \times G_1 \rightarrow S_2 \setminus f(S_1)$ is an arbitrary solution of the translation equation, is a solutions of (1).

Proof. The proof of the “only if” part is evident.

The function F_2 defined by (12) is well-defined due to (11) and it is a solution of (1). Moreover, for $u = f(x)$, $v = h(t)$, $w = h(s)$, we have

$$\begin{aligned} F_2(F_2(u, v), w) &= F_2[F_2(f(x), h(t)), h(s)] = F_2[f(F_1(x, t)), h(s)] \\ &= f[F_1(F_1(x, t), s)] = f[F_1(x, t + s)] = F_2(u, v + w). \end{aligned}$$

□

Example. 7. The condition (11) is satisfied for $h(t) = t$ and by every $f(x) = \varepsilon \exp(a + tx)$ ($\varepsilon = -1, 0, 1$) in the equation in Example 1

- a/ since the function h and the functions f for $\varepsilon \neq 0$ are injective and for $f = 0$ ($\varepsilon = 0$) the condition (11) is evident or
- b/ since there exist function f, h, F_1, F_2 which satisfy equation (7).

If we want to have F_2 defined on $S_2 \times G_2$, then we must extend the solution F_2 of the translation equation on $S_2 \times h(G_1)$ to the solution of the translation equation on $S_2 \times G_2$. This extension is not always possible (see Example 8 below). For the condition when this extension is possible see [6] and [8].

Examples. 8. Let $S_1 = S_2 = \mathbb{Z}$, $G_1 = G_2 = \mathbb{Z}$ (the additive group of integers), $f(x) = x$, $h(t) = 2t$, $F_1(x, t) = x + t$. These functions evidently satisfy (11). Assume that there exists a solution $F_2 : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ of the translation equation such that $F_2(x, u) = F(x, t)$ for $u = 2t$, i.e., such that $F_2(x, 2t) = F_1(x, t) = x + t$ for $(x, t) \in \mathbb{Z} \times \mathbb{Z}$. From here we have for $x \in \mathbb{Z}$

$$\mathbb{Z} = F_1(x, \mathbb{Z}) = F_2(x, 2\mathbb{Z}) \subset F_2(x, \mathbb{Z}) \subset \mathbb{Z}.$$

Thus $F_2(x, \mathbb{Z}) = \mathbb{Z}$. Since F_2 is the solution of the translation equation and $F_2(x, 0) = x$ for $x \in \mathbb{Z}$, $F_2(x, t) = b_1^{-1}(b_1(x) + t)$, where b_1 is a bijection from $\mathbb{Z}$ to $\mathbb{Z}/\mathbb{Z}^*$ for a subgroup $\mathbb{Z}^*$ of $\mathbb{Z}$ [7]. We have $b_1(0) = \mathbb{Z}^* + a$ for some $a \in \mathbb{Z}$. We obtain for $b(x) := b_1(x) - a$ that

$$F_2(x, t) = b_1^{-1}(b_1(x) + t) = b^{-1}(b(x) + t) \text{ and } b(0) = \mathbb{Z}^*.$$

This yields that $b^{-1}(b(x) + 2) = F_2(x, 2) = F_1(x, 1) = x + 1$. From here $b(x) + 2 = b(x + 1)$, thus $b(1) = b(0) + \mathbb{Z}^*$. This implies that $b(x) = \mathbb{Z}^* + 2x$ for $x \in \mathbb{Z}$. Let $c \geq 0$ be the generator of $\mathbb{Z}^*$, i.e., $\mathbb{Z}^* = \{nc : n \in \mathbb{Z}\}$.

- i/ If $c = 0$, i.e., $\mathbb{Z}^* = \{0\}$, then $b(x) = \{0\} + 2x$. From here b is not a function onto $\mathbb{Z}/\mathbb{Z}^*$.
 - ii/ If $c > 0$, then $b(c) = \mathbb{Z}^* + 2c = \mathbb{Z}^* = \mathbb{Z}^* + 4c = b(2c)$. Thus the function b is not injective.
9. Let $S_1 = S_2 = G_1 = G_2 = G$ be a semigroup and $f(x) = x$ for $x \in G$. Let $h: G \rightarrow G$ be a homomorphism such that $h(G) \neq G$ and $F_1(x, t) = x + h(t)$ for $x, t \in G$. The function F_1 is the solution of the translation equation and F_1 satisfies (11). The function F_2 , defined on $G \times h(G)$ as in (12) (with $f(x) = x$), is of the form $F_2(x, h(t)) = F_1(x, t) = x + h(t)$. Thus the function $x + t$ is an extension of F_2 on the set $G \times G$.

