Embeddability of pairs of weighted quasi-arithmetic means into a semiflow

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Dedicated to Professor János Aczél on his 95th birthday

Abstract. We determine the form of all semiflows of pairs of weighted quasi-arithmetic means, those over positive dyadic numbers as well as the continuous ones. Then the iterability of such pairs is characterized: necessary and sufficient conditions for a given pair of weighted quasi-arithmetic means to be embeddable into a continuous semiflow are given. In particular, it turns out that surprisingly the existence of a square iterative root in the class of such pairs implies the embeddability.

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1. Introduction

One crucial problem of dynamical system is the embeddability of an individual function into a flow or, more generally, into a semiflow. Given an arbitrary set X and a subgroup T of the additive group of reals [subsemigroup T of the additive semigroup of positive reals] any function $F : X \times T \rightarrow X$, satisfying the celebrated translation equation

$$F(F(x, s), t) = F(x, s + t),$$

is called a T -flow [T -semiflow] in X . In the case when X is endowed with a topology a flow $F : X \times \mathbb{R} \rightarrow X$ [semiflow $F : X \times (0, +\infty) \rightarrow X$] is said to be *continuous* if the function $F(x, \cdot)$ is continuous for each $x \in X$. In iteration theory continuous flows and semiflows are usually called *continuous iteration groups* and *semigroups*, respectively. A function $f : X \rightarrow X$ is said to be *embeddable into a T -flow [T -semiflow]* if there exists a flow [semiflow] $F : X \times T \rightarrow X$ such that $f = F(\cdot, 1)$.

Embeddability problems have been discussed by many authors in different settings. The following ones can serve as examples of such research. In particular, in 1968 Karlin and McGregor published two papers on this: [7] concerning the embeddability of branching processes and [8] dealing with such a problem for analytic functions with two fixed points. Embeddability of homeomorphisms into differential flows was studied by Ping Fun Lam [12] in 1976. A number of papers devoted to different variants of the embedding problem was written by Zdun. He answered, among others, the questions of how to embed continuous strictly monotonic functions defined on an interval [15], homeomorphisms of the circle [16], two commuting functions [17] and, jointly with Solarz, diffeomorphisms of the plane in a regular iteration semigroup [20]. Also some problems close to embedding were considered, like approximative embedding (see [18] by Zdun) and embeddability of homeomorphisms of the circle into set-valued flows (see [9] by Krassowska and Zdun). Embedding problems have been discussed also in a number of monographs and surveys, e.g. [10] by Kuczma, [14] by Targonski, [11] by Kuczma, Choczewski and Ger, [2] by Belitskii and Tkachenko, [19] by Zdun and Solarz.

In 2010 the third present author and Matkowski considered the embeddability problem for pairs of homogeneous symmetric means; see [6]. Given any real interval I a *two-variable mean* M on I is any function $M : I^2 \rightarrow I$ satisfying

$$\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}, \quad x, y \in I.$$

M is *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$. In the case when $I = (0, +\infty)$ or $I = \mathbb{R}$ the mean M is called *homogeneous* if

$$M(tx, ty) = tM(x, y), \quad x, y \in I,$$

for all $t \in (0, +\infty)$ or for all $t \in \mathbb{R}$, respectively. Here we study the embeddability of pairs of *weighted quasi-arithmetic means*, that is means A_p^f of the form

$$A_p^f(x, y) = f^{-1}(pf(x) + (1-p)f(y)),$$

where $f : I \rightarrow \mathbb{R}$ is any continuous strictly monotonic function and $p \in (0, 1)$. Observe that weighted quasi-arithmetic means are, in general, neither symmetric, nor homogeneous. Clearly, A_p^f is symmetric if and only if $p = 1/2$. On the other hand, provided $I = (0, +\infty)$ or $I = \mathbb{R}$ it follows from [3, Cor. 5.1] that A_p^f is homogeneous if and only if it is either the *weighted Hölder mean* of the form $(px^\alpha + (1-p)y^\alpha)^{1/\alpha}$ with a non-zero real α , or the *weighted geometric mean* of the form $x^p y^{1-p}$; in the case when $p = 1/2$ the form of homogeneous means A_p^f was known for Hardy, Littlewood and Pólya already in 1934 (see [5]). So the research reported here substantially differs from that presented in [6].

