New refinements of the discrete Jensen's inequality generated by finite or infinite permutations

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Abstract. In this paper some new refinements of the discrete Jensen's inequality are obtained in real vector spaces. The idea comes from some former refinements determined by cyclic permutations. We essentially generalize and extend these results by using permutations of finite sets and bijections of the set of positive numbers. We get refinements of the discrete Jensen's inequality for infinite convex combinations in Banach spaces. Similar results are rare. Finally, some applications are given on different topics.

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1. Introduction

Different variants of Jensen's inequality and other inequalities have their origin in the notion of convexity. A real function f defined on a convex subset C of a real vector space is called convex if it satisfies

$$f(\alpha v_1 + (1 - \alpha) v_2) \leq \alpha f(v_1) + (1 - \alpha) f(v_2)$$

for all $v_1, v_2 \in C$ and all $\alpha \in [0, 1]$.

The set of positive integers will be denoted by $\mathbb{N}_+$.

The following versions of Jensen's inequality are well known.

Theorem 1.1. (discrete Jensen's inequalities, see [11] and [13]) *(a) Let C be a convex subset of a real vector space V , and let $f : C \rightarrow \mathbb{R}$ be a convex function. If $p_1, \dots, p_n$ are nonnegative numbers with $\sum_{i=1}^n p_i = 1$, and $v_1, \dots, v_n \in C$, then*

$$f\left(\sum_{i=1}^n p_i v_i\right) \leq \sum_{i=1}^n p_i f(v_i). \quad (1.1)$$

(b) Let C be a closed convex subset of a real Banach space V , and let $f : C \rightarrow \mathbb{R}$ be a convex function. If $p_1, p_2, \dots$ are nonnegative numbers with $\sum_{i=1}^{\infty} p_i = 1$, and $v_1, v_2, \dots \in C$ such that the series $\sum_{i=1}^{\infty} p_i v_i$ and $\sum_{i=1}^{\infty} p_i f(v_i)$ are absolutely convergent, then

$$f\left(\sum_{i=1}^{\infty} p_i v_i\right) \leq \sum_{i=1}^{\infty} p_i f(v_i). \quad (1.2)$$

To give refinements of the discrete Jensen's inequality (1.1) is an extensively investigated theme with numerous methods and results (see e.g. the book [8] and references therein), and applications (see e.g. [5] and [6]). Then again, to the best of my knowledge, there are no refinements of the discrete Jensen's inequality (1.2) in such generality. There are some refinements of (1.2) when C is an interval of $\mathbb{R}$: either one estimates formulas in (1.2) in a suitable way (see [12]) or one can obtain results from refinements of integral Jensen's inequality (see [7]).

The following refinement of (1.1) can be found in [9] (see also [1]).

Theorem 1.2. Let $2 \leq k \leq n$ be integers, and let $p_1, \dots, p_n$ and $\lambda_1, \dots, \lambda_k$ be positive numbers with $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^k \lambda_i = 1$. If C is a convex subset of a real vector space V , $f : C \rightarrow \mathbb{R}$ is a convex function, and $v_1, \dots, v_n \in C$, then

$$\begin{aligned} f\left(\sum_{i=1}^n p_i v_i\right) &\leq C_{dis} = C_{dis}(f, \mathbf{v}, \mathbf{p}, \boldsymbol{\lambda}) \\ &:= \sum_{i=1}^n \left(\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}\right) f\left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j} v_{i+j}}{\sum_{j=0}^{k-1} \lambda_{j+1} p_{i+j}}\right) \\ &\leq \sum_{i=1}^n p_i f(v_i), \end{aligned} \quad (1.3)$$

where $i + j$ means $i + j - n$ in case of $i + j > n$.

It is easy to think that the previous result cannot be generalized for infinite sums, but we can observe that the middle term C_{dis} in (1.3) can be rewritten in the following form

$$C_{dis} = \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)}\right) f\left(\frac{\sum_{j=1}^k \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j p_{\pi_j(i)}}\right), \quad (1.4)$$

where π_j ($j = 1, \dots, k$) is the $(j-1)$ -cyclic permutation of the set $\{1, \dots, n\}$ to the right (all elements are moved to the right $j-1$ times with elements overflowing from the right being inserted to the left).

