On meaningful transformation equations

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Abstract. The meaningfulness condition, applied to scientific or geometric laws, requires that the mathematical form of an equation does not change when we change the units of its ratio scale variables. Suitably formalized, this condition considerably limits the possible form of a law. In this paper, we give five new examples of such restricted representations. We use the meaningfulness condition on the five transformation equations displayed in the left column of the table below, in which x , y , z , u and v are real numbers and F , K , G , H , M and N are real valued functions. We show that under relatively weak general conditions (such as continuity, symmetry, monotonicity, homogeneity), each transformation equation must have, up to some real constants, the representation in the right column.

Transformation equations	Representations
$F(F(x, y), z) = F(x, K(y, z))$	$F(x, y) = x e^{\frac{x}{\kappa}}$ $K(y, z) = \left(y^{\frac{1}{\theta}} + z^{\frac{1}{\theta}} \right)^{\theta}$
$F(G(x, y), z) = F(G(x, z), y)$	$F(x, y) = \phi x y^{\gamma}$
$F(G(x, y), z) = G(F(x, z), F(y, z))$	$F(x, z) = \phi x z^{\gamma}$ $G(x, y) = (x^{\theta} + y^{\theta})^{\frac{1}{\theta}}$
$F(G(x, y), z) = H(x, K(y, z))$	$F(G(x, y), z)$ $= \phi x y z^{\gamma} = H(x, K(y, z))$
$F(G(x, y), H(u, v)) = K(M(x, u), N(y, v))$	$F(G(x, y), H(u, v)) = (x^{\theta} + y^{\theta} + u^{\theta} + v^{\theta})^{\frac{1}{\theta}}$ $= K(M(x, u), N(y, v))$

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Meaningfulness is a fundamental condition for scientific or geometric laws expressed in terms of ratio scale variables: it formally requires that the mathematical form of an equation does not change when we change the units of

its variables. This condition is essential because the units of laws have no representation in nature. As shown by the examples of this paper, the meaningfulness condition, suitably formalized, may be a powerful tool in a search for the exact form of a scientific or geometric law because it entails the analysis of the relevant phenomenon in terms of some critical features which are either empirically intuitive (such as continuity, monotonicity, symmetry, or homogeneity) or can be independently gauged or tested. (We expand on this in the last section of this paper. For general introductions and other examples, see [2, 3]).

The idea that the mathematical form of a scientific equation should not depend upon the units of its variables is also the focal concept of dimensional analysis. However, in dimensional analysis, this invariance concept is embedded in the notation. It is not an axiom, as it is here. The difference is a critical one: in dimensional analysis, the implications of the invariance are not taken advantage of as much as they could be.

In the functional equation literature, the equation

$$F(F(x, y), z) = F(x, K(y, z)) \quad (\text{with } x, y, z \in I, \text{ a real interval}) \quad (1)$$

is referred to as the *transformation equation* [1, Section 7.1, page 315]. Setting $K(y, z) = y + z$, we get the special case:

$$F(F(x, y), z) = F(x, y + z), \quad (2)$$

which is called the *translation equation* [1, page 245].

Under a meaningfulness assumption (see Definition 5 in Sect. 2) and a couple of other general conditions, Eq. (2) takes the form of *Beer's Law*.¹ In other words, for some positive constant κ , we must have²

$$F(x, y) = x e^{\frac{y}{\kappa}} \quad (\text{with } x, y \in \mathbb{R}_{++}) \quad (3)$$

(cf. Theorem 26, in [2]).

We prove here that a similar result holds for Eq. (1). Under some general conditions, the same meaningfulness assumption implies that for some real parameter θ , we must have

$$K(y, z) = (y^\theta + z^\theta)^{\frac{1}{\theta}}, \quad (4)$$

$$F(x, y) = x e^{\frac{y^\theta}{\kappa}}. \quad (5)$$

Equation (5) is identical to Eq. (3), bar the exponent θ . What is remarkable about this result is that it follows from relatively weak additional conditions on the function K in Eq. (1): namely, K is homogeneous and solvable (see Definition 1).

¹Beer's Law describes the attenuation of light caused by traversing an absorbing medium (see Sect. 9).

²In this paper, we write $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ for the sets of real numbers, non-negative real numbers, and positive real numbers, respectively.

We also give four other examples of meaningful derivations in this paper.

One is the *quasi-permutability equation* [1, page 270]:

$$F(G(x, y), z) = F(G(x, z), y). \quad (6)$$

Our Theorem 13 states that, together with some general conditions, the quasi-permutability equation implies that we must have

$$F(x, y) = \phi x y^\gamma \quad (\text{with } \phi, \gamma > 0 \text{ constants}). \quad (7)$$

Theorem 13 is related to Theorem 7.2.1 in [3], arising from obtaining Eq. (7) by assuming that a single code is permutable with respect to the initial code.

Our next example is the *distributivity equation* [1, page 318]:

$$F(G(x, y), z) = G(F(x, z), F(y, z)), \quad (8)$$

which is shown to imply, for some real parameters γ and θ , the joint representation

$$\begin{aligned} F(x, z) &= \xi x z^\gamma \\ G(x, y) &= (x^\theta + y^\theta)^{\frac{1}{\theta}}. \end{aligned} \quad (9)$$

We then turn to what we call the *four-functions-transformation equation*

$$F(G(x, y), z) = H(x, K(y, z)), \quad (10)$$

which we sometimes refer to as the *4-FT equation* for short. We show how a meaningful derivation from this equation leads to the representation

$$F(G(x, y), z) = \phi x y z^\gamma = H(x, K(y, z)) \quad (11)$$

for some real parameters ϕ and γ .

Our final example is the *six-functions-transformations equation* (*6-FT equation*)

$$F(G(x, y), H(u, v)) = K(M(x, u), N(y, v)). \quad (12)$$

We prove how, under meaningfulness and various assumptions (e.g. monotonicity, homogeneity, symmetry, associativity), this equation satisfies, for some real parameter θ , the representation

$$F(G(x, y), H(u, v)) = (x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} = K(M(x, u), N(y, v)), \quad (13)$$

which can be regarded as a generalization of the Pythagorean Theorem (up to the exponent; see Fig. 1).

In Sect. 1, we recall some functional equations concepts and results relevant to this paper. Section 2 contains the definition of ‘meaningfulness’ for a collection of functions with varying ratio scale units, applicable to the transformation equations dealt with in this paper. Section 3 contains a number of ‘propagation’ lemmas stating that, under meaningfulness, certain conditions

on one function in a meaningful collection of functions extend to all such functions. This propagation property also applies to equations containing several functions, such as Eqs. (8), (10), or (12).

In Sect. 4, we show how the joint representations Eqs. (4)–(5) result from a meaningful derivation of the transformation equation (1). Sections 5, 6, 7 and 8 are devoted to similar meaningfulness derivations from the quasi-permutability Eq. (6), the distributivity Eq. (8), the 4-FT Eq. (10), and the 6-FT Eq. (12), respectively. Section 9 ends up with some remarks about future developments.

1. Some functional equations concepts and results

Definition 1. Suppose that J is a non-negative interval of positive length. A function $F : J \times J \rightarrow J$ is:

$$\text{symmetric} \quad \text{if} \quad F(x, y) = F(y, x) \quad (14)$$

$$\text{homogeneous (of degree one)} \quad \text{if} \quad F(\theta x, \theta y) = \theta F(x, y) \quad (15)$$

$$\text{left-homogeneous} \quad \text{if} \quad F(\theta x, y) = \theta F(x, y) \quad (16)$$

$$\text{associative} \quad \text{if} \quad F(F(x, y), z) = F(x, F(y, z)) \quad (17)$$

$$\text{translatable} \quad \text{if} \quad F(F(x, y), z) = F(x, y + z) \quad (18)$$

$$\text{permutable} \quad \text{if} \quad F(F(x, y), z) = F(F(x, z), y) \quad (19)$$

for all $x, y, z \in J$, and for (15) and (16), for all $\theta > 0$ and all $x, y \in J$ such that $\theta x, \theta y \in J$.

Let J, J' and H be three non-negative intervals of positive length. Two functions $F : J \times J' \xrightarrow{\text{onto}} H$ and $G : J \times J' \xrightarrow{\text{onto}} H'$ are *comonotonic* if

$$F(x, s) \leq F(y, t) \iff G(x, s) \leq G(y, t), \quad (x, y \in J; s, t \in J'). \quad (20)$$

In such a case, the equation

$$M(F(x, s)) = G(x, s) \quad (x \in J; s \in J') \quad (21)$$

defines a strictly increasing continuous function $M : H \xrightarrow{\text{onto}} H'$. We say then that G is *M-comonotonic* with F . Note that the comonotonicity relation is transitive.

Definition 2. With the interval J as above, a function $F : J \times J \rightarrow J$ is *quasi-permutable* if for some function $G : J \times J \rightarrow J$ comonotonic with F , we have

$$F(G(x, y), z) = F(G(x, z), y). \quad (22)$$

We say in such a case that F is *permutable with respect to G* , or *G-permutable* for short. When $G = F$, that is, F is permutable with respect to itself, we simply say that F is *permutable*, a terminology consistent with Aczél [1, Chapter

6, page 270]. It is easily shown that a code F is G -permutable only if G is permutable.

A function $F : J \times J' \rightarrow H$ is *solvable* if it satisfies the following two conditions.

[S1] If $F(x, t) < p \in H$, there exists $w \in J$ such that $F(w, t) = p$.

[S2] The function F is *1-point right solvable*, that is, there exists a point $x_0 \in J$ such that for every $p \in H$, there is $v \in J'$ satisfying $F(x_0, v) = p$. In such a case, we may say that F is x_0 -solvable.