Remarks. It is possible that the functions of the form (12) are not all the solutions of Eq. (1). E.g., if $S_1 = S_2 = \mathbb{R}$, $G_1 = G_2 = \mathbb{R}$ with the usual addition, $h(t) = t$, $f(x) = \exp x$, $F_1(x, t) = x$, $F_2(u, v) = u$ for $u > 0, v \in \mathbb{R}$ and $F_2(u, v) = 1$ for $u \leq 0, v \in \mathbb{R}$, then the solution of the translation equation F_2 is the solution of (1), too. This function F_2 is not of the form (12) since $0 \in S_2 \setminus f(S_1) = (-\infty, 0]$ and $F_2(0, v) = 1 \notin S_2 \setminus f(S_1)$.

If $f(S_1) = S_2$, then F_2 must be of the form $F_2(u, v) = f[F_1(x, t)]$ for $u = f(x)$, $v = h(t)$, thus Eq. (1) has only one solution. In this case if there exists t_0 such that $F_1(x, t_0) = x$ for $x \in S_1$, then $F_2(u, h(t_0)) = f(x) = u$ for $u \in S_2$. From here if F_1 is the transformation law for a geometric object, then F_2 is the same.

If $S_1 = S_2$, then for the function

$$F_2(u, v) = \begin{cases} f[F_1(x, t)] & \text{for } u = f(x), v = h(t), \\ u & \text{for } u \in S_2 \setminus f(S_1), v \in G_2 \end{cases}$$

we have the same situation as above.

The condition (11) and the equation

$$f[F_1(x, t)] = F_1(f(x), h(t))$$

are not equivalent. A solution of this equation is a solution of (11) but not vice versa. E.g., every constant function f is a solution of (11) and $f(x) = c$ is a solution of the above equation if and only if $F_1(c, h(t)) = c$ for $t \in G$.

4. Unknown function F_1

If the function f is injective, then the function $F_1(x, t) = f^{-1}[F_2(f(x), h(t))]$ is the only solution of (1).

There exists solutions of (1) for a non-injective function f .

Examples. 10. If $f(x) = c$, F_2 is a solution of the translation equation such that c is a fixed point of F_2 ($F_2(c, v) = c$ for $v \in G_2$) and h is an arbitrary homomorphism, then every function $F_1 : S_1 \times G_1 \rightarrow S_1$ is a solution of (1).

A function $F_1 : S_1 \times G_1 \rightarrow S_1$ is the solution of (1) only if $f(x) = f(y) \Rightarrow f(F_1(x, t)) = f(F_1(y, t))$ for $x, y \in S_1$, $t \in G_1$.

Note that the solution $F_1(x, t) = c$ for $(x, t) \in S_1 \times G_1$ of the above condition is a solution of (1) only if $F_2(f(x), h(t)) = f(c)$ for $(x, t) \in S_1 \times G_1$.

11. Let $F_i : S_i \times G_i \rightarrow S_i$ for $i = 1, 2$ be arbitrary functions not necessarily a solutions of the translation equation. Consider a special case of equation (11) namely if F_2 is such that $F_2(u, v) = u$ on $f(S_1) \times h(G_1)$. Then Eq. (11) has the form

$$f(F_1(x, t)) = f(x). \quad (13)$$

In this case the function F_1 , such that $x \in F_1(x, G_1)$ for $x \in S_1$, is a solution of (11) if and only if the set $F_1(x, G_1)$ is included in some level of the function f for every $x \in S_1$.

From here the function F_1 of the form as in (2) satisfies Eq. (13) if and only if for every $k \in K_1$ the set S_{1k} is included in some level of f .

E.g., let $S_1 = \mathbb{R}$ and $f(x) = |x|$. The levels of f have the form $\{-a, a\}$ for $a \in \mathbb{R}$. Assume that the group G_1 is non-orientable, i.e., there exists no subgroup of G_1 with index 2 (e.g., the additive group $\mathbb{R}$ is non-orientable). Thus it is impossible that $S_{1k} = \{-a, a\}$ for $a \neq 0$. From here every S_{1k} in the form F_1 in (2) is a singleton. This yields that the constant functions $F_1(x, t) = c \in \mathbb{R}$ are the only solutions of (13) in this case.