In this paper we show that formulas like (1.4) refine both (1.1) and (1.2) by using either permutations of the set $\{1, \dots, n\}$ or bijections from $\mathbb{N}_+$ onto

itself. On the one hand, an essential generalization of Theorem 1.2 is given, on the other hand, refinements of (1.2) are developed without assuming that V is a special Banach space. Finally, we give some applications concerning information theory, the norm function, Hölder's inequality and the inequality of arithmetic and geometric means.

2. Main results

The positive part f^+ and the negative part f^- of a real valued function f are defined in the usual way.

Let the set I denote either $\{1, \dots, n\}$ for some $n \geq 1$ or $\mathbb{N}_+$. We say that the numbers $(p_i)_{i \in I}$ represent a (positive) discrete probability distribution if $(p_i > 0)$ $p_i \geq 0$ ($i \in I$) and $\sum_{i \in I} p_i = 1$. A permutation π of I refers to a bijection from I onto itself.

We need the following hypotheses which are partitioned into two classes:

- (H₁) Let $k, n \geq 2$ be integers, and let $p_1, \dots, p_n$ and $\lambda_1, \dots, \lambda_k$ represent positive probability distributions.
- (H₂) For each $j = 1, \dots, k$ let π_j be a permutation of the set $\{1, \dots, n\}$.
- (H₃) Let C be a convex subset of a real vector space V , and $f : C \rightarrow \mathbb{R}$ be a convex function.
- (C₁) Let the set J denote either $\{1, \dots, k\}$ for some $k \geq 2$ or $\mathbb{N}_+$. Let $p_1, p_2, \dots$ and $(\lambda_j)_{j \in J}$ represent positive probability distributions.
- (C₂) For each $j \in J$ let π_j be a permutation of the set $\mathbb{N}_+$.
- (C₃) Let C be a closed convex subset of a real Banach space $(V, \|\cdot\|)$, and $f : C \rightarrow \mathbb{R}$ be a convex function.

Theorem 2.1. (a) Assume (H₁), (H₂) and (H₃). If $v_1, \dots, v_n \in C$, then

$$\begin{aligned} f\left(\sum_{i=1}^n p_i v_i\right) &\leq C_{per} = C_{per}(f, \mathbf{v}, \mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{\pi}) \\ &:= \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)}\right) f\left(\frac{\sum_{j=1}^k \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j p_{\pi_j(i)}}\right) \\ &\leq \sum_{i=1}^n p_i f(v_i). \end{aligned} \quad (2.1)$$

(b) Assume (C₁), (C₂) and (C₃). If $v_1, v_2, \dots \in C$ such that the series $\sum_{i=1}^{\infty} p_i v_i$ and $\sum_{i=1}^{\infty} p_i f(v_i)$ are absolutely convergent, then

$$f\left(\sum_{i=1}^{\infty} p_i v_i\right) \leq C_{per} = C_{per}(f, \mathbf{v}, \mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{\pi})$$

$$\begin{aligned}
&:= \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right) \\
&\leq \sum_{i=1}^{\infty} p_i f(v_i). \tag{2.2}
\end{aligned}$$

Proof. (a) By using Theorem 1.1 (a) and the fact that π_j is a permutation of the set $\{1, \dots, n\}$,

$$\begin{aligned}
C_{per} &\leq \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right) = \sum_{j=1}^k \lambda_j \left(\sum_{i=1}^n p_{\pi_j(i)} f(v_{\pi_j(i)}) \right) \\
&= \left(\sum_{j=1}^k \lambda_j \right) \left(\sum_{i=1}^n p_i f(v_i) \right) = \sum_{i=1}^n p_i f(v_i).
\end{aligned}$$

The left hand side inequality can be proved similarly. Since

$$\sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} \right) = \left(\sum_{j=1}^k \lambda_j \right) \left(\sum_{i=1}^n p_i \right) = 1,$$

the discrete Jensen's inequality implies that

$$C_{per} \geq f \left(\sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} v_{\pi_j(i)} \right) \right) = f \left(\sum_{i=1}^n p_i v_i \right).$$

(b) The proof is divided into four parts.

