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Remarks on (F, t)-convex functions

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Abstract. In this work we discuss counterparts of some classical results connected with convex functions for a new class of functions, namely for (F, t)-convex functions. We obtain Bernstein–Doetsch, Ostrowski and Sirpiński type theorems for them. A version of a Kuhn type result is also presented.

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1. Introduction

Let D be a convex subset of a real vector space X and $F : X \to \mathbb{R}$ be a fixed function. A function $f : D \to \mathbb{R}$ is called F-*convex* if

$$f\,(tx + (1-t)y) \le tf(x) + (1-t)f(y) - t(1-t)F(x-y), \tag{1}$$

for all $x, y \in D$ and $t \in (0, 1)$. A function $f : D \to \mathbb{R}$ is called F-*midconvex* if in the above inequality the parameter t is fixed and equals $1/2$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y), \tag{2}$$

for all $x, y \in D$. Observe that for the zero function F they become standard convex and midconvex (or Jensen convex) functions, respectively (see, for instance, [4,6,14]). Moreover, if X is a real normed space, then substituting the function F with the function $c\,\|\cdot\|^2$, where c is a fixed positive real number, we get strongly convex functions with modulus c and strongly midconvex functions with modulus c, respectively (see e.g. [4,14]). Strongly convex functions were introduced by Polyak [13] who used them for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics. For instance, a rich collection of properties and applications of strongly convex functions can be found in [5,8–13,15]. The concept of F-convex and F-midconvex functions appears in [1], where the author generalizes the results presented in [11]. As in the case of strong convexity (see [2]), condition (2) is much weaker than condition (1). However, as it is presented in [2], if X is a real normed space and $F(x) = c\|x\|^2$ (with a fixed positive real number c), then condition (2) becomes (1) if the function f satisfies some additional assumptions. In particular, the authors obtain Berstein–Doetsch and Sierpiński type results.

The aim of this work is to present counterparts of Berstein–Doetsch and Sierpiński type results for F-convexity. It could be interesting and helpful for possible applications that under weak regularity assumptions the class of functions satisfying (2) is the same as that satisfying (1).

2. Main result

We start with the following definition unifying the cases of convexity mentioned earlier.

Definition. Let D be a convex subset of a real vector space X, $F : X \to \mathbb{R}$ be a given function and t be a fixed number in $(0,1)$. A function $f : D \to \mathbb{R}$ we will call (F,t)-*convex* if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) - t(1-t)F(x-y),$$

for all $x, y \in D$.

Of course, in this notation a function is convex, midconvex, strongly convex with modulus c, strongly midconvex with modulus c, F-convex, F-midconvex iff it is $(0,t)$-convex for all $t \in (0,1)$, $(0,\frac{1}{2})$-convex, $(c\|\cdot\|^2, t)$-convex for all $t \in (0,1)$, $(c\|\cdot\|^2, \frac{1}{2})$-convex, (F,t)-convex for all $t \in (0,1)$ and $(F,\frac{1}{2})$-convex, respectively. Also if F is a nonegative even function and homogenous of degree 2 for some $t \in (0,1)$, then (F,t)-convexity gives the functions considered in [7].

As we know, if a function is $(0,\frac{1}{2})$-convex, then, in particular, it is $(0,t)$-convex with all dyadic parameters $t \in (0,1)$ (see [6,14]). It appears that for F-convexity it is generally not true. Let's look at the following example.

Example 1. Let $F : \mathbb{R} \to \mathbb{R}$ be a constant function equal to -4 and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by the formula

$$f(x) = \begin{cases} 0 & \text{for } x \text{ dyadic} \\ 2 & \text{otherwise} \end{cases}.$$

We can verify that f is $(F,\frac{1}{2})$-convex. Moreover, for a fixed $t \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ we can choose a dyadic number x and a non-dyadic number y such that $tx+(1-t)y$

and $(1-t)x + ty$ are non-dyadic numbers (we have this possibility because the set of dyadic numbers is countable and the set of non-dyadic numbers is uncountable). From this we conclude that f is not (F, t)-convex for each $t \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

In many areas of optimization theory and mathematical economics we use continuous and also convex (strongly convex) functions. Thus the above example may be less interesting because f is discontinuous and F is negative, but a similar result with a continuous function f and a nonegative function F, which is zero only in zero, is presented in the next example.

Example 2. Define functions $f, f^* : [-2, 2] \to \mathbb{R}$ by the formulas

$$f(x) = x^2 \text{ and } f^*(x) = \begin{cases} x^{\frac{3}{2}} & \text{for } x \in [0,1] \\ x^2 & \text{otherwise} \end{cases}$$

and a function $F^* : [-4, 4] \to \mathbb{R}$ as follows

$$F^*(x) = 4 \inf_{u,v \in [-2,2], u-v=x} \left\{ \frac{f^*(u) + f^*(v)}{2} - f^*\left(\frac{u+v}{2}\right) \right\}.$$

Note that the function f^* is $(F^*, \frac{1}{2})$-convex, $F^* > 0$ except for zero and f satisfies the equation

$$f\left(tx + (1-t)y\right) = tf(x) + (1-t)f(y) - t(1-t)F(x-y),$$

for all $x, y \in [-2, 2]$ and $t \in (0, 1)$ with $F(x) = x^2$.

