Invariant means and iterates of mean-type mappings

JANUSZ MATKOWSKI AND PAWEŁ PASTECZKA

Dedicated to Professor Janos Aczél on the occasion of his 95th birthday.

Abstract. A classical result states that for two continuous, strict means $M, N: I^2 \rightarrow I$ (I is an interval) there exists a unique (M, N) -invariant mean $K: I^2 \rightarrow I$, i.e. such a mean that $K \circ (M, N) = K$ and, moreover, the sequence of iterates $((M, N)^n)_{n=1}^\infty$ converge to (K, K) pointwise. Recently it was proved that continuity assumption cannot be omitted in general. We show that if K is a unique (M, N) -invariant mean then, without continuity assumption, $(M, N)^n \rightarrow (K, K)$.

Mathematics Subject Classification. 26E60, 26D15.

Keywords. Mean, Invariant mean, Iteration.

1. Introduction

It is known that if M and N are continuous bivariate means in an interval I , and the mean-type mapping (M, N) is diagonally contractive, that is,

$$|M(x, y) - N(x, y)| < |x - y|, \quad x, y \in I, \quad x \neq y, \quad (1)$$

then there is a unique mean $K: I^2 \rightarrow I$ that is (M, N) -invariant and, moreover the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping (M, N) converges to (K, K) ([5, Theorem 4.5 (i),(ii)]).

For the results of this type, with more restrictive assumptions, see [1]. In particular, instead of (1) it was assumed that both means are strict; in [2] it was assumed that at most one mean is not strict, and condition (1) appeared first in [4] (see also [6]). Moreover, in all these papers the uniqueness of the invariant mean was obtained under the condition that it is continuous. In [5, Theorem 4.5] it was shown that this condition can be relaxed. In Proposition 2.8 it is shown that condition (1) holds if M is right-strict in both variables

and N is left-strict in both variables (or vice versa), that is essentially weaker condition than the strictness of the means.

In Sect. 3 we give conditions under which the uniqueness of the invariant mean guarantee the relevant convergence of the sequence of iterates of the mean-type mapping (see Theorem 3.3). If $K, M, N : I^2 \rightarrow I$ are arbitrary means and K is a unique (M, N) -invariant mean, then the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping (M, N) converges to (K, K) pointwise in I^2 .

In Sect. 4 we deliver certain conditions which guarantee uniqueness of invariant means.

2. Preliminaries

Let $I \subset \mathbb{R}$ be an interval. Recall that a function $M : I^2 \rightarrow \mathbb{R}$ is called a *mean* in I , if it is internal, i.e. if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

The mean M is called *strict*, if these inequalities are sharp unless $x = y$; *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$.

For two means $M, N : I^2 \rightarrow I$, a mean $K : I^2 \rightarrow I$ is called *invariant* with respect to the mean-type mapping $(M, N) : I^2 \rightarrow I^2$ or, briefly, (M, N) -*invariant*, if $K \circ (M, N) = K$ [3].

Remark 2.1. If $M, N : I^2 \rightarrow I$ are two means in an interval I then, for all $n \in \mathbb{N}$, the mapping $(M_n, N_n) := (M, N)^n$ is a mean-type mapping. Moreover every (M, N) -invariant mean is (M_n, N_n) -invariant for all $n \in \mathbb{N}$.

Remark 2.2. ([3]) If $K : I^2 \rightarrow I$ is a continuous symmetric and strictly increasing in both variables, then for every mean $M : I^2 \rightarrow I$ there exists a unique function $N : I^2 \rightarrow I$ such that $K \circ (M, N) = K$, and N is a mean in I . Moreover, the mean N is called *complementary* to M with respect to K .

The following result is a consequence of bivariable version of Corollary 1 in [5] (see also [1, 2, 4]).

