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Aequationes Mathematicae

An extension of the Hermite–Hadamard inequality for convex and s-convex functions

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Abstract. The Hermite–Hadamard inequality was extended using iterated integrals by Retkes [Acta Sci Math (Szeged) 74:95–106, 2008]. In this paper we further extend the main results of the above paper for convex and also for s-convex functions in the second sense.

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1. Introduction

For a convex function $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$, the well-known Hermite–Hadamard inequality states the following for any $a < x_1 < x_2 < b$:

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\, dx \leq \frac{f(x_1) + f(x_2)}{2}.$$

This inequality was extended by Retkes [6].

Theorem 1.1. [6] *Let $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, $x_i \in (a, b)$, $i = 1, \ldots, n$, such that $x_i \neq x_j$ if $1 \leq i < j \leq n$. Then the following inequality holds:*

$$\sum_{i=1}^{n} \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \ldots, x_n)} \leq \frac{1}{n!} \sum_{i=1}^{n} f(x_i)$$

where $F^{(j)}$ is the j-th iterated integral of f and

$$\Pi_i(x_1, \ldots, x_n) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j).$$

In the concave case "$\leq$" is changed to "$\geq$".

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The Hermite–Hadamard inequality has been extended in several ways besides the previous theorem. One way is to change the convexity condition to general convexity (see for example [5,8]). We consider the well-studied concept of
s-convex functions, which are defined on an interval of $\mathbb{R}_0^+ = [0, \infty)$.

Definition 1.2. [1,4] Let $s \in (0, 1]$ be a fixed number. A function $f : [a, b) \subseteq \mathbb{R}_0^+ \to \mathbb{R}$ is said to be s-convex (in the second sense) if

$$f(px_1 + (1-p)x_2) \le p^s f(x_1) + (1-p)^s f(x_2)$$

holds for any $x_1, x_2 \in [a, b)$ and $p \in [0, 1]$.

The appropriate extension regarding this type of functions is due to Dragomir and Fitzpatrick.

Theorem 1.3. [3] *Suppose that $f : [a, b) \subseteq \mathbb{R}_0^+ \to \mathbb{R}$ is an s-convex function, where $s \in (0, 1]$, and let $x_1, x_2 \in [a, b)$ with $x_1 < x_2$. Then the following inequality holds:*

$$2^{s-1} f\left(\frac{x_1 + x_2}{2}\right) \le \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\, dx \le \frac{f(x_1) + f(x_2)}{s+1}.$$

2. Main results

We extend Theorem 1.3 in the spirit of Theorem 1.1.

Theorem 2.1. *Let $f : [a, b) \subseteq \mathbb{R}_0^+ \to \mathbb{R}$ be an s-convex function, where $s \in (0, 1]$, and $x_i \in (a, b)$, $i = 1, \ldots, n$, such that $x_i \neq x_j$ if $1 \le i < j \le n$. Then the following inequality holds:*

$$\frac{n^s}{n!} f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \sum_{i=1}^n \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \ldots, x_n)} \le \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^n f(x_i). \tag{2.1}$$

In the s-concave case, "$\le$" are changed to "$\ge$".

In the case of $s = 1$, an extension of Theorem 1.1 is obtained.

Corollary 2.2. *Let $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, $x_i \in (a, b)$, $i = 1, \ldots, n$, such that $x_i \neq x_j$ if $1 \le i < j \le n$. Then*

$$\frac{1}{(n-1)!} f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \sum_{i=1}^n \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \ldots, x_n)} \le \frac{1}{n!} \sum_{i=1}^n f(x_i).$$

In the concave case, "$\le$" are changed to "$\ge$".

3. Proofs

To prove our theorems, we need three lemmas.

Lemma 3.1. [2] *For any $f : [a, b) \subseteq [0, \infty) \to \mathbb{R}$, the following statements are equivalent:*

(1) *f is an s-convex function on $[a, b)$.*
(2) *For every $x_1, \dots, x_n \in [a, b)$ and non-negative real numbers $p_1, \dots, p_n$ with $\sum_{i=1}^{n} p_i = 1$, we have that*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i^s f(x_i).$$

Lemma 3.2. [6, Lemma] *Let $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function and $x_i \in (a, b)$, $i = 1, \dots, n$, such that $x_i \neq x_j$ if $1 \le i < j \le n$. Then*

$$\int\limits_{H_{n-1}}\cdots\int f\left(x_n - \sum_{i=1}^{n-1} p_i (x_n - x_i)\right) dp_1 \dots dp_{n-1} = \sum_{i=1}^{n} \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \dots, x_n)},$$

where

$$H_n = \left\{(p_1, \dots, p_n) \in \mathbb{R}^n : 0 \le p_i \le 1, i = 1, \dots n \wedge \sum_{i=1}^{n} p_i \le 1\right\}.$$