The results below are due to [5–7] (cf. also [1]).

Lemma 3. *Let J , J' and H be non-negative real intervals of positive length, and let $F : J \times J' \rightarrow H$ be a solvable function (cf. Definition 2).*

(i) *The function $F : J \times J' \rightarrow H$ is quasi-permutable if and only if there exist three continuous functions $m : \{f(y) + g(y) \mid x \in J, y \in J'\} \rightarrow H$, $f : J \rightarrow \mathbb{R}$, and $g : J' \rightarrow \mathbb{R}$, with m and f strictly increasing and g strictly monotonic, such that*

$$F(x, y) = m(f(x) + g(y)). \quad (23)$$

(ii) *A solvable code $G : J \times J' \rightarrow J$ is a permutable code if and only if, with f and g as above, we have*

$$G(x, y) = f^{-1}(f(x) + g(y)). \quad (24)$$

(iii) *If a solvable code $G : J \times J \rightarrow J$ is a symmetric function, then G is permutable if and only if there exists a strictly increasing and continuous function $f : J \rightarrow J$ satisfying*

$$G(x, y) = f^{-1}(f(x) + f(y)). \quad (25)$$

2. Defining meaningfulness

The concept of meaningfulness applies to a collection of functions, with each function being specified by a particular choice of units for its variables. We begin by giving a general definition of this type of collection, applicable in the cases of the five transformation equations dealt with in this paper—Eqs. (1), (6), (8), (10), and (12)—each of which only contains functions of two real positive variables.

Definition 4. Let J , J' and H be non-negative real intervals of positive length and suppose that $F : J \times J' \rightarrow H$ is a strictly monotonic function, continuous in its two arguments, and strictly increasing in the first argument. In this paper, we call such a function a (*two dimensional*) *code*. Note that $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote, respectively: the set of real numbers, the set non-negative real numbers, and the set of positive real numbers.

Let $\mathcal{F} = \{F_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{R}_{++}\}$ be a collection of codes. The indices α and β specify the units of the variables x and y of $F_{\alpha,\beta}(x, y)$. The function $F = F_{1,1} \in \mathcal{F}$ stands for the *initial code*. The indices $1,1$ signify that its variables are expressed in the *initial units*.³ The domain and the range of any code $F_{\alpha,\beta}$ may be affected by the units.

A collection of codes $\mathcal{F}$ is *self-transforming* if the unit of any code (the unit of the range of the code) is the same as the unit of its first variable.⁴ If the two variables of the codes in a self-transforming collection $\mathcal{F}$ have the same unit, then $J = J'$ and we write $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_{++}\}$. We say then that $\mathcal{F}$ is a *single unit* collection. By convention, a single unit collection is self-transforming.

We now define the concept of *meaningfulness* for a collection of codes. The first definition below is the special case of the definition given in Falmagne ([2], see Definition 2) in which the collections of codes are self-transforming. We then deal with the case of codes having the same unit for both variables.

Definition 5. A collection of codes $\mathcal{F}$ is *self-transforming meaningful*, or *ST-meaningful* for short, if for every $\alpha, \beta, \mu, \nu \in \mathbb{R}_{++}$, we have with $F_{\alpha,\beta}, F_{\mu,\nu} \in \mathcal{F}$,

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x, \beta y) = \frac{1}{\mu} F_{\mu,\nu}(\mu x, \nu y) \quad (x \in J, y \in J') \quad (26)$$

yielding

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x, \beta y) = F_{1,1}(x, y) = F(x, y). \quad (27)$$

The last equation introduces the simplified notation. If $\frac{x}{\alpha} \in J$ and $\frac{y}{\beta} \in J'$, this implies

$$F_{\alpha,\beta}(x, y) = \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right). \quad (28)$$

The collection of codes corresponding to Beer's Law, stated in Eq. (3), is self-transforming. Indeed, because the constant c in the ratio $\frac{y}{c}$ in (3) is expressed in the same units as the variable y , we have

$$\begin{aligned} \frac{1}{\alpha} F_{\alpha,\beta}(\alpha x, \beta y) &= \frac{1}{\alpha} (\alpha x) e^{\frac{\beta y}{\beta c}} = x e^{\frac{y}{c}} \\ &= \frac{1}{\mu} F_{\mu,\nu}(\mu x, \nu y) = F(x, y). \end{aligned}$$

A single unit collection $\mathcal{F} = \{F_\alpha\}$ is *single-unit meaningful*, or *SU-meaningful*, if for $x, y \in J = J', \alpha, \mu \in \mathbb{R}_{++}$, we have

$$\frac{1}{\alpha} F_\alpha(\alpha x, \alpha y) = \frac{1}{\mu} F_\mu(\mu x, \mu y) = F_1(x, y) = F(x, y) \quad (29)$$

³The choice of the initial units is arbitrary.

⁴Many scientific or geometric laws satisfy this restriction.

implying

$$F_\alpha(x, y) = \alpha F\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) \quad \left(\frac{x}{\alpha}, \frac{y}{\alpha} \in J\right). \quad (30)$$

3. Propagation Lemmas

A remarkable property of meaningfulness is that it allows the ‘*propagation*’ of some conditions holding for one code of a meaningful collection to all the codes of that collection. The results used here are stated in the form of six lemmas.

First Propagation Lemma. *Suppose that $\mathcal{F} = \{F_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{R}_{++}\}$ is a ST-meaningful collection of codes. Suppose that some code $F_{\alpha,\beta}$ of that collection satisfies any of the following six properties:*

- (i) $F_{\alpha,\beta}$ is solvable;
- (ii) $F_{\alpha,\beta}$ is quasi-permutable;
- (iii) $F_{\alpha,\beta} = F_\alpha$ is a symmetric function;
- (iv) $F_{\alpha,\beta}$ is a symmetric homogeneous function: $F_\alpha(\gamma x, \gamma y) = \gamma F_\alpha(x, y)$ for any $\gamma \in \mathbb{R}_{++}$, and for all $x, y \in J$ such that $\gamma x, \gamma y \in J$;
- (v) $F_{\alpha,\beta}$ is strictly monotonic in both of its variables and continuous in its first variable;
- (vi) $F_{\alpha,\beta}$ is left-homogeneous: $F_{\alpha,\beta}(ax, y) = aF_{\alpha,\beta}(x, y)$ for any $a \in \mathbb{R}_{++}$ and any $x \in J$ such that $ax \in J$.

Then all the codes in $\mathcal{F}$ satisfy the same property. Moreover, Condition (iv) implies that $F_\nu(x, y) = F(x, y)$ for all $\nu > 0$ and $x, y \in \mathbb{R}_{++}$.

For a proof of the implications of Conditions (i)–(iii) see Falmagne [2, Lemma 16]. In the case of Conditions (iv) and (v), the result follows immediately from the ST-meaningfulness, and so for Condition (vi): with $F_{\alpha,\beta} = F$ the initial code, we have by the ST-meaningfulness and left-homogeneity for the initial code, for any code $F_{\mu,\nu}$,

$$F_{\mu,\nu}(ax, y) = \mu F\left(\frac{ax}{\mu}, \frac{y}{\nu}\right) = \mu a F\left(\frac{x}{\mu}, \frac{y}{\nu}\right) = a F_{\mu,\nu}(x, y).$$

The propagation property illustrated by the above lemma may also apply to equations involving several functions. The next five lemmas are examples.

Definition 6. Let J , J' and H be non-negative intervals of positive length. When the transformation equation

$$F(F(x, y), z) = F(x, K(y, z)) \quad (31)$$

holds for two functions $F : J \times J' \rightarrow H$ and $K : J' \times J' \rightarrow J'$, we will sometimes say for short that the function F is K -transformable.

Second Propagation Lemma. Suppose that $\mathcal{F} = \{F_{\alpha,\beta}\}$ is a *ST-meaningful* collection of codes, and that $\mathcal{K} = \{K_\alpha\}$ is a *SU-meaningful* collection of codes. If for some pair of codes $(F_{\mu,\nu}, K_\nu)$, with $F_{\mu,\nu} \in \mathcal{F}$ and $K_\nu \in \mathcal{K}$, the code $F_{\mu,\nu}$ is K_ν -transformable, then for any pair $(F_{\alpha,\beta}, K_\beta)$ with $F_{\alpha,\beta} \in \mathcal{F}$ and $K_\beta \in \mathcal{K}$, the code $F_{\alpha,\beta}$ is K_β -transformable, that is: the pair $(F_{\alpha,\beta}, K_\beta)$ satisfies the transformation equation:

$$F_{\alpha,\beta}(F_{\alpha,\beta}(x, y), z) = F_{\alpha,\beta}(x, K_\beta(y, z)) \quad (\alpha, \beta \in \mathbb{R}_{++}).$$

Proof. Without loss of generality, we assume that the hypothesis holds for the pair of initial codes (F, K) : with $F_{\mu,\nu} = F$ and $K_\nu = K$, the initial code F is K -transformable:

$$\begin{aligned} F_{\alpha,\beta}(F_{\alpha,\beta}(x, y), z) &= \alpha F \left(\alpha \frac{F \left(\frac{x}{\alpha}, \frac{y}{\beta} \right)}{\alpha}, \frac{z}{\beta} \right) && (\mathcal{F} \text{ is ST-meaningful, applied twice}) \\ &= \alpha F \left(F \left(\frac{x}{\alpha}, \frac{y}{\beta} \right), \frac{z}{\beta} \right) && (\text{simplifying}) \\ &= \alpha F \left(\frac{x}{\alpha}, K \left(\frac{y}{\beta}, \frac{z}{\beta} \right) \right) && (F \text{ is } K\text{-transformable}) \\ &= \alpha F \left(\frac{x}{\alpha}, \left(\frac{1}{\beta} \right) \beta K \left(\frac{y}{\beta}, \frac{z}{\beta} \right) \right) && (\text{rewriting}) \\ &= \alpha F \left(\frac{x}{\alpha}, \frac{K_\beta(y, z)}{\beta} \right) && (K \text{ is SU-meaningful}) \\ &= F_{\alpha,\beta}(x, K_\beta(y, z)) && (\mathcal{F} \text{ is ST-meaningful}). \end{aligned}$$

□

The next propagation lemma deals with the distributivity equation.