Let S_1 and f be as above. Assume that G_1 is orientable by the subgroup G_1^* (like the additive group of integers by even integers). We have in this case,

except for the constant solutions, the non-constant solutions of (13), too, e.g., the solution of the form

$$F_1(x, t) = \begin{cases} -1 & \text{for } x = -1, t \in G_1^* \text{ or } x = 1, t \in G_1 \setminus G_1^*, \\ 1 & \text{for } x = -1, t \in G_1 \setminus G_1^* \text{ or } x = 1, t \in G_1^*, \\ c \in \mathbb{R} & \text{for } x \in \mathbb{R} \setminus \{-1, 1\}, t \in G_1. \end{cases}$$

5. Unknown homomorphism h

Theorem 4. *If a function $F: S \times G \rightarrow S$ is of the form (6) and function f is of the form (8), then the equation*

$$f[F(x, t)] = F(f(x), h(t)) \quad (14)$$

has $h(t) = t$ as the only solution if $\gamma(N_1) \not\subset N_2$.

If $\gamma(N_1) \subset N_2$, then every function $h: G \rightarrow G$ is a solution of (14).

Proof. If $\gamma(N_1) \not\subset N_2$, then there exists $m \in N_1$ such that $\gamma(m) \in N_1$. Assume that the function h is a solution of (1). Then for $x \in I_m$ we have $f(x) = b_{\gamma(m)}^{-1}[b_m(x) + \delta(m)] \in S_{\gamma(m)}$, $F(x, t) = b_m^{-1}[b_m(x) + t] \in S_m$. From here (14) has the form

$$b_{\gamma(m)}^{-1}[b_m(x) + t + \delta(m)] = b_{\gamma(m)}^{-1}[b_m(x) + \delta(m) + h(t)],$$

thus $h(t) = t$. This homomorphism is a solution of (1) by Example 3.

If $\gamma(N_1) \subset N_2$, then $\gamma(n) \in N_1$ for every $n \in N$. From here for every function $h: G \rightarrow G$

- 1/ if $x \in N_2$, then $f(x) = \gamma(x)$ and we obtain $f[F(x, t)] = f(x) = \gamma(x) = F(\gamma(x), h(t))$,
- 2/ if $x \in I_n$, then $f(x) = \gamma(n)$ and since $F(x, t) \in I_n$ we obtain $f[F(x, t)] = \gamma(x) = F(\gamma(x), h(t)) = F(f(x), h(t))$.

□

Proposition. *Let functions F_i be as in (2). If there exists $x_0 \in S_1$ such that $F_1(x_0, \cdot) : G_1 \rightarrow S_1$ is a constant function and $F_2(f(x_0), \cdot) : G_2 \rightarrow S_2$ is an injection, then $h(t) = 0$ for $t \in G_1$ is the only solution of (1).*

The function $h(t) = 0$ for $t \in G_1$ is a solution of (1) if and only if the function f is constant on every S_{1k} for $k \in K_1$. In this case other solutions h can exist.

Proof. We have $f[F_1(x_0, t)] = f[F_1(x_0, 0)] = f(x_0)$. For $f(x_0) \in S_{2k}$ for some $k \in K_2$ we obtain $F_2(f(x_0), h(t)) = b_{2k}^{-1}[b_{2k}(f(x_0) + h(t))]$, where b_{2k} is a bijection from S_{2k} onto G_2 . From here $b_{2k}(f(x_0)) = b_{2k}(f(x_0)) + h(t)$. This yields that $h(t) = 0$ for $t \in G_1$.

If $h(t) = 0$, then $f[F_1(x, t)] = F_2(f(x), 0) = f(x)$. Since $F_1(x, G_1) = S_{1k}$ for $x \in S_{1k}$, f is constant on S_{1k} . The inverse implication is evident. If $f(x) = c \in S_2$ for $x \in S_1$ and $F_2(c, t) = c$ for $t \in G_2$, the every function $h: G_1 \rightarrow G_2$ is a solution of (1). □

Lemma 1. Let $F: S \times G \rightarrow S$, where G is a group, be a solution of the translation equation such that $F(x, 0) = x$ for $x \in S$. A function $g: G \rightarrow G$ (not necessarily a homomorphism) is a solution of the equation

$$F(x, t) = F(x, g(t)) \quad (15)$$

if and only if $g(t) - t \in \cap_{x \in S} \{s \in G: F(x, s) = x\}$ for $t \in G$.