I. We first prove that the series

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \|v_{\pi_j(i)}\| \right) \tag{2.3}$$

and

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} |f(v_{\pi_j(i)})| \right) \tag{2.4}$$

are convergent and

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)} \right) = \sum_{i=1}^{\infty} p_i v_i, \tag{2.5}$$

and

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right) = \sum_{i=1}^{\infty} p_i f(v_i).$$

For each $j \in J$ the series

$$\sum_{i=1}^{\infty} p_{\pi_j(i)} v_{\pi_j(i)} \quad (2.6)$$

is a rearrangement of the absolutely convergent series $\sum_{i=1}^{\infty} p_i v_i$, and hence it is also absolutely convergent and

$$\sum_{i=1}^{\infty} p_{\pi_j(i)} v_{\pi_j(i)} = \sum_{i=1}^{\infty} p_i v_i. \quad (2.7)$$

(i) If $J = \{1, \dots, k\}$, then it follows trivially from (2.7) that

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} v_{\pi_j(i)} \right) = \sum_{i=1}^{\infty} p_i v_i.$$

(ii) Assume $J = \mathbb{N}_+$. The property of absolute convergence of (2.6) implies that

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \lambda_j p_{\pi_j(i)} \|v_{\pi_j(i)}\| \right) &= \sum_{j=1}^{\infty} \lambda_j \left(\sum_{i=1}^{\infty} p_{\pi_j(i)} \|v_{\pi_j(i)}\| \right) \\ &= \sum_{i=1}^{\infty} p_i \|v_i\| < \infty. \end{aligned}$$

Therefore, as it is well known, the order of summation can be interchanged in the double sum

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right),$$

and hence

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)} \right) = \sum_{i=1}^{\infty} p_i v_i.$$

The series in (2.4) can be handled in a similar way.

II. (i) If $J = \{1, \dots, k\}$, then Theorem 1.1 (a) gives us that for every $n \in \mathbb{N}_+$

$$\sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} \right) f \left(\frac{\sum_{j=1}^k \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j p_{\pi_j(i)}} \right) \leq \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right). \quad (2.8)$$

(ii) Assume $J = \mathbb{N}_+$. For each $i \in \mathbb{N}_+$ the series $\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}$ is obviously convergent and

$$\frac{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} \|v_{\pi_j(i)}\|}{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}} \leq \frac{1}{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}} \sum_{j=1}^{\infty} p_{\pi_j(i)} \|v_{\pi_j(i)}\|$$

$$= \frac{1}{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}} \sum_{i=1}^{\infty} p_i \|v_i\| < \infty,$$

and hence the series

$$\frac{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} \|v_{\pi_j(i)}\|}{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}}$$

is absolutely convergent. Further, we know from part I that the series

$$\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}), \quad i \in \mathbb{N}_+$$

is also absolutely convergent.

Consequently, Theorem 1.1 (b) shows that

$$\sum_{i=1}^n \left(\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} \right) f \left(\frac{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)}} \right) \leq \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \lambda_j p_{\pi_j(i)} f(v_{\pi_j(i)}) \right). \quad (2.9)$$

We have seen in part I that the series (2.4) is convergent and its sum is $\sum_{i=1}^{\infty} p_i f(v_i)$. Thus by (2.8) or by (2.9), the second inequality in (2.2) will be proved if we succeed in showing that the series

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right) \quad (2.10)$$

is convergent.

It is known that the positive part of f is also convex. The convergence of (2.4) implies the convergence of the series

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} f^+(v_{\pi_j(i)}) \right).$$

It now follows that we can copy the proofs of (2.8) and (2.9) with f^+ instead of f . Taking account of the nonnegativity of f^+ , we obtain

$$\begin{aligned} & \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f^+ \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right) \\ & \leq \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} f^+(v_{\pi_j(i)}) \right) < \infty. \end{aligned} \quad (2.11)$$

From this it follows that the series (2.10) will be convergent (absolutely) if and only if

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f^- \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right) < \infty. \quad (2.12)$$

III. In this step we show that the series (2.12) is convergent and the first inequality in (2.2) holds assuming f is bounded below, that is $f(v) \geq c$ ($v \in C$) for some nonpositive number c .