Definition 7. Let J be a non-negative interval of positive length. When the distributivity equation

$$F(G(x, y), z) = G(F(x, z), F(y, z)) \quad (32)$$

holds for two functions $F : J \times J \rightarrow J$ and $G : J \times J \rightarrow J$, we will sometimes say that the function G is *F-distributive*.

Third Propagation Lemma. Suppose that $\mathcal{F} = \{F_{\mu,\nu}\}$ is a *ST-meaningful* collection of codes, and that $\mathcal{G} = \{G_\mu\}$ is a *SU-meaningful* collection of codes. If for some pair of codes $(F_{\mu,\nu}, G_\mu)$, with $F_{\mu,\nu} \in \mathcal{F}$ and $G_\mu \in \mathcal{G}$, the code G_μ is $F_{\mu,\nu}$ -distributive, then for any pair $(F_{\alpha,\beta}, G_\alpha)$ with $F_{\alpha,\beta} \in \mathcal{F}$ and $G_\alpha \in \mathcal{G}$, G_α is $F_{\alpha,\beta}$ -distributive, that is: the pair $(F_{\alpha,\beta}, G_\alpha)$ satisfies the distributivity equation:

$$F_{\alpha,\beta}(G_\alpha(x, y), z) = G_\alpha(F_{\alpha,\beta}(x, z), F_{\alpha,\beta}(y, z)) \quad (\alpha, \beta \in \mathbb{R}_{++}).$$

Proof. Again, without loss of generality, we suppose that for the initial pair of codes (F, G) , the code G is F -distributive. We obtain successively

$$\begin{aligned}
 F_{\alpha,\beta}(G_{\alpha}(x, y), z) &= \alpha F\left(\frac{G_{\alpha}(x, y)}{\alpha}, \frac{z}{\alpha}\right) && (\mathcal{F} \text{ is ST-meaningful}) \\
 &= \alpha F\left(\left(\frac{1}{\alpha}\right) \alpha G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), \frac{z}{\alpha}\right) && (\mathcal{G} \text{ is SU-meaningful}) \\
 &= \alpha F\left(G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), \frac{z}{\alpha}\right) && (\text{simplifying}) \\
 &= \alpha G\left(F\left(\frac{x}{\alpha}, \frac{z}{\alpha}\right), F\left(\frac{y}{\alpha}, \frac{z}{\alpha}\right)\right) && (G \text{ is } F\text{-distributive}) \\
 &= \alpha G\left(\frac{1}{\alpha} F_{\alpha,\beta}(x, z), \frac{1}{\alpha} F_{\alpha,\beta}(y, z)\right) && (\mathcal{F} \text{ is ST-meaningful}) \\
 &= G_{\alpha}\left(F_{\alpha,\beta}(x, z), F_{\alpha,\beta}(y, z)\right) && (\mathcal{G} \text{ is SU-meaningful}).
 \end{aligned}$$

□

We recall that, for a quadruple of codes $(F_{\mu,\nu}, G_{\mu,\nu}, H_{\mu,\nu}, K_{\nu})$, the *4-FT equation* is short for the ‘four-functions-transformation equation’, that is the equation

$$F_{\mu,\nu}(G_{\mu,\nu}(x, y), z) = H_{\mu,\nu}(x, K_{\nu}(y, z)).$$

Fourth Propagation Lemma. *Let $\mathcal{F} = \{F_{\mu,\nu}\}$, $\mathcal{G} = \{G_{\mu,\nu}\}$ and $\mathcal{H} = \{H_{\mu,\nu}\}$ be three ST-meaningful collections of codes, and let $\mathcal{K} = \{K_{\nu}\}$ be a SU-collection of codes. If the 4-FT equation is satisfied for some quadruple of codes $(F_{\mu,\nu}, G_{\mu,\nu}, H_{\mu,\nu}, K_{\nu})$, then this equation will also hold for any quadruple of codes $(F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, K_{\beta})$, with $F_{\alpha,\beta} \in \mathcal{F}$, $G_{\alpha,\beta} \in \mathcal{G}$, $H_{\alpha,\beta} \in \mathcal{H}$ and $K_{\beta} \in \mathcal{K}$; that is:*

$$F_{\alpha,\beta}(G_{\alpha,\beta}(x, y), z) = H_{\alpha,\beta}(x, K_{\beta}(y, z)) \quad (\alpha, \beta \in \mathbb{R}_{++}).$$

Proof. Suppose that the quadruple of initial codes (F, G, H, K) satisfies the 4-FT equation. Take any quadruple of codes $(F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, K_{\beta})$. Successively, we get

$$\begin{aligned}
 F_{\alpha,\beta}(G_{\alpha,\beta}(x, y), z) &= \alpha F\left(\frac{G_{\alpha,\beta}(x, y)}{\alpha}, \frac{z}{\beta}\right) && (\mathcal{F} \text{ is ST-meaningful}) \\
 &= \alpha F\left(\left(\frac{1}{\alpha}\right) \alpha G\left(\frac{x}{\alpha}, \frac{y}{\beta}\right), \frac{z}{\beta}\right) && (\mathcal{G} \text{ is ST-meaningful}) \\
 &= \alpha F\left(G\left(\frac{x}{\alpha}, \frac{y}{\beta}\right), \frac{z}{\beta}\right) && (\text{rewriting}) \\
 &= \alpha H\left(\frac{x}{\alpha}, K\left(\frac{y}{\beta}, \frac{z}{\beta}\right)\right) && \left((F, G, H, K) \text{ satisfies the 4-FT equation}\right) \\
 &= \alpha H\left(\frac{x}{\alpha}, \left(\frac{1}{\beta}\right) \beta K\left(\frac{y}{\beta}, \frac{z}{\beta}\right)\right) && (\text{rewriting}) \\
 &= \alpha H\left(\frac{x}{\alpha}, \frac{K_{\beta}(y, z)}{\beta}\right) && (\mathcal{K} \text{ is SU-meaningful}) \\
 &= H_{\alpha,\beta}(x, K_{\beta}(y, z)) && (\mathcal{H} \text{ is ST-meaningful}).
 \end{aligned}$$

□

We recall that, for a sextuple of codes $(F_\mu, G_\mu, H_{\mu,\nu}, K_\mu, M_\mu, N_\mu)$, the *6-FT equation* is short for the ‘*six-functions-transformation equation*’, that is the equation

$$F_\mu(G_\mu(x, y), H_\mu(u, v)) = K_\mu(M_\mu(x, u), N_\mu(y, v)).$$

Fifth Propagation Lemma. *Let $\mathcal{F} = \{F_\mu\}$, $\mathcal{G} = \{G_\mu\}$, $\mathcal{H} = \{H_\mu\}$, $\mathcal{K} = \{K_\mu\}$, $\mathcal{M} = \{M_\mu\}$ and $\mathcal{N} = \{N_\mu\}$ be six SU-meaningful collections of codes, and suppose that each of these six collections contains a symmetric code. If the 6-FT equation is satisfied by some sextuple of codes $(F_\mu, G_\mu, H_\mu, K_\mu, M_\mu, N_\mu)$, then the 6-FT equation will also hold for any sextuple of codes $(F_\alpha, G_\alpha, H_\alpha, K_\alpha, M_\alpha, N_\alpha)$, with $F_\alpha \in \mathcal{F}$, $G_\alpha \in \mathcal{G}$, $H_\alpha \in \mathcal{H}$, $K_\alpha \in \mathcal{K}$, $M_\alpha \in \mathcal{M}$ and $N_\alpha \in \mathcal{N}$; that is:*

$$F_\alpha(G_\alpha(x, y), H_\alpha(u, v)) = K_\alpha(M_\alpha(x, u), N_\alpha(y, v)) \quad (\alpha \in \mathbb{R}_{++}).$$

Proof. Without loss of generality, we suppose that the 6-FT equation is satisfied by the sextuple of initial codes (F, G, H, K, M, N) . Successively, for any sextuple of codes $(F_\alpha, G_\alpha, H_\alpha, K_\alpha, M_\alpha, N_\alpha)$, we have:

$$\begin{aligned} & F_\alpha(G_\alpha(x, y), H_\alpha(u, v)) \\ &= \alpha F\left(\frac{G_\alpha(x, y)}{\alpha}, \frac{H_\alpha(u, v)}{\alpha}\right) && (\mathcal{F} \text{ is SU-meaningful}) \\ &= \alpha F\left(\left(\frac{1}{\alpha}\right) \alpha G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), \frac{H_\alpha(u, v)}{\alpha}\right) && (\mathcal{G} \text{ is SU-meaningful}) \\ &= \alpha F\left(\left(\frac{1}{\alpha}\right) \alpha G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), \left(\frac{1}{\alpha}\right) \alpha H\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right)\right) && (\mathcal{H} \text{ is SU-meaningful}) \\ &= \alpha F\left(G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), H\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right)\right) && (\text{simplifying}) \\ &= \alpha K\left(M\left(\frac{x}{\alpha}, \frac{u}{\alpha}\right), N\left(\frac{y}{\alpha}, \frac{v}{\alpha}\right)\right) && \left(\begin{array}{l} \text{the sextuple } (F, G, H, K, M, N) \\ \text{satisfies the 6-FT equation} \end{array}\right) \\ &= \alpha K\left(\frac{M_\alpha(x, y)}{\alpha}, \frac{N_\alpha(u, v)}{\alpha}\right) && \left(\begin{array}{l} \mathcal{M} \text{ is SU-meaningful, and} \\ \mathcal{N} \text{ is ST-meaningful} \end{array}\right) \\ &= K_\alpha(M_\alpha(x, u), N_\alpha(y, v)) && (\mathcal{K} \text{ is SU-meaningful}). \end{aligned}$$

So we get

$$F_\alpha(G_\alpha(x, y), H_{\alpha,\beta}(u, v)) = K_\alpha(M_\alpha(x, u), N_\alpha(y, v)). \quad \square$$

In the last propagation lemma, the propagation of F -permutability is obtained from the joint assumption of ST or SU-meaningfulness, with the maintenance of a ‘proportionality’ condition across changes of units of the codes.