Theorem 2.3. *If $M, N : I^2 \rightarrow I$ are continuous means such that the mean-type mapping (M, N) is diagonally-contractive, then there is a unique mean $K : I^2 \rightarrow I$ which is (M, N) -invariant, and the sequence of iterates $((M_n, N_n))_{n \in \mathbb{N}}$ converges (uniformly on compact subsets of I^2) to (K, K) .*

To formulate some simple sufficient conditions guarantying (1) let us introduce the following notions:

Definition 2.4. ([7]) Let $M : I^2 \rightarrow \mathbb{R}$ be a mean in an interval $I \subset \mathbb{R}$ and let $x_0, y_0 \in I$. The mean $M : I^2 \rightarrow \mathbb{R}$ is called:

- left-strict in the first variable at the point x_0 , if for all $t \in I$,

$$t < x_0 \implies M(x_0, t) < x_0;$$

- right-strict in the first variable at the point x_0 , if for all $t \in I$,

$$x_0 < t \implies x_0 < M(x_0, t);$$

- left-strict in the second variable at the point y_0 , if for all $t \in I$,

$$t < y_0 \implies M(t, y_0) < y_0;$$

- right-strict in the second variable at the point y_0 , if for all $t \in I$,

$$y_0 < t \implies y_0 < M(t, y_0).$$

The mean M is called *left-strict in the first variable in a set $A \subset I$* , if it is left-strict in the first variable at every point $x_0 \in A$; and M is called *left-strict in the first variable*, if it is left-strict in the first variable at every point. (Analogously one can introduce the remaining notions).

We adapt the convention that if a mean is left-strict (right-strict) in both variables then we call it simply *left-strict* (*right-strict*). Then

M is left-strict if and only if $(x \neq y \implies M(x, y) < \max(x, y))$;

M is right-strict if and only if $(x \neq y \implies M(x, y) > \min(x, y))$.

Example 2.5. The projective mean $P_1: \mathbb{R}^2 \ni (x, y) \mapsto x$, is left and right-strict in the second variable, but it is neither left nor right strict at any point in the first variable. Similarly, the projective mean $P_2: \mathbb{R}^2 \ni (x, y) \mapsto y$, is left and right-strict in the first variable, but it is neither left nor right strict at any point in the second variable.

The mean-type mapping (P_1, P_2) coincides with the identity of $\mathbb{R}^2$, and of course, the sequence of its iterates $((P_1, P_2)^n : n \in \mathbb{N})$ converges to (P_1, P_2) . Note that here $P_1 \neq P_2$, moreover, every mean $K: \mathbb{R}^2 \rightarrow \mathbb{R}$ is (P_1, P_2) -invariant.

Example 2.6. Consider the extreme mean $\min: \mathbb{R}^2 \rightarrow \mathbb{R}$. Since, $t < x_0 \implies \min(x_0, t) < x_0$ for every t ; the mean $\min$ is left-strict in the first variable and in the second variable, but it is not right-strict in the first and the second variable. Moreover, the extreme mean $\max: \mathbb{R}^2 \rightarrow \mathbb{R}$ is right-strict in the first and the second variable, but it is not left-strict in the first and the second variable.

For every $n \in \mathbb{N}$, the n th iterate $(\min, \max)^n$ of mean-type mapping $(\min, \max)$ has the form

$$(\min, \max)^n(x, y) = \begin{cases} (x, y) & \text{if } x \leq y \\ (y, x) & \text{if } x > y \end{cases}$$

and, of course, the sequence of iterates of $((\min, \max)^n : n \in \mathbb{N})$ converges.

On the other hand, one can check that a mean is invariant with respect to the mean-type mapping $(\min, \max)$ if and only if it is symmetric.

Remark 2.7. If $M : I^2 \rightarrow \mathbb{R}$ is a strict mean then it is left and right-strict in each of the two variables.