Since

$$\begin{aligned} \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f^- \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right) &\leq -c \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) \\ &= -c \sum_{j \in J} \lambda_j \left(\sum_{i=1}^{\infty} p_{\pi_j(i)} \right) = -c, \end{aligned}$$

the series (2.12) is convergent.

According to (2.11) and (2.12) the series (2.10) is absolutely convergent.

Since the series (2.3) and (2.10) are absolutely convergent and (2.5) holds, we can apply Theorem 1.1 (b), and obtain the first inequality in (2.2).

IV. At this point we abandon the lower boundedness hypothesis on f .

Let the function $f_n : C \rightarrow \mathbb{R}$ be defined by

$$f_n(v) = \max(f(v), -n), \quad n \in \mathbb{N}_+.$$

Then f_n ($n \in \mathbb{N}_+$) is convex and bounded below, and $f_n \geq f$ ($n \in \mathbb{N}_+$). From this and from the results of part III, we get that for each $n \in \mathbb{N}_+$ the series

$$\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f_n \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right)$$

is absolutely convergent and

$$f \left(\sum_{i=1}^{\infty} p_i v_i \right) \leq f_n \left(\sum_{i=1}^{\infty} p_i v_i \right) \leq \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) f_n \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right).$$

Since the sequence $(f_n)_{n \in \mathbb{N}_+}$ is decreasing and $\lim_{n \rightarrow \infty} f_n = f$ pointwise, the previous two assertions imply that B. Levi's theorem can be applied, and it gives that the series (2.10) is absolutely convergent and the first inequality in (2.2) holds.

The proof is complete. $\square$

Remark 2.2. It can be seen that Theorem 2.1 (a) contains Theorem 1.2 as a special case.

3. Applications

We begin with some inequalities corresponding to information theory.

The following notion was introduced by Csiszár in [2] and [3].

Definition 3.1. Let $f :]0, \infty[\rightarrow]0, \infty[$ be a convex function, and let $\mathbf{r} := (r_1, \dots, r_n) \in]0, \infty[^n$ and $\mathbf{q} := (q_1, \dots, q_n) \in]0, \infty[^n$. The f -divergence functional is

$$I_f(\mathbf{r}, \mathbf{q}) := \sum_{i=1}^n q_i f\left(\frac{r_i}{q_i}\right).$$

Based on this concept, we have introduced a new functional in [10], and this functional can be further generalized:

Definition 3.2. Let C be a convex subset of a real vector space V , and $f : C \rightarrow \mathbb{R}$ be a convex function. If $\mathbf{w} := (w_1, \dots, w_n) \in V^n$ and $\mathbf{q} := (q_1, \dots, q_n) \in]0, \infty[^n$ such that

$$\frac{w_i}{q_i} \in C, \quad i = 1, \dots, n, \quad (3.1)$$

then define

$$I_f(\mathbf{w}, \mathbf{q}) := \sum_{i=1}^n q_i f\left(\frac{w_i}{q_i}\right).$$

Proposition 3.3. Let $k, n \geq 2$ be integers, and let $\lambda_1, \dots, \lambda_k$ represent a positive probability distribution. Assume (H_2) and (H_3) . If $\mathbf{w} := (w_1, \dots, w_n) \in V^n$ and $\mathbf{q} := (q_1, \dots, q_n) \in]0, \infty[^n$ such that

$$\frac{w_i}{q_i} \in C, \quad i = 1, \dots, n,$$

then

$$\begin{aligned} I_f(\mathbf{w}, \mathbf{q}) &= \sum_{i=1}^n q_i f\left(\frac{w_i}{q_i}\right) \\ &\geq \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j q_{\pi_j(i)} \right) f\left(\frac{\sum_{j=1}^k \lambda_j w_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j q_{\pi_j(i)}} \right) \\ &\geq \left(\sum_{i=1}^n q_i \right) \cdot f\left(\frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n q_i} \right). \end{aligned} \quad (3.2)$$