Definition 8. A code $F_{\alpha,\beta} : J \times J' \rightarrow H$ in a ST-meaningful collection of codes $\mathcal{F}$ is *unit-proportional* if for any $x, z, \alpha x, \alpha z \in J$ and $y, w, \beta y, \beta w \in J'$ we have

$$F_{\alpha,\beta}(x, y) = F_{\alpha,\beta}(z, w) \iff F_{\alpha,\beta}(\alpha x, \beta y) = F_{\alpha,\beta}(\alpha z, \beta w).$$

A collection of codes $\mathcal{F}$ is *unit-proportional* if all its codes are unit-proportional. A similar definition applies to the codes of a SU-meaningful family $\mathcal{F}$.

Sixth Propagation Lemma. *If one code in a unit-proportional ST or SU-meaningful collection $\mathcal{F}$ is G -permutable, for some function G comonotonic with the code, then all the codes are G -permutable.*

Proof. Suppose that some code $F_{\alpha,\beta}$ in a ST-meaningful collection $\mathcal{F}$ is G -permutable. Successively, we have, for any code $F_{\mu,\nu} \in \mathcal{F}$:

$$\begin{aligned}
 & F_{\alpha,\beta}(G(x,y),z) = F_{\alpha,\beta}(G(x,z),y) && (F_{\alpha,\beta} \text{ is } G\text{-permutable}) \\
 \iff & F_{\alpha,\beta}(\alpha G(x,y),\beta z) = F_{\alpha,\beta}(\alpha G(x,z),\beta y) && (F_{\alpha,\beta} \text{ is unit-proportional}) \\
 \iff & \frac{1}{\alpha} F_{\alpha,\beta}(\alpha G(x,y),\beta z) = \frac{1}{\alpha} F_{\alpha,\beta}(\alpha G(x,z),\beta y) && \left(\text{multiplying both sides by } \frac{1}{\alpha} \right) \\
 \iff & \frac{1}{\mu} F_{\mu,\nu}(\mu G(x,y),\nu z) = \frac{1}{\mu} F_{\mu,\nu}(\mu G(x,z),\nu y) && (\mathcal{F} \text{ is ST-meaningful}) \\
 \iff & F_{\mu,\nu}(\mu G(x,y),\nu z) = F_{\mu,\nu}(\mu G(x,z),\nu y) && \left(\text{cancelling } \frac{1}{\mu} \right) \\
 \iff & F_{\mu,\nu}(G(x,y),z) = F_{\mu,\nu}(G(x,z),y) && (\mathcal{F} \text{ is unit-proportional})
 \end{aligned}$$

So $F_{\mu,\nu}$ is G -permutable. The proof in the case of SU-meaningfulness is similar. $\square$

In the rest of this paper, we will encounter several examples of unit-proportional codes, and one counterexample.

Two examples and one counterexample 9. The first example below occurs as Cases A and B of Theorem 13. Example 2 occurs as Case C. (The same examples also occur elsewhere in the paper.)

1. The code $F_{\alpha,\beta}(x,y) = \phi xy^\gamma$ is unit-proportional, for we have:

$$\begin{aligned}
 F_{\alpha,\beta}(x,y) = F_{\alpha,\beta}(z,w) & \iff \phi xy^\gamma = \phi zw^\gamma \\
 & \iff \alpha\beta^\gamma \times \phi xy^\gamma = \alpha\beta^\gamma \times \phi zw^\gamma \\
 & \iff \phi(\alpha x)(\beta y)^\gamma = \phi(\alpha z)(\beta w)^\gamma \\
 & \iff F_{\alpha,\beta}(\alpha x, \beta y) = F_{\alpha,\beta}(\alpha z, \beta w).
 \end{aligned}$$

2. In the case of a SU-meaningful collection the unit-proportionality condition becomes

$$F_\alpha(x,y) = F_\alpha(z,w) \iff F_\alpha(\alpha x, \alpha y) = F_\alpha(\alpha z, \alpha w).$$

The code $F_\alpha(x,y) = (x^\theta + y^\theta)^{\frac{1}{\theta}}$ is unit-proportional. Indeed:

$$\begin{aligned}
 F_\alpha(x,y) = F_\alpha(z,w) & \iff (x^\theta + y^\theta)^{\frac{1}{\theta}} = (z^\theta + w^\theta)^{\frac{1}{\theta}} \\
 & \iff \alpha (x^\theta + y^\theta)^{\frac{1}{\theta}} = \alpha (z^\theta + w^\theta)^{\frac{1}{\theta}} \\
 & \iff ((\alpha x)^\theta + (\alpha y)^\theta)^{\frac{1}{\theta}} = ((\alpha z)^\theta + (\alpha w)^\theta)^{\frac{1}{\theta}}
 \end{aligned}$$

$$\Longleftrightarrow F_{\alpha}(\alpha x, \alpha y) = F_{\alpha}(\alpha z, \alpha w).$$

3. The next example is the code $F_{\alpha, \beta}(x, y) = xe^{\frac{1}{\kappa}(\frac{y}{\beta})^{\theta}}$ obtained in Theorem 12. This code is not unit-proportional. Unit-proportionality would require:

$$\begin{aligned} F_{\alpha, \beta}(x, y) = F_{\alpha, \beta}(z, w) &\Longleftrightarrow xe^{\frac{1}{\kappa}(\frac{y}{\beta})^{\theta}} = ze^{\frac{1}{\kappa}(\frac{w}{\beta})^{\theta}} \\ &\Longleftrightarrow F_{\alpha, \beta}(\alpha x, \beta y) = F_{\alpha, \beta}(\alpha z, \beta w) \quad (\text{by unit-proportionality}) \\ &\Longleftrightarrow \alpha xe^{\frac{1}{\kappa}(\frac{\beta y}{\beta})^{\theta}} = \alpha ze^{\frac{1}{\kappa}(\frac{\beta w}{\beta})^{\theta}} \\ &\Longleftrightarrow xe^{\frac{y^{\theta}}{\kappa}} = ze^{\frac{w^{\theta}}{\kappa}} \quad (\text{after simplification}). \end{aligned}$$

This is false. We cannot have the equivalence:

$$xe^{\frac{1}{\kappa}(\frac{y}{\beta})^{\theta}} = ze^{\frac{1}{\kappa}(\frac{w}{\beta})^{\theta}} \Longleftrightarrow xe^{\frac{y^{\theta}}{\kappa}} = ze^{\frac{w^{\theta}}{\kappa}}.$$

For example, with $\theta = \kappa = z = y = 1$, $w = 2$, $x = e$ and $\beta = 2$, we get:

$$xe^{\frac{y^{\theta}}{\kappa}} = ze^{\frac{w^{\theta}}{\kappa}} = e^2 \quad \text{while} \quad xe^{\frac{1}{\kappa}(\frac{y}{\beta})^{\theta}} = e^{1.5} \neq ze^{\frac{1}{\kappa}(\frac{w}{\beta})^{\theta}} = e.$$

4. The transformation equation

We don't apply the meaningfulness condition to the transformation equation (31) directly because, under remarkably weak conditions, using a well known functional equations result, we can obtain general forms for the functions F and K in Eq. (1).

Theorem 10. [1, Theorem 4, page 316]. *Let J and J' be two non-negative intervals of positive length. Suppose that in the transformation equation:*

$$F(F(x, y), z) = F(x, K(y, z)) \quad (x, F(x, y) \in J, ; y, z, K(y, z) \in J') \quad (33)$$

the functions F and K are strictly monotonic in both variables, K is continuous, and F is continuous in its first variable. Then there exist two strictly monotonic functions f and k such that

$$F(x, y) = f(f^{-1}(x) + k^{-1}(y)) \quad (34)$$

$$K(y, z) = k(k^{-1}(y) + k^{-1}(z)). \quad (35)$$

We have encountered these two equations before: in Lemma 3, Eqs. (24) and (25). It is clear that the function K of Eq. (35) is symmetric. It is also easy to show that it is associative: $K(K(x, y), z) = K(x, K(y, z))$. Indeed, we have:

$$\begin{aligned} K(K(x, y), z) &= k(k^{-1}(K(x, y)) + k^{-1}(z)) \\ &= k((k^{-1} \circ k)(k^{-1}(x) + k^{-1}(y)) + k^{-1}(z)) \\ &= k(k^{-1}(x) + k^{-1}(y) + k^{-1}(z)) \end{aligned}$$

$$\begin{aligned}
&= k(k^{-1}(x) + (k^{-1} \circ k)(k^{-1}(y) + k^{-1}(z))) \\
&= k(k^{-1}(x) + k^{-1}(K(y, z))) \\
&= K(x, K(y, z)).
\end{aligned}$$

If we assume that the function K is also homogeneous and solvable—cf. Definition 1, Eq. (15) and Conditions [S1] and [S2]—the following result follows.