Using the definition of one-side strict means we get the following result

Proposition 2.8. *Let $M, N : I^2 \rightarrow I$ be two means in an interval I . If at every point one of the means M and N is strict both left and right then property (1) holds.*

Proof. Suppose to the contrary that there exists $x, y \in I$, $x \neq y$ such that

$$|M(x, y) - N(x, y)| \geq |x - y| = |\max(x, y) - \min(x, y)|.$$

Binding this inequality with

$$\min(x, y) \leq M(x, y) \leq \max(x, y) \text{ and } \min(x, y) \leq N(x, y) \leq \max(x, y)$$

we obtain that $M(x, y)$ and $N(x, y)$ are two different endpoints of the interval $[\min(x, y), \max(x, y)]$. Thus $\{M(x, y), N(x, y)\} = \{\min(x, y), \max(x, y)\}$.

Consequently, either

$$M(x, y) = \min(x, y) \quad \text{and} \quad N(x, y) = \max(x, y)$$

or

$$M(x, y) = \max(x, y) \quad \text{and} \quad N(x, y) = \min(x, y).$$

Assume that $x < y$ (which can be done without any loss of generality), we have that either $M(x, y) = x$ and $N(x, y) = y$ or $M(x, y) = y$ and $N(x, y) = x$.

In the first case, putting $x_0 = x$, $t := y$ we have $x_0 < t$ and $M(x_0, t) = x_0$ so, in view of Definition 2.4, M is not right-strict in the first variable at the point x_0 . Moreover, putting $t := x$, $y_0 := y$ we have $t < y_0$ and $N(t, y_0) = y_0$, so N is not left-strict in the second variable at y_0 .

We omit similar considerations in the second case. $\square$

2.1. Weakly contractive mean-type mappings

In this section we deal with some relaxation of the assumption (1). For two means $M, N : I^2 \rightarrow I$, the mean-type mapping (M, N) is *weakly contractive* if for every elements $x, y \in I$ with $x \neq y$ there is a positive integer $n = n(x, y)$ such that

$$|M_n(x, y) - N_n(x, y)| < |x - y|. \quad (2)$$

Next lemma provide a complete characterization of weak-contractivity in terms of (M_2, N_2) .

Lemma 2.9. *Let $M, N : I^2 \rightarrow I$ be means. Mean-type mapping (M, N) is weakly contractive if and only if mean-type mapping (M_2, N_2) is contractive.*

Proof. We prove that if inequality (2) is satisfied for some triple (x, y, n) then it is also valid for $(x, y, 2)$. As M and N are means we get

$$|M(x, y) - N(x, y)| \leq |x - y|.$$

Therefore

$$\text{either } |M(x, y) - N(x, y)| < |x - y| \quad \text{or} \quad |M(x, y) - N(x, y)| = |x - y|.$$

In the first case we obtain validity of (2) for the triple $(x, y, 1)$ which, by mean property, implies its validity for $(x, y, 2)$.

In the second case we get, by mean property, $\{M(x, y), N(x, y)\} = \{x, y\}$. If $(M, N)(x, y) = (x, y)$ then $(M, N)^n(x, y) = (x, y)$ for all $n \in \mathbb{N}$ contradicting the assumption. Thus we obtain $M(x, y) = y$ and $N(x, y) = x$. Applying the same argumentation to the pair (y, x) we get that either

$$|M_2(x, y) - N_2(x, y)| = |M(y, x) - N(y, x)| < |x - y|,$$

which implies (2) for the triple $(x, y, 2)$ or

$$(M_2, N_2)(x, y) = (M, N)(y, x) = (x, y).$$

However in the second subcase we get

$$(M_n, N_n)(x, y) = \begin{cases} (y, x) & \text{if } n \text{ is odd,} \\ (x, y) & \text{if } n \text{ is even,} \end{cases}$$

which implies that (2) is valid for no $n \in \mathbb{N}$ contradicting the assumption.

As the converse implication is trivial, the proof is complete $\square$

Proposition 2.10. *Let $M, N : I^2 \rightarrow I$ be two means in an interval I . If all the following conditions are valid:*

- (i) *M is right-strict in the first variable or N is left-strict in the second variable,*
- (ii) *M is left-strict in the first variable or N right-strict in the second variable,*
- (iii) *M is left-strict in the second variable or M is right-strict in the second variable or N is left-strict in the first variable or N is right-strict in the first variable,*

then the mapping (M, N) is weakly contractive.