Proof. By applying Theorem 2.1 (a) with

$$p_i := \frac{q_i}{\sum_{i=1}^n q_i}, \quad v_i := \frac{w_i}{q_i}, \quad i = 1, \dots, n$$

we have

$$\begin{aligned}
 \sum_{i=1}^n q_i f\left(\frac{w_i}{q_i}\right) &= \left(\sum_{i=1}^n q_i\right) \cdot \sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i} f\left(\frac{w_i}{q_i}\right) \\
 &\geq \left(\sum_{i=1}^n q_i\right) \cdot \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j \frac{q_{\pi_j(i)}}{\sum_{i=1}^n q_i}\right) f\left(\frac{\sum_{j=1}^k \lambda_j \frac{q_{\pi_j(i)}}{\sum_{i=1}^n q_i} \frac{w_{\pi_j(i)}}{q_{\pi_j(i)}}}{\sum_{j=1}^k \lambda_j \frac{q_{\pi_j(i)}}{\sum_{i=1}^n q_i}}\right) \\
 &= \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j q_{\pi_j(i)}\right) f\left(\frac{\sum_{j=1}^k \lambda_j w_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j q_{\pi_j(i)}}\right) \\
 &\geq \left(\sum_{i=1}^n q_i\right) \cdot f\left(\frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n q_i}\right).
 \end{aligned}$$

The proof is complete. $\square$

Remark 3.4. (a) It was proved in [4] that if $f :]0, \infty[\rightarrow]0, \infty[$ is a convex function, and $\mathbf{r} := (r_1, \dots, r_n) \in]0, \infty[^n$ and $\mathbf{q} := (q_1, \dots, q_n) \in]0, \infty[^n$, then

$$I_f(\mathbf{r}, \mathbf{q}) \geq \sum_{i=1}^n q_i f\left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n q_i}\right).$$

From Proposition 3.3 we can obtain the following refinement of this inequality:

$$I_f(\mathbf{r}, \mathbf{q}) \geq \sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j q_{\pi_j(i)}\right) f\left(\frac{\sum_{j=1}^k \lambda_j r_{\pi_j(i)}}{\sum_{j=1}^k \lambda_j q_{\pi_j(i)}}\right) \geq \sum_{i=1}^n q_i f\left(\frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n q_i}\right). \quad (3.3)$$

(b) Let $f := -\log$, where the base of log is greater than 1, $\mathbf{r} := (1, \dots, 1)$, and $\mathbf{q} := (q_1, \dots, q_n)$ represent a positive probability distribution. Then (3.3) gives

$$\begin{aligned}
H(\mathbf{q}) &:= -\sum_{i=1}^n q_i \log(q_i) \\
&\leq -\sum_{i=1}^n \left(\sum_{j=1}^k \lambda_j q_{\pi_j(i)} \right) \log \left(\sum_{j=1}^k \lambda_j q_{\pi_j(i)} \right) \leq \log(n),
\end{aligned}$$

which is a refinement of a remarkable inequality for the Shannon entropy.

Next we establish inequalities for the norm function.

Proposition 3.5. *Assume (C_1) and (C_2) , and assume $(V, \|\cdot\|)$ is a Banach space. If $v_1, v_2, \dots \in V$ such that the series $\sum_{i=1}^{\infty} p_i \|v_i\|^\alpha$ is absolutely convergent for some $\alpha \geq 1$, then*

$$\left\| \sum_{i=1}^{\infty} p_i v_i \right\|^\alpha \leq \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} \right) \left\| \frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \right\|^\alpha \leq \sum_{i=1}^{\infty} p_i \|v_i\|^\alpha.$$

Proof. Since the function $f : V \rightarrow \mathbb{R}$ defined by $f(v) = \|v\|^\alpha$ is convex, and the series $\sum_{i=1}^{\infty} p_i v_i$ is also absolutely convergent, the result follows from Theorem 2.1 (b). $\square$

Now we get a refinement of the discrete Hölder's inequality.