Denote by $\mathcal{CM}(I)$ the class of continuous strictly monotonic functions mapping the interval I into $\mathbb{R}$. We say that functions $f, g \in \mathcal{CM}(I)$ are *equivalent* if there are numbers $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$g(x) = af(x) + b, \quad x \in I;$$

then we write $f \sim g$. Clearly, $\sim$ is an equivalence relation in the set $\mathcal{CM}(I)$. Using this notion one can formulate the following result solving the so-called equality problem for weighted quasi-arithmetic means (see [1, Sec. 6.4.3, Theorem 2] also [13] by Maksa and Páles; for quasi-arithmetic means the answer was known already in 1934 and can be found in the book [5]).

Theorem A. *Let $f, g \in \mathcal{CM}(I)$ and $p, q \in (0, 1)$. Then $A_p^f = A_q^g$ if and only if $f \sim g$ and $p = q$.*

2. Posing the problem

We start with the definition of iterability. Given an additive semigroup $T \subset (0, +\infty)$ containing 1, a set X and a family $\mathcal{A}$ of self-mappings of X , a function from $\mathcal{A}$ is said to be *T -iterable in $\mathcal{A}$* if it is embeddable into a T -semiflow $\Phi : X \times T \rightarrow X$ such that $\Phi(\cdot, t) \in \mathcal{A}$ for every $t \in T$. In the case when X is a topological space a function from $\mathcal{A}$ is called *continuously iterable in $\mathcal{A}$* if it is embeddable into a continuous semiflow $\Phi : X \times (0, +\infty) \rightarrow X$ with $\Phi(\cdot, t) \in \mathcal{A}$ for every $t \in T$.

Denote by $\mathcal{A}$ the set of pairs of weighted quasi-arithmetic means on an interval I :

$$\mathcal{A} := \{(A_p^f, A_q^g) : f, g \in \mathcal{CM}(I), p, q \in (0, 1)\}.$$

Our first aim is to find the form of all continuous semiflows of pairs from $\mathcal{A}$. As a consequence we obtain the main result giving a characterization of the continuous iterability of an arbitrary pair $(A_p^f, A_q^g) \in \mathcal{A}$ in $\mathcal{A}$. The main tool used in the proofs presented in Sects. 3, 4 and 5 is the following result originating from the paper [4]. Here

$$\Delta = \{(p, q) \in (0, 1)^2 : p \geq q\}$$

and

$$\Gamma = \{(p, q) \in \Delta : q > p(1 - p) \text{ and } p < 1 - (1 - q)q\}.$$

Theorem B. *Let $f, g, \varphi, \psi \in \mathcal{CM}(I)$ and $p, q, \mu, \nu \in (0, 1)$. Then the pair $(A_\mu^\varphi, A_\nu^\psi)$ is a square iterative root of (A_p^f, A_q^g) if and only if*

$$f \sim g, \quad \varphi \sim f, \quad \psi \sim g$$

and one of the following cases holds:

(i) $(p, q) \in \Gamma$ and either

$$\mu = \frac{p + \sqrt{p - q}}{1 + \sqrt{p - q}} \quad \text{and} \quad \nu = \frac{q}{1 + \sqrt{p - q}}, \quad (1)$$

or

$$\mu = \frac{p - \sqrt{p - q}}{1 - \sqrt{p - q}} \quad \text{and} \quad \nu = \frac{q}{1 - \sqrt{p - q}};$$

(ii) $(p, q) \in \Delta \setminus \Gamma$ and condition (1) is satisfied.

3. The form of continuous semiflows of pairs of weighted quasi-arithmetic means

To prove Theorems 4 and 6, which are the main results of this section, we need some auxiliary facts. The first one reduces determining semiflows of pairs from $\mathcal{A}$ to solving a system of functional equations.

Proposition 1. *Let $T \subset (0, +\infty)$ be an additive semigroup, $\mu, \nu: T \rightarrow (0, +\infty)$ and let $f \in \mathcal{CM}(I)$. The function $F: I^2 \times T \rightarrow I^2$, defined by*

$$F(\cdot, t) = \left(A_{\mu(t)}^f, A_{\nu(t)}^f \right),$$

is a semiflow if and only if the pair (μ, ν) satisfies the system of equations

$$\begin{cases} \mu(s+t) = \mu(t)\mu(s) + (1-\mu(t))\nu(s) \\ \nu(s+t) = \nu(t)\mu(s) + (1-\nu(t))\nu(s). \end{cases} \quad (2)$$

In the proof we use the following lemma which can be easily verified.