Lemma 3.3. *Let $f : [a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function and $x_i \in (a, b)$, $i = 1, \dots, n$, such that $x_i \neq x_j$ if $1 \le i < j \le n$. Then*

$$\int\limits_{H_{n-1}}\cdots\int p_i^s \, dp_1 \dots dp_{n-1} = \frac{\Gamma(s+1)}{\Gamma(s+n)} \tag{3.1}$$

and

$$\int\limits_{H_{n-1}}\cdots\int \left(1 - \sum_{i=1}^{n-1} p_i\right)^s dp_1 \dots dp_{n-1} = \frac{\Gamma(s+1)}{\Gamma(s+n)}. \tag{3.2}$$

Proof. Due to symmetry, without loss of generality, we can assume that in (3.1) we have $i = 1$. By Fubini's theorem,

$$\begin{aligned}
&\int\limits_{H_{n-1}}\cdots\int p_i^s \, dp_1 \dots dp_{n-1} \\
&= \int_0^1 \int_0^{1-p_{n-1}} \cdots \int_0^{1-p_{n-1}-\dots-p_2} (1 - p_{n-1} - \dots - p_1)^s \, dp_1 \dots dp_{n-1} \\
&= \frac{1}{s+1} \int_0^1 \int_0^{1-p_{n-1}} \cdots \int_0^{1-p_{n-1}-\dots-p_3} (1 - p_{n-1} - \dots - p_2)^{s+1} \, dp_2 \dots dp_{n-1}
\end{aligned}$$

$$= \frac{1}{(s+1)(s+2)} \int_0^1 \int_0^{1-p_{n-1}} \cdots \int_0^{1-p_{n-1}-\ldots-p_4} (1-p_{n-1}-\ldots-p_3)^{s+2}\, dp_3 \ldots dp_{n-1}$$

$$\vdots$$

$$= \frac{1}{(s+1)\cdots(s+n-2)} \int_0^1 (1-p_{n-1})^{s+n-2}\, dp_{n-1}$$

$$= \frac{\Gamma(s+1)}{\Gamma(s+n)}.$$

Similarly,

$$\int_{H_{n-1}} \cdots \int \left(1 - \sum_{i=1}^{n-1} p_i\right)^s dp_1 \ldots dp_{n-1}$$

$$= \int_0^1 \int_0^{1-p_{n-1}} \cdots \int_0^{1-p_{n-1}-\ldots-p_2} (1-p_{n-1}-\ldots-p_1)^s\, dp_1 \ldots dp_{n-1}$$

$$= \frac{1}{s+1} \int_0^1 \int_0^{1-p_{n-1}} \cdots \int_0^{1-p_{n-1}-\ldots-p_3} (1-p_{n-1}-\ldots-p_2)^{s+1}\, dp_2 \ldots dp_{n-1},$$

and we can obtain (3.2). □

Proof of Theorem 2.1. We start with the left hand side of (2.1). From the s-convexity of f, by Lemma 3.1, we have for any $y_i \in [a,b)$, $i = 1,\ldots,n$ that

$$f\left(\frac{\sum_{i=1}^n y_i}{n}\right) \le \frac{\sum_{i=1}^n f(y_i)}{n^s}. \tag{3.3}$$

Putting

$$y_1 := p_1x_2 + p_2x_3 + \ldots + p_{n-1}x_n + (1-p_1-\ldots-p_{n-1})x_1$$
$$y_2 := p_1x_3 + p_2x_4 + \ldots + p_{n-1}x_1 + (1-p_1-\ldots-p_{n-1})x_2$$
$$\vdots$$
$$y_n := p_1x_1 + p_2x_2 + \ldots + p_{n-1}x_{n-1} + (1-p_1-\ldots-p_{n-1})x_n$$

and integrating both sides of (3.3) with respect to $p_1,\ldots,p_{n-1}$ over H_{n-1} yields the following. For the left hand side:

$$\begin{aligned}
&\int_{H_{n-1}}\!\!\cdots\int f\left(\frac{\sum_{i=1}^n y_i}{n}\right) dp_1 \dots dp_{n-1} \\
&\quad = \int_{H_{n-1}}\!\!\cdots\int f\left(\frac{\sum_{i=1}^n x_i}{n}\right) dp_1 \dots dp_{n-1} \\
&\quad = V(H_{n-1})\, f\left(\frac{\sum_{i=1}^n x_i}{n}\right) = \frac{1}{(n-1)!}\, f\left(\frac{\sum_{i=1}^n x_i}{n}\right),
\end{aligned}$$

since $V(H_n)$, the volume of the n-dimensional tetrahedron H_n, is equal to $\frac{1}{n!}$ ([6]). For the right hand side, using Lemma 3.2, we get

$$\begin{aligned}
&\int_{H_{n-1}}\!\!\cdots\int \frac{\sum_{i=1}^n f(y_i)}{n^s}\, dp_1 \dots dp_{n-1} \\
&\quad = \frac{1}{n^s}\sum_{i=1}^n \int_{H_{n-1}}\!\!\cdots\int f(y_i)\, dp_1 \dots dp_{n-1} \\
&\quad = \frac{1}{n^s}\sum_{i=1}^n \int_{H_{n-1}}\!\!\cdots\int f\left(x_i - \sum_{j=1}^{n-1} p_j (x_i - x^*_{i+j})\right) dp_1 \dots dp_{n-1} \\
&\quad = \frac{n}{n^s}\sum_{i=1}^n \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \dots, x_n)},
\end{aligned}$$

where x^*_k denotes x_l for which $k \equiv l \pmod n$. Thus

$$\frac{1}{(n-1)!}\, f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \le \frac{n}{n^s}\sum_{i=1}^n \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \dots, x_n)},$$

and the left hand side of (2.1) is obtained.