Moreover, observe that for $t \in (\frac{1}{2}, \frac{3}{4})$ we have

$$\begin{aligned} f^*\left(t2 + (1-t)(-2)\right) > f\left(t2 + (1-t)(-2)\right) = tf(2) + (1-t)f(-2) \\ - t(1-t)F(4) = tf^*(2) + (1-t)f^*(-2) - t(1-t)F^*(4). \end{aligned}$$

This means, in particular, that for the dyadic number $\frac{5}{8}$ the function f^* is not $(F^*, \frac{5}{8})$-convex.

The following theorem provides the answer to when $(F, \frac{1}{2})$-convexity implies (F, t)-convexity for all dyadic numbers $t \in (0, 1)$.

Theorem 1. *Let D be a convex subset of a real vector space X and $F : X \to \mathbb{R}$ be a given function such that $F(tx) \geq t^2 F(x)$ for all dyadic numbers $t \in (0, 1)$ and $x \in X$. If a function $f : D \to \mathbb{R}$ is $(F, \frac{1}{2})$-convex, then it is (F, t)-convex for all dyadic numbers $t \in (0, 1)$.*

Proof. We have to show that inequality (1) is true for all $x, y \in D$ and t of the form $t = \frac{k}{2^n}$, where $k, n \in \mathbb{N}$ and $k < 2^n$. It will be shown by induction on n. Fix $x, y \in D$. For $n = 1$ it is obviously true from the definition. Assume that f is $\left(F, \frac{k}{2^n}\right)$-convex for some $n \in \mathbb{N}$ and $k < 2^n$. Take $k < 2^{n+1}$, then $k < 2^n$

or $k > 2^n$ or $k = 2^n$. Suppose that $k < 2^n$. Therefore, by $(F, \frac{1}{2})$-convexity, the induction assumption and the identity

$$\frac{k}{2^{n+1}}x + \left(1 - \frac{k}{2^{n+1}}\right)y = \frac{1}{2}\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) + \frac{1}{2}y$$

we get

$$\begin{aligned}
&f\left(\frac{k}{2^{n+1}}x + \left(1 - \frac{k}{2^{n+1}}\right)y\right) = f\left(\frac{1}{2}\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) + \frac{1}{2}y\right) - \frac{1}{4}F(x-y)\\
&\leq \frac{1}{2}f\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y\right) + \frac{1}{2}f(y) - \frac{1}{4}F\left(\frac{k}{2^n}x + \left(1 - \frac{k}{2^n}\right)y - y\right)\\
&\leq \frac{1}{2}\left[\frac{k}{2^n}f(x) + \left(1 - \frac{k}{2^n}\right)f(y) - \frac{k}{2^n}\left(1 - \frac{k}{2^n}\right)F(x-y)\right]\\
&\quad + \frac{1}{2}f(y) - \frac{1}{4}F\left(\frac{k}{2^n}(x-y)\right)\\
&\leq \frac{k}{2^{n+1}}f(x) + \left(1 - \frac{k}{2^{n+1}}\right)f(y) - \frac{k}{2^{n+1}}\left(1 - \frac{k}{2^n}\right)F(x-y) - \frac{1}{4}\frac{k^2}{2^{2n}}F(x-y)\\
&\quad = \frac{k}{2^{n+1}}f(x) + \left(1 - \frac{k}{2^{n+1}}\right)f(y) - \left(\frac{k}{2^n}\left(1 - \frac{k}{2^n}\right) + \frac{1}{4}\frac{k^2}{2^{2n}}\right)F(x-y)\\
&\quad = \frac{k}{2^{n+1}}f(x) + \left(1 - \frac{k}{2^{n+1}}\right)f(y) - \frac{k}{2^{n+1}}\left(1 - \frac{k}{2^n} + \frac{1}{4}\frac{k}{2^{n-1}}\right)F(x-y)\\
&\quad = \frac{k}{2^{n+1}}f(x) + \left(1 - \frac{k}{2^{n+1}}\right)f(y) - \frac{k}{2^{n+1}}\left(1 - \frac{k}{2^{n+1}}\right)F(x-y).
\end{aligned}$$

If $k > 2^n$ then the proof is similar but we use the identity

$$\frac{k}{2^{n+1}}x + \left(1 - \frac{k}{2^{n+1}}\right)y = \frac{1}{2}x + \frac{1}{2}\left(\frac{k - 2^n}{2^n}x + \frac{2^{n+1} - k}{2^n}y\right).$$

For $k = 2^n$ it is obvious, which ends the proof. □

As a consequence of Theorem 1 and the density of the set of dyadic numbers in the real line we obtain the following corollary.