Lemma 2. *If $f \in \mathcal{CM}(I)$ and $p_1, p_2, q_1, q_2 \in (0, 1)$, then*

$$(A_{p_2}^f, A_{q_2}^f) \circ (A_{p_1}^f, A_{q_1}^f) = \left(A_{p_2 p_1 + (1-p_2)q_1}^f, A_{q_2 p_1 + (1-q_2)q_1}^f \right).$$

Proof of Proposition 1. For all $s, t \in T$, by Lemma 2, we have

$$\begin{aligned} F(\cdot, t) \circ F(\cdot, s) &= \left(A_{\mu(t)}^f, A_{\nu(t)}^f \right) \circ \left(A_{\mu(s)}^f, A_{\nu(s)}^f \right) \\ &= \left(A_{\mu(t)\mu(s) + (1-\mu(t))\nu(s)}^f, A_{\nu(t)\mu(s) + (1-\nu(t))\nu(s)}^f \right) \\ &= \left(A_{\mu(s+t)}^f, A_{\nu(s+t)}^f \right) = F(\cdot, s+t), \end{aligned}$$

and thus $F(\cdot, s+t) = F(\cdot, t) \circ F(\cdot, s)$ if and only if the equalities in (2) hold. $\square$

The next result will be derived from Theorems B and A.

Proposition 3. *If $(F_n)_{n \in \mathbb{N}_0}$ is a sequence of pairs of weighted quasi-arithmetic means on I such that $F_n \circ F_n = F_{n-1}$ for all $n \in \mathbb{N}$, then there exist a function $f \in \mathcal{CM}(I)$ and numbers $p, q \in (0, 1)$ such that $p \geq q$ and*

$$F_n = (A_{p_n}^f, A_{q_n}^f), \quad n \in \mathbb{N}_0, \quad (3)$$

where

$$p_n = \frac{q + (1-p)(p-q)^{\frac{1}{2^n}}}{1 - (p-q)} \quad \text{and} \quad q_n = \frac{q - q(p-q)^{\frac{1}{2^n}}}{1 - (p-q)}, \quad n \in \mathbb{N}_0. \quad (4)$$

Proof. By Theorems B and A there exist a function $f \in \mathcal{CM}(I)$, numbers $p, q \in (0, 1)$ and sequences $(p_n)_{n \in \mathbb{N}_0}$ and $(q_n)_{n \in \mathbb{N}_0}$ of numbers from $(0, 1)$ such that $p_0 = p$, $q_0 = q$, $p_n \geq q_n$ for all $n \in \mathbb{N}_0$ and (3) holds true; moreover, either

$$p_{n+1} = \frac{p_n + \sqrt{p_n - q_n}}{1 + \sqrt{p_n - q_n}} \quad \text{and} \quad q_{n+1} = \frac{q_n}{1 + \sqrt{p_n - q_n}} \quad (5)$$

or

$$p_{n+1} = \frac{p_n - \sqrt{p_n - q_n}}{1 - \sqrt{p_n - q_n}} \quad \text{and} \quad q_{n+1} = \frac{q_n}{1 - \sqrt{p_n - q_n}} \quad (6)$$

for every $n \in \mathbb{N}_0$. Suppose that the conditions in (5) do not hold for some $n \in \mathbb{N}_0$. Then the equalities in (6) are true, hence

$$0 \leq p_{n+1} - q_{n+1} = \frac{-\sqrt{p_n - q_n} + (p_n - q_n)}{1 - \sqrt{p_n - q_n}} = -\sqrt{p_n - q_n}.$$

Consequently, $p_n = q_n$, also $p_{n+1} = q_{n+1}$, and thus (5) follows contrary to the supposition. This implies that the equalities in (5) hold for all $n \in \mathbb{N}_0$.

In view of (5), where n is replaced by $n-1$, we have

$$p_n - q_n = \frac{\sqrt{p_{n-1} - q_{n-1}} + (p_{n-1} - q_{n-1})}{1 + \sqrt{p_{n-1} - q_{n-1}}} = (p_{n-1} - q_{n-1})^{\frac{1}{2}}, \quad n \in \mathbb{N},$$

which implies the condition

$$p_n - q_n = (p - q)^{\frac{1}{2^n}}, \quad n \in \mathbb{N}_0. \quad (7)$$

Now, using the first equality in (5), we prove by induction the first part of (4) and then, applying (7), also the second one. $\square$

Now we are in a position to describe all $\mathbb{D}_+$ -semiflows of pairs from the family $\mathcal{A}$, where $\mathbb{D}_+$ stands for the set of all positive dyadic numbers.