Proof. Fix $x, y \in I$, $x < y$. It suffices to prove that

$$\{(M, N)^n(x, y) : n \in \mathbb{N}\} \not\subset \{(x, y), (y, x)\}.$$

Suppose to the contrary that $\{(M, N)^n(x, y) : n \in \mathbb{N}\} \subset \{(x, y), (y, x)\}$. Applying property (i) we obtain that $(M, N)(x, y) \neq (x, y)$. In view of property (ii) we get $(M, N)(y, x) \neq (y, x)$.

The only remaining case is that $(M, N)(x, y) = (y, x)$ and $(M, N)(y, x) = (x, y)$. Equivalently,

$$M(x, y) = y, \quad N(x, y) = x, \quad M(y, x) = x, \quad \text{and} \quad N(y, x) = y.$$

However these equalities cannot be simultaneously valid as they are excluded by consecutive alternatives in (iii), respectively.

The case $x > y$ is completely analogous. $\square$

3. Uniqueness of invariant means and convergence of iterates of mean-type mappings

Definition 3.1. Let $M, N : I^2 \rightarrow I$ be means in an interval I . Define the diagonal basin $\mathcal{A}_{M,N} \subset I^2$ by

$$\left\{ (x, y) \in I^2 : ((M_n, N_n)(x, y))_{n=1}^{\infty} \text{ converges to some point on the diagonal} \right\}.$$

Theorem 3.2. Let $M, N : I^2 \rightarrow I$ be means in an interval I . Then $\mathcal{A}_{M,N}$ is the maximal subset of I^2 such that all (M, N) -invariant means are equal to each other.

Proof. Fix $x, y \in I$. Let $a = (a_n)$ be a sequence which is a shuffling of M_n and N_n , i.e.

$$a = (x, y, M_1(x, y), N_1(x, y), M_2(x, y), N_2(x, y), M_3(x, y), N_3(x, y), \dots).$$

By [6, p. 413] we obtain that

$$L(x, y) := \liminf_{n \rightarrow \infty} a_n \quad \text{and} \quad U(x, y) := \limsup_{n \rightarrow \infty} a_n$$

are the smallest and the biggest (M, N) -invariant means, respectively.

It is necessarily and also sufficient to prove that $L(x, y) = U(x, y)$ if and only if $(x, y) \in \mathcal{A}_{M,N}$.

If $L(x, y) = U(x, y)$ then we obtain that the sequence a converges, whence $(M_n(x, y))_{n \in \mathbb{N}}$ and $(N_n(x, y))_{n \in \mathbb{N}}$ converge to a common limit, i.e. $(x, y) \in \mathcal{A}_{M,N}$.

Conversely, if $(x, y) \in \mathcal{A}_{M,N}$ then we get

$$\lim_{n \rightarrow \infty} M_n(x, y) = \lim_{n \rightarrow \infty} N_n(x, y),$$

which implies that the sequence a is convergent, and therefore $L(x, y) = U(x, y)$. As L and U is the smallest and the greatest (M, N) -invariant mean, we have that all (M, N) -invariant means have the same value at the point (x, y) . $\square$

Theorem 3.3. Let $K, M, N : I^2 \rightarrow I$ be arbitrary means. K is a unique (M, N) -invariant mean if and only if the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping (M, N) converges to (K, K) pointwise in I^2 .

Proof. For $x, y \in I$. Let a be a sequence which is a shuffling of M_n and N_n , i.e

$$a = (x, y, M_1(x, y), N_1(x, y), M_2(x, y), N_2(x, y), M_3(x, y), N_3(x, y), \dots).$$

By [6] we obtain that

$$L(x, y) := \liminf_{n \rightarrow \infty} a_n, \quad U(x, y) := \limsup_{n \rightarrow \infty} a_n$$

are the smallest and the biggest (M, N) -invariant means, respectively.