Theorem 11. *Let $\mathcal{K} = \{K_\nu \mid \nu \in \mathbb{R}_{++}, K_\nu : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}\}$ be a SU-meaningful collection of codes. Moreover, suppose that one of these codes, say K_α , is strictly increasing in both variables, homogeneous, and associative. Then any code $K_\nu \in \mathcal{F}$ must have the form*

$$K_\nu(x, y) = (x^\theta + y^\theta)^{\frac{1}{\theta}} = K(x, y), \quad (36)$$

for some positive constant θ .

For a proof, see Falmagne and Doble [3, Theorem 7.1.1].⁵ The main result of this section follows.

Theorem 12. *Given a ST-meaningful collection of codes*

$$\mathcal{F} = \{F_{\alpha, \beta} \mid F_{\alpha, \beta} : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}, \alpha, \beta \in \mathbb{R}_{++}\}$$

and a SU-meaningful collection of codes

$$\mathcal{K} = \{K_\beta \mid K_\beta : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}, \beta \in \mathbb{R}_{++}\}$$

suppose that the following conditions are satisfied for some pair of codes $(F_{\mu, \nu}, K_\nu)$, with $F_{\mu, \nu} \in \mathcal{F}$ and $K_\nu \in \mathcal{K}$:

- (a) $F_{\mu, \nu}$ is K_ν -transformable;
- (b) $F_{\mu, \nu}$ is left-homogeneous;
- (c) K_ν is homogeneous and associative;
- (d) K_ν is strictly increasing in both variables.

Then the following two equations hold for the initial codes $F \in \mathcal{F}$ and $K \in \mathcal{K}$. For some positive constants θ and κ , we have:

$$F(x, y) = x e^{\frac{y^\theta}{\kappa}}, \quad (37)$$

$$K(x, y) = (x^\theta + y^\theta)^{\frac{1}{\theta}}; \quad (38)$$

and for any codes $F_{\alpha, \beta} \in \mathcal{F}$ and $K_\beta \in \mathcal{K}$,

$$F_{\alpha, \beta}(x, y) = x e^{\frac{1}{\kappa} \left(\frac{y}{\beta}\right)^\theta}, \quad (39)$$

$$K_\beta(x, y) = \left(x^{\frac{1}{\theta}} + y^{\frac{1}{\theta}}\right)^\theta. \quad (40)$$

⁵Theorem 7.1.1 in [3] assumes that $(\frac{1}{2}, \frac{1}{2})$ -meaningfulness holds, which is equivalent to SU-meaningfulness.

Proof. By the Second Propagation Lemma, Condition (a) implies that any code $F_{\alpha,\beta}$ is K_β -transformable; that is, we have

$$F_{\alpha,\beta}(F_{\alpha,\beta}(x, y), z) = F_{\alpha,\beta}(x, K_\beta(y, z)).$$

By the First Propagation Lemma, Condition (b) holds for any code $F_{\alpha,\beta} \in \mathcal{F}$. Theorem 10 implies the existence of two strictly monotonic functions $f_{\alpha,\beta}$ and k_β such that

$$F_{\alpha,\beta}(x, y) = f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + k_\beta(y)). \quad (41)$$

The function K_β satisfies all the conditions of Theorem 11, so we have for some constant $\xi \in \mathbb{R}_{++}$ and any $\beta \in \mathbb{R}_{++}$,

$$K_\beta(y, z) = (y^\xi + z^\xi)^{\frac{1}{\xi}} = K(y, z). \quad (42)$$

By Eq. (35) in Theorem 10, we also have

$$K_\beta(y, z) = (y^\xi + z^\xi)^{\frac{1}{\xi}} = k_\beta(k_\beta^{-1}(y) + k_\beta^{-1}(z))$$

yielding, with $u = y^\xi$, $v = z^\xi$ and $h_\beta(u) = k_\beta^{-1}(u^{\frac{1}{\xi}})$, the Cauchy equation

$$h_\beta(u + v) = h_\beta(u) + h_\beta(v). \quad (43)$$

As the function $u \mapsto k_\beta^{-1}(u^{\frac{1}{\xi}}) = h_\beta(u)$ is strictly monotonic, the solution of the Cauchy equation (43) is

$$h_\beta(u) = d(\beta)u = k_\beta^{-1}(u^{\frac{1}{\xi}}) = k_\beta^{-1}(y) = d(\beta)y^\xi$$

for some $d(\beta) > 0$. So, we have

$$k_\beta(y) = \left(\frac{1}{d(\beta)} \right)^{\frac{1}{\xi}} y^{\frac{1}{\xi}}$$

and with $c(\beta) = \left(\frac{1}{d(\beta)} \right)^{\frac{1}{\xi}}$,

$$k_\beta(y) = c(\beta)y^{\frac{1}{\xi}}. \quad (44)$$

Replacing $k_\beta(y)$ in (41) by its expression in (44) gives

$$F_{\alpha,\beta}(x, y) = f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + c(\beta)y^{\frac{1}{\xi}}). \quad (45)$$

By the left-homogeneity of $F_{\alpha,\beta}$, we thus have for any $a \in \mathbb{R}_{++}$:

$$F_{\alpha,\beta}(ax, y) = f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(ax) + c(\beta)y^{\frac{1}{\xi}}) = aF_{\alpha,\beta}(x, y). \quad (46)$$

From (45) and (46), we get

$$f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(ax) + c(\beta)y^{\frac{1}{\xi}}) = af_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + c(\beta)y^{\frac{1}{\xi}}). \quad (47)$$

Set $f_{\alpha,\beta}^{-1}(x) = w$, and so $f_{\alpha,\beta}(w) = x$. Applying $f_{\alpha,\beta}^{-1}$ on both sides of (47), we get

$$\left(f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}\right)(w) + c(\beta)y^{\frac{1}{\xi}} = \left(f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}\right)(w + c(\beta)y^{\frac{1}{\xi}}).$$

Fixing temporarily α and β and defining $v = c(\beta)y^{\frac{1}{\xi}}$, the above equation becomes

$$\left(f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}\right)(w) + v = \left(f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}\right)(w + v),$$

a Pexider equation. This implies that the function $f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}$ must be of the form

$$\left(f_{\alpha,\beta}^{-1} \circ a f_{\alpha,\beta}\right)(w) = w + B_{\alpha,\beta}(a)$$

for some function $B_{\alpha,\beta} : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, or equivalently

$$f_{\alpha,\beta}^{-1}(ax) = f_{\alpha,\beta}^{-1}(x) + B_{\alpha,\beta}(a), \quad (48)$$

another Pexider equation in the variables a and x [1, Chapter 3, Section 3.1]. Since $f_{\alpha,\beta}^{-1}$ is strictly monotonic, so is $B_{\alpha,\beta}$. Equation (48) implies, for some constants $\gamma_1(\alpha, \beta) > 0$ and $\gamma_2(\alpha, \beta)$,

$$\begin{aligned} f_{\alpha,\beta}^{-1}(x) &= \gamma_1(\alpha, \beta) \ln x + \gamma_2(\alpha, \beta) \\ B_{\alpha,\beta}(a) &= \gamma_1(\alpha, \beta) \ln a, \end{aligned} \quad (49)$$

which gives us

$$f_{\alpha,\beta}(w) = e^{\frac{w - \gamma_2(\alpha, \beta)}{\gamma_1(\alpha, \beta)}}. \quad (50)$$

Replacing in (46) $f_{\alpha,\beta}^{-1}$ and $f_{\alpha,\beta}$ by their expressions in (49) and (50), we get successively

$$\begin{aligned} F_{\alpha,\beta}(x, y) &= f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + c(\beta)y^{\frac{1}{\xi}}) \\ &= f_{\alpha,\beta}(\gamma_1(\alpha, \beta) \ln x + \gamma_2(\alpha, \beta) + c(\beta)y^{\frac{1}{\xi}}) \\ &= e^{\frac{\gamma_1(\alpha, \beta) \ln x + \gamma_2(\alpha, \beta) + c(\beta)y^{\frac{1}{\xi}} - \gamma_2(\alpha, \beta)}{\gamma_1(\alpha, \beta)}} \\ &= x e^{\frac{c(\beta)y^{\frac{1}{\xi}}}{\gamma_1(\alpha, \beta)}} \end{aligned}$$

yielding

$$F_{\alpha,\beta}(x, y) = x e^{\frac{c(\beta)y^{\frac{1}{\xi}}}{\gamma_1(\alpha, \beta)}}.$$

The left homogeneity of $F_{\alpha,\beta}$ and the ST-meaningfulness of the collection $\mathcal{F}$ imply

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x, \beta y) = \left(\frac{1}{\alpha}\right) \alpha x e^{\frac{c(\beta)(\beta y)^{\frac{1}{\xi}}}{\gamma_1(\alpha, \beta)}}$$

$$\begin{aligned}
&= x e^{\frac{c(\beta)\beta^{\frac{1}{\xi}} y^{\frac{1}{\xi}}}{\gamma_1(\alpha, \beta)}} \\
&= F(x, y).
\end{aligned}$$

The last equation shows that the ratio $\frac{c(\beta)\beta^{\frac{1}{\xi}}}{\gamma_1(\alpha, \beta)}$ does not depend upon α or β . Defining

$$\kappa = \frac{\gamma_1(\alpha, \beta)}{c(\beta)\beta^{\frac{1}{\xi}}} \quad \text{and} \quad \theta = \frac{1}{\xi},$$

we finally obtain

$$F(x, y) = x e^{\frac{y^\theta}{\kappa}}. \quad (51)$$

Accordingly, for any code $F_{\alpha, \beta} \in \mathcal{F}$, we obtain

$$F_{\alpha, \beta}(x, y) = \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right) = x e^{\frac{1}{\kappa} \left(\frac{y}{\beta}\right)^\theta}. \quad (52)$$

Gathering Equations (51), (52) and (42), the proof of the theorem is complete. $\square$

5. The quasi-permutability equation

Theorem 13, the main result of this section, is a representation theorem for meaningful quasi-permutable transformations in the style of Theorem 12 for the transformation equation. This theorem is closely related to a previous result of Falmagne and Doble [3, Theorem 7.2.1, page 88]. The essential difference is that their theorem assumes that all the codes in the ST-meaningful collection are permutable with respect to the initial code, while the key assumption of Theorem 13 relies on the joint condition that one code is permutable with respect to the initial code together with the unit-proportionality condition introduced in Definition 8.