Proof. If $g(t) - t \in \cap_{x \in S} \{s \in G: F(x, s) = x\}$, then

$$F(x, g(t)) = F(F(x, g(t) - t), t) = F(x, t).$$

If $F(x, g(t)) = F(x, t)$, then $g(t) - t \in \cap_{x \in S} \{s \in G: F(x, s) = x\}$ since

$$F(x, g(t) - t) = F(F(x, g(t)), -t) = F(F(x, t), -t) = F(x, 0) = x.$$

□

Lemma 2. Let a function F be of the form $F(x, t) = b_k^{-1}(b_k(x) + t)$ for $x \in S_k$, $k \in K$, $t \in G$, where $S = \cup_{k \in K} S_k$, S_k are disjoint and b_k is a bijection from S_k to G/G_k (see (2)). Then

$$\cap_{x \in S} \{t \in G: F(x, t) = x\} = \cap_{u \in G} \cap_{k \in K} (-u + G_k + u).$$

Proof. Let $s \in \{t \in G: F(x, t) = x\}$ for every $x \in S$. We have $G_k + u \in G/G_k$ for every $u \in G$, $k \in K$. There exists $y \in S_k$ such that $b_k(y) = G_k + u$. Since $s \in \{t \in G: F(y, t) = y\}$, $y = F(y, s) = b_k^{-1}(b_k(y) + s)$. From here $G_k + u = G_k + u + s$. This yields that $-u + G_k + u = [-u + G_k + u] + s$, thus $s \in [-u + G_k + u]$ for every $u \in G$, $k \in K$.

Let $s \in (-u + G_k + u)$ for every $u \in G$, $k \in K$ and $x \in S_k$. Then $b_k(x) = G_k + a(x)$ for some $a(x) \in G$. From here $b_k(x) + s = a(x) + [-a(x) + G_k + a(x)] + s = b_k(x)$, thus $F(x, s) = b_k^{-1}(b_k(x) + s) = x$. □

Note that if $\cap_{u \in G} \cap_{k \in K} (-u + G_k + u) = \{0\}$, then $g(t) = t$ for $t \in G$ is the only solution of (15).

If the group G is abelian, then the function g in Lemma 1 is a homomorphism if and only if the function $g(t) - t$ is a homomorphism. We have thus by Lemmas 1 and 2 the following theorem.

Theorem 5. Let F be as in Lemma 2. Assume that the group G is abelian. The homomorphism $h: G \rightarrow G$ is a solution of Eq. (15) if and only if $h(t) = h_1(t) + t$ for $t \in G$, where h_1 is an arbitrary homomorphism from G to $\cap_{k \in K} G_k$.

Remarks. 1/ If the group G is not abelian, then Theorem 5 is not true since the difference of two homomorphism may not be a homomorphism. E.g., for the smallest non-abelian group given in the table below and for $h(t) = -a + t + a$ the function $h_1(t) = h(t) - t$ is not a homomorphism. Indeed, $h_1(a + b) = c \neq d = h_1(a) + h_1(b)$.

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	c	b	e	d
b	b	d	0	e	a	c
c	c	e	a	d	0	b
d	d	b	e	0	c	a
e	e	c	d	a	b	0

The assumption that G is abelian in the above Theorem 5 can not be replaced by the assumption that G_k are abelian in F for $k \in K$. We give an example below.

Let $S = \{1, 2\}$ and $K = \{1\}$. Let G be a group as in the table above. Put $b_1(1) = \{0, c, d\}$, $b_1(2) = \{a, b, e\}$ and $F(x, t) = b_1^{-1}(b_1(x) + t)$ for $x \in S$, $t \in G$. Then $G_1 = \{0, c, d\}$ is an abelian group, the homomorphism $h(t) = -a + x + a$ is a solution of (15) and $h(t) - t$ is not a homomorphism.

The tables of these functions $F(x, t)$, $h(t)$ and $F(x, h(t))$ are shown below.