Theorem 4. *A function $F : I^2 \times \mathbb{D}_+ \rightarrow I^2$ is a semiflow of pairs of weighted quasi-arithmetic means if and only if there exist a function $f \in \mathcal{CM}(I)$ and numbers $p, q \in (0, 1)$ such that $p \geq q$ and*

$$F(\cdot, t) = \left(A_{\mu(t)}^f, A_{\nu(t)}^f \right), \quad t \in \mathbb{D}_+, \quad (8)$$

where the functions $\mu, \nu : \mathbb{D}_+ \rightarrow (0, 1)$ are given by

$$\mu(t) = \frac{q + (1-p)(p-q)^t}{1 - (p-q)} \quad \text{and} \quad \nu(t) = \frac{q - q(p-q)^t}{1 - (p-q)}. \quad (9)$$

In the proof of Theorem 4 we need the following useful lemma.

Lemma 5. *Let $p, q \in (0, 1)$, $p \geq q$. Then the pair of functions $\mu, \nu : \mathbb{D}_+ \rightarrow \mathbb{R}$, given by (9), satisfies system (2). Moreover, $\mu(t) - \nu(t) = (p-q)^t$ for all $t \in \mathbb{D}_+$. If $p = q$, then $\mu(t) = \nu(t) = p = q$, $t \in \mathbb{D}_+$. If $p > q$, then μ is strictly decreasing and maps $\mathbb{D}_+$ into $\left(\frac{q}{1-p+q}, 1\right)$, and ν is strictly increasing and maps $\mathbb{D}_+$ into $\left(0, \frac{q}{1-p+q}\right)$. In particular, μ and ν take all their values in $(0, 1)$.*

Proof. For any $t \in \mathbb{D}_+$ we have

$$\begin{aligned} \mu(t) - \nu(t) &= \frac{q + (1-p)(p-q)^t}{1 - (p-q)} - \frac{q - q(p-q)^t}{1 - (p-q)} \\ &= \frac{[1 - (p-q)](p-q)^t}{1 - (p-q)} = (p-q)^t. \end{aligned}$$

Hence, if $s, t \in \mathbb{D}_+$, then

$$\begin{aligned} \mu(t)\mu(s) + (1 - \mu(t))\nu(s) &= \mu(t)(\mu(s) - \nu(s)) + \nu(s) = \mu(t)(p-q)^s + \nu(s) \\ &= \frac{q(p-q)^s + (1-p)(p-q)^{s+t} + q - q(p-q)^s}{1 - (p-q)} \\ &= \frac{q + (1-p)(p-q)^{s+t}}{1 - (p-q)} = \mu(s+t) \end{aligned}$$

and

$$\begin{aligned} \nu(t)\mu(s) + (1 - \nu(t))\nu(s) &= \nu(t)(\mu(s) - \nu(s)) + \nu(s) = \nu(t)(p-q)^s + \nu(s) \\ &= \frac{q(p-q)^s - q(p-q)^{s+t} + q - q(p-q)^s}{1 - (p-q)} \\ &= \frac{q - q(p-q)^{s+t}}{1 - (p-q)} = \nu(s+t). \end{aligned}$$

To prove the remaining properties it is enough to notice that if $p > q$, then the function $\mathbb{D}_+ \ni t \mapsto (p-q)^t$ is strictly decreasing and

$$\lim_{t \rightarrow 0} (p-q)^t = 1 \quad \text{and} \quad \lim_{t \rightarrow +\infty} (p-q)^t = 0.$$

□

Proof of Theorem 4. Assume that $F : I^2 \times \mathbb{D}_+ \rightarrow I^2$ is of the form (8) with some function $f \in \mathcal{CM}(I)$, numbers $p, q \in (0, 1)$ such that $p \geq q$, and the functions $\mu, \nu : \mathbb{D}_+ \rightarrow \mathbb{R}$ defined by (9). Then Lemma 5 implies that $\mu(\mathbb{D}_+), \nu(\mathbb{D}_+) \subset (0, 1)$ and (μ, ν) satisfies system (2). Therefore, on account of Proposition 1, the function F is a semiflow.