($\Rightarrow$) As K is a unique (M, N) -invariant mean we get $K = L = U$. It implies that the sequence (a_n) is convergent to $K(x, y)$. Therefore both M_n and N_n converge to K pointwise on I^2 . In particular its Cartesian product $(M, N)^n = (M_n, N_n)$ converges to (K, K) pointwise on the same set.

($\Leftarrow$) We know that the sequence (a_n) is convergent to $K(x, y)$, in particular $K(x, y) = L(x, y) = U(x, y)$. As x, y were taken arbitrarily we have $K = L = U$. It implies that the smallest and the biggest (M, N) -invariant mean coincide providing the uniqueness. $\square$

4. Mean-type mappings with diagonally-contractive iterates and invariant means

Theorem 4.1. *If $M, N : I^2 \rightarrow I$ are continuous means such that (M, N) is weakly contractive then there exists a unique mean $K : I^2 \rightarrow I$ which is (M, N) -invariant and, moreover, the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping (M, N) converges (uniformly on compact subsets of I^2) to (K, K) .*

Proof. In view of Theorem 3.3 we get that (M_2, N_2) is a contractive mean-type mapping, which is also continuous. Applying the result from [5] (see the beginning of this paper) we obtain that there exists a unique (M_2, N_2) -invariant mean. As every (M, N) -invariant mean is also (M_2, N_2) -invariant we get that there exists at most one (M, N) -invariant mean.

On the other hand, applying some general results from [6], we have that (without any assumptions for the means M and N) there exists at least one (M, N) -invariant mean. Therefore there exists a unique (M, N) -invariant mean; say $K : I^2 \rightarrow I$.

We can now apply Theorem 3.3 to obtain that $((M, N)^n)_{n \in \mathbb{N}}$ converges to (K, K) pointwise on I^2 . Moreover, in view of [5, Corollary 4.4 (ii)], this convergence is uniform on every compact subset of I^2 . $\square$

Remark 4.2. Let us emphasize that the continuity assumption in the previous theorem cannot be omitted; see [6, section 3.1] for the details.

Theorem 4.3. *Let $M, N : I^2 \rightarrow I$ be (not necessarily continuous) means and $c \in [0, 1)$. If for every point $(x, y) \in I^2$, $x \neq y$, there is a positive integer $n = n(x, y)$ such that*

$$|M_n(x, y) - N_n(x, y)| < c|x - y|,$$

where $(M_n, N_n) := (M, N)^n$, then there is a unique mean $K : I^2 \rightarrow I$ that is (M, N) -invariant and, moreover the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping (M, N) converges (uniformly on compact subsets of I^2) to (K, K) .

Proof. Take an arbitrary $(x, y) \in I^2$. Since M and N are means in I , for every $n \in \mathbb{N}$, we have

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad \min(x, y) \leq N(x, y) \leq \max(x, y),$$

hence

$$\begin{aligned} \min(x, y) &\leq \min(M(x, y), N(x, y)) \\ &\leq M(M(x, y), N(x, y)) \leq \max(M(x, y), N(x, y)) \leq \max(x, y), \end{aligned}$$

and

$$\begin{aligned} \min(x, y) &\leq \min(M(x, y), N(x, y)) \\ &\leq N(M(x, y), N(x, y)) \leq \max(M(x, y), N(x, y)) \leq \max(x, y), \end{aligned}$$

whence

$$\begin{aligned} \min(x, y) &\leq \min(M_1(x, y), N_1(x, y)) \leq \min(M_2(x, y), N_2(x, y)) \\ &\leq \max(M_2(x, y), N_2(x, y)) \leq \max(M_1(x, y), N_1(x, y)) \\ &\leq \max(x, y), \end{aligned}$$

and, by induction, for every $n \in \mathbb{N}$,

$$\begin{aligned} \min(x, y) &\leq \min(M_n(x, y), N_n(x, y)) \leq \min(M_{n+1}(x, y), N_{n+1}(x, y)) \\ &\leq \max(M_{n+1}(x, y), N_{n+1}(x, y)) \leq \max(M_n(x, y), N_n(x, y)) \\ &\leq \max(x, y). \end{aligned}$$