$F(x, t)$	0	a	b	c	d	e
1	1	2	2	1	1	2
2	2	1	1	2	2	1

t	0	a	b	c	d	e
$h(t)$	0	a	e	d	c	b
$F(x, h(t))$	0	a	b	c	d	e
1	1	2	2	1	1	2
2	2	1	1	2	2	1

2/ If a homomorphism $h: G \rightarrow G$ is a solution of (15), then

$$h(t) = h(s) \Rightarrow F(x, t) = F(x, s) \text{ for } x \in S, t, s \in G. \quad (16)$$

Examples. 12. An equation of the form (15) can have a solution which is not the identity function. In fact, let $S = G = \mathbb{R}$ and let b_1, b_2 be different elements of the Hamel base H of $\mathbb{R}$ over $\mathbb{Q}$. Let $a, h: \mathbb{R} \rightarrow \mathbb{R}$ be additive functions such that $a(b_1) = a(b_2) = b_1$, $h(b_1) = b_2$, $h(b_2) = b_1$, $a(t) = h(t) = 0$ for $t \in H \setminus \{b_1, b_2\}$. We have for $t = q_1 b_1 + q_2 b_2 + q_3 b_3 + \dots + q_n b_n$, where $q_i \in \mathbb{Q}$, $b_i \in H$ for $i = 1, \dots, n$, that

$$\begin{aligned} a(h(t)) &= a(q_1 h(b_1) + q_2 h(b_2)) = a(q_1 b_2 + q_2 b_1) = q_1 a(b_2) + q_2 a(b_1) = (q_1 + q_2) b_1 \\ &= q_1 a(b_1) + q_2 a(b_2) = a(q_1 b_1 + q_2 b_2 + q_3 b_3 + \dots + q_n b_n) = a(t). \end{aligned}$$

From here for $F(x,t) = x + a(t)$ we obtain $F(x,h(t)) = x + a(h(t)) = x + a(t) = F(x,t)$. The function h_1 of Theorem 6 is here of the form $h_1(t) = (q_2 - q_1)b_1 + (q_1 - q_2)b_2 - q_3b_3 - \dots - q_nb_n$ for t as above.

13. If $F_i(x,t) = x$ for $(x,t) \in (S_i, G_i)$ and $i = 1, 2$ and $f: S_1 \rightarrow S_2$ is an arbitrary function, then the arbitrary function $h: G_1 \rightarrow G_2$ is a solution of (1).
14. If functions F_1, F_2, f are as in Example 1, then Eq. (1) has the form

$$\varepsilon \exp(a + tnx + t) = \varepsilon \exp(a + tnx + h(t)).$$

Thus if $\varepsilon \neq 0$, then $h(t) = t$ is the only solution of (1) and if $\varepsilon = 0$, then every function $h: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of this equation.

15. The homomorphism $h(t) = t$ is not a solution of (1) in Example 2.
16. If $S_1 = S_2 = G_1 = G_2 = \mathbb{Z}$, $F_1(x,t) = F_2(x,t) = x + t$ and $f(x) = 2x$, then Eq. (1) has the form $2(x+t) = 2x + h(t)$. From here $h(t) = 2t$ is the only homomorphism which is a solution of (1).
17. Let G_1 be a semigroupe, let G_2 be a semigroupe with the left cancellation property and let $S_1 = G_1$, $S_2 = G_2$, $F_1(x,t) = x + t: G_1 \times G_1 \rightarrow G_1$, $F_2(u,v) = u + v: G_2 \times G_2 \rightarrow G_2$. If a function $f: G_1 \rightarrow G_2$ is a homomorphism, then $h = f$ is the only solution of Eq. (1). In fact, Eq. (1) has the form $f(x+t) = f(x) + h(t)$.

6. Conditional equation (11)

The conditional equation (11) is said to be the equation associated with the geometric concomitant equation (1). If functions f, F_1, F_2, h satisfy (1), then f, F_1, h are evidently solutions of (11) and if f, F_1, h satisfy (11), then there exists $F_2: S_2 \times h(G_1)$ such that f, F_1, h, F_2 are solutions of (1) by Theorem 3. A solution of the translation equation F_2 with this property and defined on $S_2 \times G_2$ may not exist (see Example 8). We consider the conditional equation (11) (without the index „1” in F_1) in the cases if

- (i) the homomorphism h is injective and $G_1 = G_2$
or

- (ii) the function f is injective and $S_1 = S_2$.