Theorem 13. *Let $\mathcal{F} = \{F_{\alpha, \beta} \mid \alpha, \beta \in \mathbb{R}_{++}\}$ be a ST-meaningful, unit-proportional collection of codes, with*

$$F_{\alpha, \beta} : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{\text{onto}} \mathbb{R}_{++} \quad \text{for all } \alpha, \beta \in \mathbb{R}_{++}.$$

Moreover, suppose that, in the collection $\mathcal{F}$, one arbitrarily chosen code is strictly increasing in both variables and permutable with respect to the initial code $F = F_{1,1}$. Then there are constants $\phi, \gamma > 0$ such that Case A holds:

$$\text{Case A.} \quad F(y, r) = \phi y r^\gamma, \quad (53)$$

$$F_{\alpha, \beta}(y, r) = \phi y \left(\frac{r}{\beta}\right)^\gamma \quad \text{for all } F_{\alpha, \beta} \in \mathcal{F}. \quad (54)$$

If $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_{++}\}$ is a *SU-meaningful* collection of codes, with all of the other hypotheses the same as above, then for some positive constants ϕ , γ , θ and λ and some constant η , either Case B or Case C holds:

$$\text{Case B.} \quad F(y, r) = \phi y r^\gamma, \quad (55)$$

$$F_\alpha(y, r) = \phi y \left(\frac{r}{\alpha}\right)^\gamma \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (56)$$

If some code in $\mathcal{F}$ is symmetric, then $\gamma = 1$, and if in addition some code is homogeneous, then

$$F_\alpha(y, r) = F(y, r) = \theta y r \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (57)$$

$$\text{Case C.} \quad F(y, r) = (y^\theta + \lambda r^\theta + \eta)^{\frac{1}{\theta}}, \quad (58)$$

$$F_\alpha(y, r) = (y^\theta + \lambda r^\theta + \alpha^\theta \eta)^{\frac{1}{\theta}} \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (59)$$

If some code in $\mathcal{F}$ is symmetric, then $\lambda = 1$, and if in addition some code is homogeneous, then

$$F_\alpha(y, r) = F(y, r) = (y^\theta + r^\theta)^{\frac{1}{\theta}} \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (60)$$

Proof. In Theorem 7.2.1 of Falmagne and Doble [3, page 88], the unit-proportionality condition is not used (so the Sixth Propagation Lemma is irrelevant). But it is assumed that all the codes are permutable with respect to the initial code. Other than that, Theorem 7.2.1 of [3] and Theorem 13 here are identical. Here, we assume that just one code is permutable with respect to the initial code, but we can rely on the unit-proportionality condition to ensure, via the Sixth Propagation Lemma, that all the codes satisfy that condition. Theorem 13 is thus an immediate consequence of Theorem 7.2.1 of Falmagne and Doble [3, page 88]. $\square$

Remark 14. The representation theorem for quasi-permutable transformations in Falmagne [2, Theorem 9.2] is similar to Theorem 13 in supposing that a single code in $\mathcal{F}$ is F -permutable, but without the unit-proportionality assumption. The proof of this theorem was incomplete. The assumption that a single code is permutable with respect to the initial code is insufficient, without an additional condition such as unit-proportionality, to ensure the required representation in the style of Theorem 13. To see this, let $\mathcal{F} = \{F_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{R}_{++}\}$, with $F_{\alpha,\beta} : [0, \infty[\times [0, \infty[\xrightarrow{\text{onto}} [0, \infty[$ defined by $F_{\alpha,\beta}(x, y) = x f\left(\frac{y}{\beta}\right)$, where $f : [0, \infty[\rightarrow [0, \infty[$ is strictly increasing and continuous. Then $\mathcal{F}$ is a ST-meaningful collection (see Eq. (28)):

$$F_{\alpha,\beta}(x, y) = \alpha \frac{x}{\alpha} f\left(\frac{y}{\beta}\right) = \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right).$$

Also, the code $F_{\alpha,1}$, which has the form $F_{\alpha,1}(x, y) = x f(y)$, is strictly increasing in both variables and permutable with respect to the initial code F :

$$F_{\alpha,1}(F(x, y), z) = F_{\alpha,1}(x f(y), z) = x f(y) f(z) = F_{\alpha,1}(F(x, z), y).$$

But the codes $F_{\alpha,\beta}$ of the collection $\mathcal{F}$ do not have one of the forms listed as Cases A, B, or C when f is defined by, for example, $f(x) = e^x$.

6. The distributivity equation

The mechanics of our meaningful derivation from the distributivity equation

$$F(G(x, y), z) = G(F(x, z), F(y, z)) \quad (61)$$

are similar to those of the previous sections: we don't apply the meaningfulness condition to Eq. (61) straightaway. Rather, we first derive representations for the functions F and G by the functional equation result below, which is closely related to a theorem in Aczél [1, page 319]. The difference resides in the hypotheses about the function G , which is bounded above in Aczél's result. Our proof is the same as in Aczél's volume but relies on Lemma 3(iii), which is due to [5–7].

Lemma 15. *The general solution of the equation*

$$F(G(x, y), z) = G(F(x, z), F(y, z)) \quad (62)$$

with $F : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $G : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with G symmetric, continuous, strictly increasing and permutable, is the pair of equations

$$F(x, z) = h^{-1}(h(x)g(z)) \quad (63)$$

$$G(x, y) = h^{-1}(h(x) + h(y)), \quad (64)$$

with g arbitrary continuous and h strictly increasing, continuous, and such that $h(0) = 0$.

Proof. By Lemma 3(iii) there exists a strictly increasing and continuous function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$G(x, y) = h^{-1}(h(x) + h(y)). \quad (65)$$

Replacing the function G in (62) by its expression in (65) yields

$$h(F(h^{-1}(h(x) + h(y)), z)) = h(F(x, z)) + h(F(y, z)).$$

Setting $u = h(x)$, $v = h(y)$, we get

$$h(F(h^{-1}(u + v), z)) = h(F(h^{-1}(u), z)) + h(F(h^{-1}(v), z)),$$

which with $M(u, z) = h(F(h^{-1}(u), z))$ becomes

$$M(u + v, z) = M(u, z) + M(v, z).$$

For a fixed z , the above equation becomes the Cauchy equation. Since F is bounded from below and h is strictly increasing and continuous by Lemma 3(iii), we obtain

$$M(u, z) = g(z)u.$$

Since $M(u, z) = h(F(h^{-1}(u), z))$, this implies

$$F(x, z) = h^{-1}(h(x)g(z)).$$

□

Note that Eq. (63) implies

$$\begin{aligned} F(x, z) &= (h^{-1} \circ \exp) \ln(h(x)g(z)) \\ &= (h^{-1} \circ \exp)(\ln h(x) + \ln g(z)) \end{aligned}$$

and with $m = h^{-1} \circ \exp$, $f = \ln \circ h$ and $\ell = \ln \circ g$,

$$F(x, z) = m(f(x) + \ell(z)),$$

which is the equation of Case (i) in Theorem 3 (with $\ell = g$). If we assume that the function F is solvable (cf. Definition 2), then by Theorem 3 the function F is quasi-permutable. As for Eq. (64), its form is that of Eq. (25) in Case (iii) of Theorem 3. The function G is obviously symmetric. If we assume that G is also solvable, then again Theorem 3 implies that G is permutable. The importance of the quasi-permutability of F and the permutability of G is that, together with unit-proportionality, the meaningfulness condition determines then the exact form of these two functions. In other words, we can obtain the meaningful form of the distributivity equation through the meaningful forms of the two functions F and G in (61). We know what these two forms are by the Quasi-Permutability Theorem 13 of the previous section.

Theorem 16. Suppose that $\mathcal{F} = \{F_{\mu,\nu} \mid F_{\mu,\nu} : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+\}$ is a ST -meaningful unit-proportional collection of codes. Let $\mathcal{G} = \{G_\nu \mid G_\nu : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+\}$ be a SU -meaningful collection of codes. Suppose that the collection $\mathcal{F}$ contains a F -permutable code and that the following conditions are satisfied.