Proposition 3.6. *Let the set J denote either $\{1, \dots, k\}$ for some $k \geq 2$ or $\mathbb{N}_+$, and let $(\lambda_j)_{j \in J}$ represent a positive probability distribution. Let $(w_i)_{i=1}^{\infty}$ be a sequence of positive numbers, and let $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ be sequences of nonnegative numbers such that the series $\sum_{i=1}^{\infty} w_i x_i^p$ and $\sum_{i=1}^{\infty} w_i y_i^q$ are convergent, where $p > 1$ and $q > 1$ are conjugate exponents that is $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
\sum_{i=1}^{\infty} w_i x_i y_i &\leq \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j w_{\pi_j(i)} x_{\pi_j(i)}^p \right)^{\frac{1}{p}} \left(\sum_{j \in J} \lambda_j w_{\pi_j(i)} y_{\pi_j(i)}^q \right)^{\frac{1}{q}} \\
&\leq \left(\sum_{i=1}^{\infty} w_i x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} w_i y_i^q \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.4}$$

Proof. For each $s > 1$ the power function

$$f_s :]0, \infty[\rightarrow \mathbb{R}, \quad f_s(x) = x^s \tag{3.5}$$

is strictly convex. Let $v_1, v_2, \dots$ be positive numbers such that the series $\sum_{i=1}^{\infty} p_i v_i^s$ is convergent. Theorem 2.1 (b) can be applied to the function f_s and to the positive numbers $v_1, v_2, \dots$, and it yields that

$$\left(\sum_{i=1}^{\infty} p_i v_i\right)^s \leq \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)}\right)^{1-s} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}\right)^s \leq \sum_{i=1}^{\infty} p_i v_i^s. \quad (3.6)$$

If $\sum_{i=1}^{\infty} w_i y_i^q = 0$, then (3.4) is obvious.

Otherwise, from the inequality (3.6) with the choices

$$s = \frac{1}{p}, \quad p_i = \frac{w_i y_i^q}{\sum_{l=1}^{\infty} w_l y_l^q}, \quad v_i = x_i^p y_i^{-q}, \quad i \in \mathbb{N}_+$$

($-f_{1/p}$ is convex) we obtain

$$\begin{aligned} & \left(\frac{1}{\sum_{l=1}^{\infty} w_l y_l^q}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} w_i x_i^p\right)^{\frac{1}{p}} \geq \frac{1}{\sum_{l=1}^{\infty} w_l y_l^q} \\ & \cdot \sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j w_{\pi_j(i)} y_{\pi_j(i)}^q\right)^{\frac{1}{q}} \left(\sum_{j \in J} \lambda_j w_{\pi_j(i)} x_{\pi_j(i)}^p\right)^{\frac{1}{p}} \\ & \geq \frac{1}{\sum_{l=1}^{\infty} w_l y_l^q} \sum_{i=1}^n w_i x_i y_i, \end{aligned}$$

and this delivers the desired conclusion.

The proof is complete. $\square$

Finally, we apply our results to get a refinement of the inequality of generalized arithmetic and geometric means.

Proposition 3.7. Assume (C_1) and (C_2) . If $v_1, v_2, \dots$ are positive numbers such that the series $\sum_{i=1}^{\infty} p_i v_i$ and $\sum_{i=1}^{\infty} p_i \ln(v_i)$ are absolutely convergent, then

$$\sum_{i=1}^{\infty} p_i v_i \geq \prod_{i=1}^{\infty} \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}}\right)^{\sum_{j \in J} \lambda_j p_{\pi_j(i)}} \geq \prod_{i=1}^{\infty} v_i^{p_i}. \quad (3.7)$$

Proof. By applying Theorem 2.1 (b) to the convex function $-\ln$, we obtain

$$\begin{aligned} & -\ln \left(\sum_{i=1}^{\infty} p_i v_i\right) \\ & \leq -\sum_{i=1}^{\infty} \left(\sum_{j \in J} \lambda_j p_{\pi_j(i)}\right) \ln \left(\frac{\sum_{j \in J} \lambda_j p_{\pi_j(i)} v_{\pi_j(i)}}{\sum_{j \in J} \lambda_j p_{\pi_j(i)}}\right) \leq -\sum_{i=1}^{\infty} p_i \ln(v_i), \end{aligned}$$

and this is equivalent to (3.7). $\square$

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