In case (i) the functions f, h, F are solutions of (11) if and only if

$$f(x) = f(y) \Rightarrow f[F(x,t)] = f[F(y,t)] \text{ for } x, y, t \in G. \quad (17)$$

In case (ii) the functions f, h, F are solutions of (11) if and only if

$$h(t) = h(s) \Rightarrow F(x,t) = F(x,s) \text{ for } x \in S, t, s \in G. \quad (18)$$

Theorem 6. *Let G be a group. A function $f: S \rightarrow S$ is a solution of the conditional equation (17), where $F(x,t) = b^{-1}(b(x) + t)$ and $b: S \rightarrow G$ is a bijection, if and only if $f(x) = b^{-1}\Phi([b(x)])$ for $x \in S$, where $[x] \in G/G^*$ for some subgroup G^* of G and Φ is an injection from G/G^* to G .*

Proof. If $b^{-1}\Phi([b(x)]) = b^{-1}\Phi([b(y)])$, then $[b(x)] = [b(y)]$. From here $(b(x) + t) - (b(y) + t) = b(x) - b(y) \in G^*$ for every $t \in G$. This yields that $[b(x) + t] = [b(y) + t]$. This implies that

$$b^{-1}\Phi([b(F(x,t))]) = b^{-1}\Phi([b(x) + t]) = b^{-1}\Phi([b(y) + t]) = b^{-1}\Phi([b(F(y,t))]).$$

Analogously, if f is a solution of (15), then $\varphi(u) = bfb^{-1}(u)$ for $u \in G$ is a solution of the implication

$$\varphi(u) = \varphi(v) \Rightarrow \varphi(u + t) = \varphi(v + t).$$

We define a relation R on G : $uRv \Leftrightarrow [\varphi(u) = \varphi(v)]$. It is an equivalence relation and uRv implies $(u + t)R(v + t)$. From here the family of equivalence classes of R is the same as the family G/G^* for some subgroup G^* [3]. Let S be a selector of this family. Put $\Phi([u]) = \varphi(u_0)$, where $u_0 \in S \cap [u]$. Since $u_0 \in [u]$, $[u] = [u_0]$. For $u \in [u] = [u_0]$ we have $\varphi(u) = \varphi(u_0)$, thus $\varphi(u) = \Phi([u])$. From here $f(x) = b^{-1}\Phi([b(x)])$. $\square$

Example. 18. Consider the conditional equation (16) where F is a function as in (2). The function $f(x) = \Phi(k)$ for $x \in S_{1k}$, where Φ is a function from K_1 to S_1 , is a solution of this condition. In fact, if $f(x) = f(y)$, $f(x) = \Phi(k)$ for $x \in S_{1k}$ and $f(y) = \Phi(l)$ for $y \in S_{1l}$, then $\Phi(k) = \Phi(l)$. Since $F_1(x,t) \in S_{1k}$ for $t \in G_1$, $f[F_1(x,t)] = \Phi(k)$. Analogously $f[F_1(y,t)] = \Phi(l)$. From here $f[F_1(x,t)] = f[F_1(y,t)]$.

Example. 19. Let $F(x,t) = x + t$ be a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, let G be the additive group of real numbers. The function $\Phi([x]) = M(x)$ for $[x] \in \mathbb{R}/\mathbb{Z}$, where $M(x)$ is the mantissa of $x \in \mathbb{R}$, is an injection from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$. From here $M(x)$ is a solution of (15) and it is not a solution of Eq. (7).

Proposition. Let G be a group and let $F: S \times G \rightarrow S$ be a solution of the translation equation such that $F(x, 0) = x$ for $x \in S$. A homomorphism $h: G \rightarrow G$ is a solution of the conditional equation (18) if and only if

$$\text{Kern}(h) \subset \cap_{x \in S} \{s \in G : F(x, s) = x\}. \quad (19)$$

Proof. If h is a solution of (18) and $t \in \text{Kern}(h)$, then $h(t) = 0 = h(0)$, thus $F(x, t) = F(x, 0) = x$ for every $x \in S$. From here $t \in \cap_{x \in S} \{s \in G : F(x, s) = x\}$.