Now assume that $F : I^2 \times \mathbb{D}_+ \rightarrow I^2$ is a semiflow. Then, by Proposition 3 applied to the sequence $(F_n)_{n \in \mathbb{N}_0}$ given by $F_n = F(\cdot, 1/2^n)$, we see that

$$F\left(\cdot, \frac{1}{2^n}\right) = (A_{p_n}^f, A_{q_n}^f), \quad n \in \mathbb{N}_0,$$

for some $f \in \mathcal{CM}(I)$, numbers $p, q \in (0, 1)$ satisfying $p \geq q$ and the sequences $(p_n)_{n \in \mathbb{N}_0}$ and $(q_n)_{n \in \mathbb{N}_0}$ defined by (4). Therefore, according to (4),

$$F\left(\cdot, \frac{1}{2^n}\right) = \left(A_{\mu(\frac{1}{2^n})}^f, A_{\nu(\frac{1}{2^n})}^f\right), \quad n \in \mathbb{N}_0,$$

where $\mu, \nu : \mathbb{D}_+ \rightarrow \mathbb{R}$ are given by (9). It follows from Lemma 5 and Proposition 1 that the mapping $\mathbb{D}_+ \ni t \mapsto (A_{\mu(t)}^f, A_{\nu(t)}^f)$ is a semiflow, and thus, taking arbitrary numbers $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have

$$F\left(\cdot, \frac{k}{2^n}\right) = F\left(\cdot, \frac{1}{2^n}\right)^k = \left(A_{\mu(\frac{1}{2^n})}^f, A_{\nu(\frac{1}{2^n})}^f\right)^k = \left(A_{\mu(\frac{k}{2^n})}^f, A_{\nu(\frac{k}{2^n})}^f\right),$$

that is condition (8) holds. $\square$

Since $\mathbb{D}_+$ is a dense subset of $(0, +\infty)$ we get the following characterization of continuous semiflows of pairs of weighted quasi-arithmetic means as an immediate consequence of Theorem 4.

Theorem 6. *Let $F : I^2 \times (0, +\infty) \rightarrow I^2$ be a function such that $F(\cdot, t)$ is continuous for each $t \in (0, +\infty)$. The function F is a continuous semiflow of pairs of weighted quasi-arithmetic means if and only if there exist a function $f \in \mathcal{CM}(I)$ and numbers $p, q \in (0, 1)$ such that $p \geq q$ and*

$$F(\cdot, t) = (A_{\mu(t)}^f, A_{\nu(t)}^f), \quad t \in (0, +\infty),$$

where the functions $\mu, \nu : (0, +\infty) \rightarrow (0, 1)$ are given by (9).

4. Iterability of pairs of weighted quasi-arithmetic means

In the present section we formulate and prove a little bit surprising characterization of iterability of pairs (A_p^f, A_q^g) in the family $\mathcal{A}$.

Theorem 7. *Let $f, g \in \mathcal{CM}(I)$ and $p, q \in (0, 1)$. Then the following statements are pairwise equivalent:*

- (i) the pair (A_p^f, A_q^g) is continuously iterable;
- (ii) the pair (A_p^f, A_q^g) is $(0, +\infty)$ -iterable;
- (iii) the pair (A_p^f, A_q^g) is $\mathbb{D}_+$ -iterable;
- (iv) the pair (A_p^f, A_q^g) has a square iterative root being an element of the family $\mathcal{A}$;
- (v) $f \sim g$ and $p \geq q$.

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious, whereas (iv) $\Rightarrow$ (v) follows from Theorem B. To get the implication (v) $\Rightarrow$ (i) it is enough to make use of Theorems 6 and A. $\square$

5. Solutions of system (2)

We complete the paper with solving the system of equations (2). Notice that this is done as a byproduct of studying the embeddability problem.

Theorem 8. *Let $\mu, \nu : T \rightarrow (0, +\infty)$, where T is either $\mathbb{D}_+$, or $(0, +\infty)$. If $T = \mathbb{D}_+$, then the pair (μ, ν) is a solution of system (2) if and only if the equalities in (9) hold for all $t \in \mathbb{D}_+$. If $T = (0, +\infty)$, then the pair (μ, ν) is a continuous solution of system (2) if and only if the equalities in (9) hold for all $t \in (0, +\infty)$.*

Proof. If the pair (μ, ν) satisfies (2), then one can apply Proposition 1 and then Theorem 4 in the case $T = \mathbb{D}_+$ and Theorem 6 when $T = (0, +\infty)$. To obtain the converse it is enough to use Lemma 5 and, if $T = (0, +\infty)$, additionally to recall the density of $\mathbb{D}_+$ in $(0, +\infty)$ and the assumed continuity of μ and ν . $\square$

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