Corollary 1. *Let D be a convex subset of a real vector space X and $F : X \to \mathbb{R}$ be a given function such that $F(tx) \geq t^2 F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in X$. If a function $f : D \to \mathbb{R}$ is $(F, \frac{1}{2})$-convex and continuous on each segment contained in D, then it is (F,t)-convex for all numbers $t \in (0,1)$ (or shortly F-convex).*

Remark. The assumption Let "$F : X \to \mathbb{R}$ be a given function such that $F(tx) \geq t^2 F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in X$" could be replaced by Let "$F : X \to \mathbb{R}$ be a given function such that $F(nx) = n^2 F(x)$ for all $n \in \mathbb{N}$ and $x \in X$".

Recall that a function $Q : X \to R$ is said to be quadratic if it satisfies the following functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y),$$

for all $x, y \in X$. Also, in particular, it satisfies the equation $Q(\frac{1}{2}x) = \frac{1}{4}Q(x)$. Using these facts we have the next example.

Example 3. Let $a : X \to \mathbb{R}$ be an additive function and $F := -a^2$. The function $f := -a^2$ is a quadratic function, bounded from above and $(F, \frac{1}{2})$-convex (as a matter of fact, f is $(F, \frac{1}{2})$-affine, i.e. instead of inequality (2) we have an equality).

The classical Berstein–Doetsch result says that a midconvex function, defined on an open and convex subset of $\mathbb{R}^n$, locally bounded from above at a point must be convex and continuous. Observe that Examples 1 and 2 show that if we assume nothing about the function F, then we do not have a counterpart of this classical result and Example 3 leads to the same conclusion, even when we adopt the assumption from Theorem 1. A counterpart of a Berstein–Doetsch result in the case of F-convexity has the following form.

Theorem 2. *Let D be an open, convex subset of a topological real vector space X and $F : X \to \mathbb{R}$ be a given nonnegative function such that $F(tx) \geq t^2 F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in X$. If a function $f : D \to \mathbb{R}$ is F-midconvex and locally bounded from above at a point of D, then it is continuous and F-convex.*

Proof. A function f, being F-midconvex with a nonnegative function F, is also midconvex. Thus using the classical Berstein–Doetsch result, we conclude that f is continuous. Finally, from Corollary 1 the function f has to be continuous and F-convex. The proof is finished. □

Arguing as before, but using Sirpiński's result instead of Berstein–Doetsch's result, we get the following theorem.

Theorem 3. *Let D be an open, convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ be a given nonnegative function such that $F(tx) \geq t^2 F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in \mathbb{R}^n$. If a function $f : D \to \mathbb{R}$ is F-midconvex and Lebesgue measurable, then it is continuous and F-convex.*

The next result is a Kuhn type theorem for F-convexity.

Theorem 4. *Let D be a convex subset of a real vector space X and $F : X \to \mathbb{R}$ be a given function such that $F\left(\frac{1}{2}x\right) \geq \frac{1}{4}F(x)$. If a function $f : D \to \mathbb{R}$ is (F,t)-convex with some $t \in (0,1)$, then it is $\left(F, \frac{1}{2}\right)$-convex.*

Proof. Fix $x, y \in D$ and put $z := \frac{x+y}{2}$, $u := tx + (1-t)z$, $v := tz + (1-t)y$. Using Daróczy–Páles's identity (see [3]) we conclude that

$$z = (1-t)u + tv.$$

Taking the t-convexity of f into consideration we have

$$\begin{aligned} f(z) &\le (1-t)f(u) + tf(v) - t(1-t)F(u-v) \\ &= (1-t)f\left(tx + (1-t)z\right) + tf\left(tz + (1-t)y\right) - t(1-t)F(u-v) \\ &\le (1-t)\left(tf(x) + (1-t)f(z) - t(1-t)F(x-z)\right) \\ &\quad + t\left(tf(z) + (1-t)f(y) - t(1-t)F(z-y)\right) - t(1-t)F(u-v). \end{aligned}$$

Thus

$$2f(z) \le f(x) + f(y) - \left[(1-t)F(x-z) + tF(z-y) + F(u-v)\right].$$

Notice that $x - z = z - y = u - v = \frac{x-y}{2}$, then in view of the assumption $F\left(\frac{1}{2}x\right) \ge \frac{1}{4}F(x)$ we can write the last inequality in the following form

$$2f(z) \le f(x) + f(y) - \frac{1}{2}F(x-y).$$

Dividing the last inequality by 2 we get the thesis. The proof is complete. □

Using this theorem, and next Theorem 2 and Theorem 3 respectively, we get the following corollaries.

Corollary 2. *Let D be an open, convex subset of a topological real vector space X and $F : X \to \mathbb{R}$ be a given nonnegative function such that $F(tx) \ge t^2F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in X$. If a function $f : D \to \mathbb{R}$ is (F,t)-convex with some $t \in (0,1)$, and locally bounded from above at a point of D, then it is continuous and F-convex.*

Corollary 3. *Let D be an open, convex subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}$ be a given nonnegative function such that $F(tx) \ge t^2F(x)$ for all dyadic numbers $t \in (0,1)$ and $x \in \mathbb{R}^n$. If a function $f : D \to \mathbb{R}$ is (F,t)-convex with some $t \in (0,1)$, and Lebesgue measurable, then it is continuous and F-convex.*

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