Assume that (19) is true. If $h(t) = h(s)$, then $h(t-s) = 0$, thus $t-s \in \text{Kern}(h)$. This yields that $F(x, t-s) = x$ for $x \in S$. From here $F(x, t) = F(F(x, t-s), s) = F(x, s)$. $\square$

Example. 20. Let S, G, H be as in Example 10. Let $F(x + t) = x + t$. For the homomorphism

$$h(q_1b_1 + q_2b_2 + q_3b_3 + \dots + q_nb_n) = q_2b_1 + q_1b_2 + q_3b_3 + \dots + q_nb_n$$

we have $\text{Kern}(h) = \{0\}$, h satisfies (19). This implies that h is a solution of (18). This function h is evidently not a solution of (17).

Example. 21. Injection from $\mathbb{R}$ to $\mathbb{R}$ and constant function are the only solutions of (17) for the geometric object $x = v_2/v_1$, where v_1, v_2 are the coordinates of the 2-dimensional contravariant vector. Indeed, this object has the transformation law of the form $\frac{A_1^1 + A_2^1 x}{A_1^2 + A_2^2 x}$ [1]. Thus (17) is of the form

$$f(x) = f(y) \Rightarrow f\left(\frac{A_1^1 + A_2^1 x}{A_1^2 + A_2^2 x}\right) = f\left(\frac{A_1^1 + A_2^1 y}{A_1^2 + A_2^2 y}\right). \quad (20)$$

For $A_2^1 = A_1^2 = 1$ and $A_2^2 = 0$ we have

$$f(x) = f(y) \Rightarrow f(A_1^1 + x) = f(A_1^1 + y)$$

for every $x, y, A_1^1 \in \mathbb{R}$. From here the family F of the levels of f is invariant under arbitrary translations. This yields that this family $F = \mathbb{R}/\mathbb{R}^*$ for a subgroup $\mathbb{R}^*$ of $\mathbb{R}$. Assume that $\{0\} \neq \mathbb{R}^* \neq \mathbb{R}$. This implies that there exist x_1, x_2 such that $x_1 \neq 0$ and $f(0) = f(x_1) \neq f(x_2)$. We have by (20) that $f(x) = f(y) \Rightarrow f(ax) = f(ay)$ for every $x, y, a \in \mathbb{R}$. From here

$$f(0) = f\left(\frac{x_2}{x_1} \cdot 0\right) = f\left(\frac{x_2}{x_1} x_1\right) = f(x_2),$$

thus a contradiction. If $\mathbb{R}^* = \{0\}$, then f is an injection. If $\mathbb{R}^* = \mathbb{R}$, then f is a constant function.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Aczél, J., Gołąb, S.: Funktionalgleichungen der Theorie der geometrischen Objekte, Państwowe Wydawnictwo Naukowe, Warszawa, Monografie Matematyczne **39**, (1960)
- [2] Baron, K., Chojnacki, W., Jaczyk, W.: Continuity of solutions of the translation equation. *Aequ. Math.* **74**, 314–317 (2007)
- [3] Bourbaki, N.: *Éléments de mathématique*. Livre II, Algèbre, Chapitre I, Structures algébriques, Paris (1951)
- [4] Gołąb, S.: Sur quelques points concernant la notion du comitant. *Ann. Soc. Pol. Math.* **17**, 17–192 (1938)
- [5] Łojasiewicz, S.: Sur le problème d'itération. *Colloq. Math.* **3**, 176–177 (1955)

- [6] Moszner, Z.: Sur le prolongement des objets géométriques transitifs. *Tensor N.S.* **26**, 239–242 (1972)
- [7] Moszner, Z.: Structure de l'automate plein, réduit et inversible. *Aequ. Math.* **9**(1), 46–59 (1973)
- [8] Moszner, Z., Pilecka, B.: Sur le prolongement des objets géométriques non-transitifs. *Tensor N.S.* **28**, 63–66 (1974)
- [9] Moszner, Z.: Sur l'équation du comitant. *Tensor N.S.* **51**, 205–208 (1992)
- [10] Moszner, Z.: Miscelanea about the stability of functional equations. In: *Ulam Type Stability*, pp. 231–271, Springer, Berlin (2019)
- [11] Starke, P.H.: *Abstract Automata*. Nort-Holland Publishing Company, Amsterdam-London, American Elsevier Publishing Company, New York (1972)

Zenon Moszner
Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków
Poland
e-mail: zenon.moszner@e.up.krakow.pl

Received: March 16, 2019

Revised: January 4, 2020