Thus the sequence $(\min(M_n(x, y), N_n(x, y)) : n \in \mathbb{N})$ is increasing and the sequence $(\max(M_n(x, y), N_n(x, y)) : n \in \mathbb{N})$ is decreasing; and both are convergent. Clearly, if $x = y$ or $M_k(x, y) = N_k(x, y)$ for some positive integer k , then $M_n(x, y) = N_n(x, y)$ for all $n \geq k$, and consequently we have

$$\lim_{n \rightarrow \infty} M_n(x, y) = \lim_{n \rightarrow \infty} N_n(x, y).$$

Assume that

$$\lim_{n \rightarrow \infty} \min(M_n(x, y), N_n(x, y)) < \lim_{n \rightarrow \infty} \max(M_n(x, y), N_n(x, y)).$$

In particular, we have $M_n(x, y) \neq N_n(x, y)$ for every $n \in \mathbb{N}$. By the assumption, for every $n \in \mathbb{N}$ there is $k = k_n \in \mathbb{N}$ such that

$$\begin{aligned} & |M_{k_n}(M_n(x, y), N_n(x, y)) - N_{k_n}(M_n(x, y), N_n(x, y))| \\ & < c |M_n(x, y) - N_n(x, y)| \end{aligned}$$

equivalently

$$|M_{n+k_n}(x, y) - N_{n+k_n}(x, y)| < c |M_n(x, y) - N_n(x, y)|,$$

which leads to a contradiction. Thus

$$\lim_{n \rightarrow \infty} \min(M_n(x, y), N_n(x, y)) = \lim_{n \rightarrow \infty} \max(M_n(x, y), N_n(x, y)),$$

and consequently,

$$\lim_{n \rightarrow \infty} M_n(x, y) = \lim_{n \rightarrow \infty} N_n(x, y).$$

Finally, we can define K as a (common) pointwise limit of sequences (M_n) and (N_n) , as they are equal. $\square$

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Borwein, J.M., Borwein, P.B.: *Pi and the AGM*. A Wiley-Interscience Publication, New York (1987)
- [2] Matkowski, J.: Iterations of mean-type mappings and invariant means. *Ann. Math. Silesianae* **13**, 211–226 (1999)
- [3] Matkowski, J.: Invariant and complementary quasi-arithmetic means. *Aequ. Math.* **57**, 87–107 (1999)
- [4] Matkowski, J.: Iterations of the mean-type mappings. In: Sharkovsky, A.N., Susko, I.M. (eds.) *ECIT' 08*. Grazer Math. Ber., Bericht Nr., vol. 354, pp. 158–179 (2009)
- [5] Matkowski, J.: Iterations of the mean-type mappings and uniqueness of invariant means. *Ann. Univ. Sci. Budapest. Sect. Comp.* **41**, 145–158 (2013)
- [6] Pasteczka, P.: Invariant property for discontinuous mean-type mappings. *Publ. Math. Debr.* **94**, 409–419 (2019)
- [7] Toader, G., Costin, I.: *Means in Mathematical Analysis*. Academic Press, Cambridge (2018)

Janusz Matkowski
 Faculty of Mathematics, Computer Science and Econometrics
 University of Zielona Góra
 Szafrana 4a
 65-516 Zielona Góra
 Poland
 e-mail: j.matkowski@wmie.uz.zgora.pl

Paweł Pasteczka
Institute of Mathematics
Pedagogical University of Kraków
Podchorążych 2
30-084 Kraków
Poland
e-mail: pawel.pasteczka@up.krakow.pl

Received: January 22, 2019

Revised: July 19, 2019