Now we prove the right hand side of (2.1). From the s-convexity of f, by Lemma 3.1, we have that

$$f\left(\sum_{i=1}^{n-1} p_i x_i + \left(1 - \sum_{i=1}^{n-1} p_i\right) x_n\right) \le \sum_{i=1}^{n-1} p_i^s f(x_i) + \left(1 - \sum_{i=1}^{n-1} p_i\right)^s f(x_n)$$

holds for all $(p_1, ..., p_{n-1}) \in H_{n-1}$. Integrating both sides over H_{n-1}, using Lemma 3.3, gives

$$\begin{aligned}
&\int_{H_{n-1}}\!\!\cdots\int f\left(x_n - \sum_{i=1}^{n-1} p_i (x_n - x_i)\right) dp_1 \dots dp_{n-1} \\
&\quad \le \int_{H_{n-1}}\!\!\cdots\int \left\{ \sum_{i=1}^{n-1} p_i^s f(x_i) + \left(1 - \sum_{i=1}^{n-1} p_i\right)^s f(x_n)\right\} dp_1 \dots dp_{n-1}
\end{aligned}$$

$$= \sum_{i=1}^{n-1} f(x_i) \int_{H_{n-1}}\!\!\cdots\!\int p_i^s \, dp_1 \dots dp_{n-1}$$

$$+ f(x_n) \int_{H_{n-1}}\!\!\cdots\!\int \left(1 - \sum_{i=1}^{n-1} p_i\right)^s dp_1 \dots dp_{n-1}$$

$$= \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^{n} f(x_i),$$

whence Lemma 3.2 concludes the proof. □

4. Applications

We remark the consequence that for s-convex functions which are concave, the following inequalities hold due to Theorem 2.1 (and Corollary 2.2):

$$\frac{n^s}{n!} f\left(\frac{\sum_{i=1}^n x_i}{n}\right) \leq \sum_{i=1}^{n} \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \dots, x_n)} \leq \frac{1}{(n-1)!} f\left(\frac{\sum_{i=1}^n x_i}{n}\right),$$

$$\frac{1}{n!} \sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} \frac{F^{(n-1)}(x_i)}{\Pi_i(x_1, \dots, x_n)} \leq \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^{n} f(x_i).$$

For example, if we take $f : \mathbb{R}_0^+ \to \mathbb{R}$, $f(x) = x^s$ for an arbitrarily fixed $0 < s < 1$, we get

$$\frac{1}{n!} \left(\sum_{i=1}^{n} x_i\right)^s \leq \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^{n} \frac{x_i^{s+n-1}}{\Pi_i(x_1, \dots, x_n)} \leq \frac{1}{n^s (n-1)!} \left(\sum_{i=1}^{n} x_i\right)^s,$$

$$\frac{1}{n!} \sum_{i=1}^{n} x_i^s \leq \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^{n} \frac{x_i^{s+n-1}}{\Pi_i(x_1, \dots, x_n)} \leq \frac{\Gamma(s+1)}{\Gamma(s+n)} \sum_{i=1}^{n} x_i^s.$$

If we take $x_i = i$, $i = 1, \dots, n$, the inequalities read as

$$\frac{\Gamma(s+n)}{\Gamma(s+1)} \cdot \frac{n^s (n+1)^s}{2^s} \leq \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^{s+n} \leq \frac{\Gamma(s+n)}{\Gamma(s+1)} \cdot \frac{n(n+1)^s}{2^s},$$

$$\frac{\Gamma(s+n)}{\Gamma(s+1)} \sum_{i=1}^{n} i^s \leq \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} i^{s+n} \leq n! \sum_{i=1}^{n} i^s$$

since

$$\Pi_i(1, \dots, n) = (-1)^{n-i} (i-1)!(n-i)!\,.$$

If we take $x_i = \frac{1}{i}$, $i = 1, \ldots, n$, the inequalities become

$$\frac{\Gamma(s+n)}{\Gamma(s+1)n!} H_n^s \le \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{1}{i^s} \le \frac{\Gamma(s+n)}{\Gamma(s+1)n^s(n-1)!} H_n^s,$$

$$\frac{\Gamma(s+n)}{\Gamma(s+1)n!} \sum_{i=1}^{n} \frac{1}{i^s} \le \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{1}{i^s} \le \sum_{i=1}^{n} \frac{1}{i^s}$$

since

$$\Pi_i\left(\frac{1}{1}, \ldots, \frac{1}{n}\right) = (-1)^{n-1} \frac{\Pi_i(1, \ldots, n)}{i^{n-2}\, n!} = (-1)^{i-1} \frac{(i-1)!(n-i)!}{i^{n-2}\, n!} = \frac{(-1)^{i-1}}{\binom{n}{i} i^{n-1}}$$

due to [7].

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