- (a) For some pair of codes $(F_{\mu,\nu}, G_\nu)$ with $F_{\mu,\nu} \in \mathcal{F}$ and $G_\nu \in \mathcal{G}$, the pair $(F_{\mu,\nu}, G_\nu)$ satisfies the distributivity equation

$$F_{\mu,\nu}(G_\nu(x, y), z) = G_\nu(F_{\mu,\nu}(x, z), F_{\mu,\nu}(y, z));$$

- (b) one of the codes in the collection $\mathcal{F}$, say the code $F_{\alpha,\beta}$, is strictly increasing in both variables;
- (c) one of the codes in the collection $\mathcal{G}$, say the code G_β , is continuous, strictly increasing in both variables, homogeneous, and associative.

Then we have two cases.

CASE 1. The following equations hold for all $G_\beta \in \mathcal{G}$ and $F_{\alpha,\beta} \in \mathcal{F}$; for some parameters $\theta > 0$ and $\gamma > 0$:

$$\begin{aligned} G_\beta(x, y) &= (x^\theta + y^\theta)^{\frac{1}{\theta}} = G(x, y) \\ F_{\alpha,\beta}(x, z) &= \phi x \left(\frac{z}{\beta} \right)^\gamma \\ F(x, z) &= \phi x z^\gamma. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} F(G(x, y), z) &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} z^\gamma = G(F(x, z), F(y, z)) \\ F_{\alpha,\beta}(G_\beta(x, y), z) &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(\frac{z}{\beta} \right)^\gamma = G_\beta(F_{\alpha,\beta}(x, z), F_{\alpha,\beta}(y, z)). \end{aligned}$$

CASE 2. One of the codes in the collection $\mathcal{F}$ is a symmetric function. This implies that $F_{\alpha,\beta} = F_{\alpha,\alpha} = F_\alpha$ for all the codes in the SU-meaningful collection $\mathcal{F}$. Then for some parameter $\theta > 0$,

$$\begin{aligned} G_\alpha(x, y) &= (x^\theta + y^\theta)^{\frac{1}{\theta}} = G(x, y) \\ F_\alpha(x, z) &= \frac{\phi}{\alpha} xz, \end{aligned}$$

with

$$\begin{aligned} F(G(x, y), z) &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} z = G(F(x, z), F(y, z)) \quad (66) \\ F_\alpha(G_\alpha(x, y), z) &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(\frac{z}{\alpha} \right) = G_\alpha(F_\alpha(x, z), F_\alpha(y, z)). \quad (67) \end{aligned}$$

The equations for the codes in the collection $\mathcal{G}$ are thus identical in both cases, but we have the parameter $\gamma = 1$ in Case 2.

Proof.

CASE 1. Beginning with the collection of codes $\mathcal{G}$, we have by Theorem 11:

$$G_\beta(x, y) = (x^\theta + y^\theta)^{\frac{1}{\theta}} \quad (\text{for some } \theta > 0 \text{ and all } G_\beta \in \mathcal{G}) \quad (68)$$

and so

$$G(x, y) = (x^\theta + y^\theta)^{\frac{1}{\theta}}. \quad (69)$$

We turn to the collection of codes $\mathcal{F}$. Because the codes in $\mathcal{F}$ are not assumed to be symmetric functions, Case A in Theorem 13 is applicable. It implies that there are some positive constants ϕ and δ , such that:

$$F(x, z) = \phi x z^\gamma, \quad (70)$$

$$F_{\alpha,\beta}(x, z) = \phi x \left(\frac{z}{\beta} \right)^\gamma, \quad \text{for all } F_{\alpha,\beta} \in \mathcal{F}. \quad (71)$$

Using (69) and (70) in the distributivity equation, we obtain successively

$$\begin{aligned}
 F(G(x, y), z) &= \phi G(x, y) z^\gamma && \text{(by Eq. (70))} \\
 &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} z^\gamma && \text{(by Eq. (69))} \\
 &= G(F(x, z), F(y, z)) && \text{(to verify the distributivity equation)} \\
 &= (F(x, z)^\theta + F(y, z)^\theta)^{\frac{1}{\theta}} && \text{(by Eq. (69))} \\
 &= \left((\phi x z^\gamma)^\theta + (\phi y z^\gamma)^\theta \right)^{\frac{1}{\theta}} && \text{(by Eq. (70))} \\
 &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} z^\gamma && \text{(after rearranging).}
 \end{aligned}$$

We have thus

$$F(G(x, y), z) = \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} z^\gamma = G(F(x, z), F(y, z)). \quad (72)$$

Considering now any pair $(F_{\alpha, \beta}, G_\beta)$, by the Third Propagation Lemma, we have

$$F_{\alpha, \beta}(G_\beta(x, y), z) = G_\beta(F_{\alpha, \beta}(x, z), F_{\alpha, \beta}(y, z)).$$

Replacing G_β and then $F_{\alpha, \beta}$ on the l.h.s. by their expressions in (68) and (71) yields

$$\begin{aligned}
 F_{\alpha, \beta}(G_\beta(x, y), z) &= \phi G_\beta(x, y) \left(\frac{z}{\beta} \right)^\gamma \\
 &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(\frac{z}{\beta} \right)^\gamma.
 \end{aligned}$$

Dealing similarly with the r.h.s., we get

$$\begin{aligned}
 G_\beta(F_{\alpha, \beta}(x, z), F_{\alpha, \beta}(y, z)) &= (F_{\alpha, \beta}(x, z)^\theta + F_{\alpha, \beta}(y, z)^\theta)^{\frac{1}{\theta}} \\
 &= \left(\left(\phi x \left(\frac{z}{\beta} \right)^\gamma \right)^\theta + \left(\phi y \left(\frac{z}{\beta} \right)^\gamma \right)^\theta \right)^{\frac{1}{\theta}} \\
 &= \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(\frac{z}{\beta} \right)^\gamma.
 \end{aligned}$$

We obtain thus

$$F_{\alpha, \beta}(G_\beta(x, y), z) = \phi (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(\frac{z}{\beta} \right)^\gamma = G_\beta(F_{\alpha, \beta}(x, z), F_{\alpha, \beta}(y, z)). \quad (73)$$

Considering Eqs. (68), (69), (70) (71), (72), and (73), the proof of Case 1 is complete.

CASE 2. The arguments establishing the formulas for the codes in the collection $\mathcal{G}$ are the same. For the SU-meaningful collection $\mathcal{F}$, Eq. (70) still applies:

$$F(x, z) = \phi x z^\gamma,$$

and (71) is replaced by:

$$F_\alpha(x, z) = \phi x \left(\frac{z}{\alpha} \right), \quad \text{for all } F_\alpha \in \mathcal{F}.$$

Equations (66) and (67) follow by straightforward substitutions. $\square$

7. The four-functions-transformation equation

As in previous sections, we begin by stating a functional equation result allowing the transformations of each of the component functions of the 4-FT equation into workable representations.

Lemma 17. *Suppose that the following equation holds*

$$F(G(x, y), z) = H(x, K(y, z)) \quad (74)$$

with each function F , G , H and K mapping $\mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, continuous, invertible in both variables. Then, there exist six arbitrary continuous and strictly monotonic functions ℓ , f , g , k , m , and h , such that

$$(a) \quad F(x, y) = \ell(f(x) + g(y)), \quad (b) \quad G(x, y) = f^{-1}(k(x) + m(y)), \quad (75)$$

$$(c) \quad H(x, y) = \ell(k(x) + h(y)), \quad (d) \quad K(x, y) = h^{-1}(m(x) + g(y)). \quad (76)$$

Moreover, if G and H are symmetric functions, then we obtain for arbitrary continuous and strictly monotonic functions ℓ , f , g , and $k = m = h$:

$$(a) \quad F(x, y) = \ell(f(x) + g(y)), \quad (b) \quad G(x, y) = f^{-1}(k(x) + k(y)), \quad (77)$$

$$(c) \quad H(x, y) = \ell(k(x) + k(y)), \quad (d) \quad K(x, y) = k^{-1}(k(x) + g(y)). \quad (78)$$

Proof. For a proof of (75) and (76), see Aczél [1, Corollary 1, page 312].⁶ If the function G of Eq. (75)(b) is symmetric, then

$$G(x, y) = f^{-1}(k(x) + m(y)) = f^{-1}(k(y) + m(x)) = G(y, x),$$

which implies, after some manipulation

$$k(x) - k(y) = m(x) - m(y)$$

that is

$$k(x) = m(x) - c$$

⁶The Aczél statement of this result is more general, in that the set $\mathbb{R}_{++}$ is replaced by an arbitrary interval.

for some constant c which can be ‘absorbed’ in the function f^{-1} , yielding (77)(b). (For simplicity, we don’t change the notation of the function f^{-1} .) Similar arguments can be used to derive Eq. (78)(c) and (d). $\square$

Note that each of the four equations (77)(a)(b) and (78)(c)(d) is of the type of Eq. (23) of Lemma 3(i). If we assume that the four functions F , G , H , and K are solvable (cf. Definition 2), then Lemma 3(i) implies that each of these functions is quasi-permutable. This suggests an application of Theorem 13, under relevant meaningfulness and unit-proportionality conditions. We get the following result.

Theorem 18. *Let $\mathcal{F} = \{F_{\mu,\nu}\}$ and $\mathcal{K} = \{K_{\mu,\nu}\}$ be two ST -meaningful unit-proportional collections of codes, and let $\mathcal{G} = \{G_\nu\}$ and $\mathcal{H} = \{H_\nu\}$ be two SU -meaningful unit-proportional collections of codes with each code in any of these four collections mapping $\mathbb{R}_{++} \times \mathbb{R}_{++}$ into $\mathbb{R}_{++}$. Suppose that each of the four collections $\mathcal{F}$, $\mathcal{K}$, $\mathcal{G}$, and $\mathcal{H}$ contains a code that is permutable with respect to its corresponding initial code F , K , G , and H , and moreover that one particular quadruple of codes $(F_{\alpha,\beta}, G_\beta, H_\beta, K_{\alpha,\beta})$, with $F_{\alpha,\beta} \in \mathcal{F}$, $G_\beta \in \mathcal{G}$, $H_\beta \in \mathcal{H}$ and $K_{\alpha,\beta} \in \mathcal{K}$ satisfies the following conditions:*

1. *the quadruple $(F_{\alpha,\beta}, G_\beta, H_\beta, K_{\alpha,\beta})$ satisfies the 4-FT equation:*

$$F_{\alpha,\beta}(G_\beta(x, y), z) = H_\beta(x, K_{\alpha,\beta}(y, z)); \quad (79)$$

2. *each of the codes in the quadruple $(F_{\alpha,\beta}, G_\beta, H_\beta, K_{\alpha,\beta})$ is*
 - (a) *continuous,*
 - (b) *invertible in both variables,*
 - (c) *solvable;*
3. *the codes $F_{\alpha,\beta}$ and $K_{\alpha,\beta}$ are not symmetric functions;*
4. *the codes G_β and H_β are symmetric functions.*

Then the following formulas hold for the codes and for the 4-FT equation. For the quadruple of initial codes (F, G, H, K) and any quadruple of codes $(F_{\mu,\nu}, G_\nu, H_\nu, K_{\mu,\nu})$, with $F_{\mu,\nu} \in \mathcal{F}$, $G_\nu \in \mathcal{G}$, $H_\nu \in \mathcal{H}$, and $K_{\alpha,\beta} \in \mathcal{K}$, we have, for some positive constants ϕ and γ :

$$\begin{aligned} F(x, y) &= \phi xy^\gamma, & F_{\mu,\nu} &= \phi x \left(\frac{y}{\nu}\right)^\gamma \\ G(x, y) &= xy, & G_\nu &= x \left(\frac{y}{\nu}\right) \end{aligned}$$

$$\begin{aligned} H(x, y) &= \phi xy, & H_\nu &= \phi x \left(\frac{y}{\nu} \right) \\ K(x, y) &= xy^\gamma, & K_{\mu, \nu} &= x \left(\frac{y}{\nu} \right)^\gamma. \end{aligned}$$

The resulting forms of the 4-FT equations are:

$$\begin{aligned} F(G(x, y), z) &= \phi xyz^\gamma = H(x, K(y, z)) \\ F_{\mu, \nu}(G_\beta(x, y), z) &= \phi x \left(\frac{y}{\nu} \right) \left(\frac{z}{\nu} \right)^\gamma = H_\nu(x, K_{\mu, \nu}(y, z)). \end{aligned}$$

Proof. Without loss of generality, we assume that each of the codes of quadruple $(F_{\alpha, \beta}, G_\beta, H_\beta, K_{\alpha, \beta})$ satisfying the four conditions is an initial code of its collection; that is, we assume that $(F_{\alpha, \beta}, G_\beta, H_\beta, K_{\alpha, \beta}) = (F, G, H, K)$. By Condition 1, we thus have

$$F(G(x, y), z) = H(x, K(y, z)). \quad (80)$$

By Condition 4, the codes G and H are symmetric function. So, we can apply the symmetric case of Lemma 17 to the quadruple (F, G, H, K) and Eq. (80), which yields the four equations (77)(a)–(b) and (78)(c)–(d). The next step is to apply Cases (i) and (ii) of Lemma 3 to these four equations; specifically: Case (i) to (77)(a)–(b) and (78)(c), and Case (ii) to (78)(d). It is straightforward to check that all the functions satisfy the required conditions of Lemma 3. The four collections $\mathcal{F}$, $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{H}$ are unit-proportional, and each of them contains a code permutable with respect to the corresponding initial code. Moreover $\mathcal{F}$ and $\mathcal{K}$ are ST-meaningful, and $\mathcal{G}$ and $\mathcal{H}$ are SU-meaningful. Accordingly Theorem 13 is applicable in each case, and gives us the representations of the codes in any quadruple of codes $(F_{\mu, \nu}, G_\nu, H_\nu, K_{\mu, \nu})$.

The non symmetrical Case A of this theorem applies to the codes in Eqs. (77)(a) and (78)(d). For some positive real numbers ϕ_a , γ_a , ϕ_d , and γ_d (with the indices a and d referring to the two equation numbers), we get

$$F(x, y) = \phi_a x y^{\gamma_a}, \quad (81)$$

$$F_{\mu, \nu}(x, y) = \phi_a x \left(\frac{y}{\nu} \right)^{\gamma_a}, \quad \text{for all } F_{\mu, \nu} \in \mathcal{F}, \quad (82)$$

$$K(x, y) = \phi_d x y^{\gamma_d}, \quad (83)$$

$$K_{\mu, \nu}(x, y) = \phi_d x \left(\frac{y}{\nu} \right)^{\gamma_d}, \quad \text{for all } K_{\mu, \nu} \in \mathcal{K}. \quad (84)$$

The codes in the two remaining cases, Eqs. (77)(b) and (78)(a) are symmetric codes belonging to SU-meaningful collections. So Case B of Theorem 13 applies. For some constants $\phi_b > 0$ and $\phi_c > 0$, we obtain:

$$G(x, y) = \phi_b x y, \quad (85)$$

$$G_\nu(x, y) = \phi_b x \left(\frac{y}{\nu} \right), \quad \text{for all } G_\nu \in \mathcal{G}, \quad (86)$$

$$H(x, y) = \phi_c x y, \quad (87)$$

$$H_\nu(x, y) = \phi_c x \left(\frac{y}{\nu} \right), \quad \text{for all } H_\nu \in \mathcal{H}. \quad (88)$$

Linking these results for the codes F , G , H , and K to the representing equations in (77)(a)–(b) and (78)(c)–(d), yields:

$$F(x, y) = \phi_a xy^{\gamma_a} = \phi_a \exp(\ln x + \gamma_a \ln y) = \ell(f(x) + g(y)) \quad (89)$$

$$G(x, y) = \phi_b xy = \phi_b \exp(\ln x + \ln y) = f^{-1}(k(x) + k(y)) \quad (90)$$

$$H(x, y) = \phi_c xy = \phi_c \exp(\ln x + \ln y) = \ell(k(x) + k(y)) \quad (91)$$

$$K(x, y) = \phi_d xy^{\gamma_d} = \phi_d \exp(\ln x + \gamma_d \ln y) = k^{-1}(k(x) + g(y)). \quad (92)$$

Now, the fact that some of the functions occur in several places in these equations affects the values of the parameters. For example: the same term $g(y)$ occurs in (89) and (92), so we must have $\gamma_a = \gamma_d = \gamma$. Similar observations about the other parameters lead to rewriting as follows the functions for the codes in Eqs. (81)–(88):

$$F(x, y) = \phi xy^\gamma, \quad F_{\mu, \nu} = \phi x \left(\frac{y}{\nu} \right)^\gamma \quad (93)$$

$$G(x, y) = xy, \quad G_\nu = x \left(\frac{y}{\nu} \right) \quad (94)$$

$$H(x, y) = \phi xy, \quad H_\nu = \phi x \left(\frac{y}{\nu} \right) \quad (95)$$

$$K(x, y) = xy^\gamma, \quad K_{\mu, \nu} = x \left(\frac{y}{\nu} \right)^\gamma. \quad (96)$$

Using these formulas in the writing of the four-functions-transformation equation leads to

$$F(G(x, y), z) = \phi G(x, y) z^\gamma = \phi xyz^\gamma \quad (97)$$

$$H(x, K(y, z)) = \phi x K(y, z) = \phi xyz^\gamma \quad (98)$$

$$F_{\mu, \nu}(G_\nu(x, y), z) = \phi G_\nu(x, y) \left(\frac{z}{\nu} \right)^\gamma = \phi x \left(\frac{y}{\nu} \right) \left(\frac{z}{\nu} \right)^\gamma \quad (99)$$

$$H_\nu(x, K_{\mu, \nu}(y, z)) = \phi x \left(\frac{K_{\mu, \nu}(y, z)}{\nu} \right) = \phi x \left(\frac{y}{\nu} \right) \left(\frac{z}{\nu} \right)^\gamma. \quad (100)$$

Considering Eqs. (93)–(100), the proof of the theorem is complete. $\square$

8. The six-functions-transformation equation

A noticeable feature of the 6-FT equation

$$F(G(x, y), H(u, v)) = K(M(x, u), N(y, v)) \quad (101)$$

is that the positions of the two variables y and u permute between the two sides of the equation: y is the second variable of the function G , and becomes the

first variable of the function N , exchanging positions with the variable u . We refer to this feature below as the (y, u) -permutation. It suggests that the scales of y and u must be the same, which will impact the definition of meaningfulness for the 6-FT equation.

As we did before, we begin by stating a functional equation result allowing the representations of the components of the 6-FT equation by useful formulas. We use the following result from Aczél [1, Corollary 4, page 315]:

Lemma 19. *The general solution of Eq. (101), in which each of the functions of the sextuple (F, G, H, K, M, N) is a continuous, real valued mapping of the $I \times I$, with I a real interval, invertible in each variable. Then there exist nine functions $f, g, k, \ell, m, p, q, r, s$ arbitrary continuous, invertible, strictly monotonic functions satisfying the equations:*

$$\begin{aligned} (a) \quad & F(x, y) = k(f(x) + g(y)), & (b) \quad & G(x, y) = f^{-1}(p(x) + q(y)), \\ (c) \quad & H(x, y) = g^{-1}(r(x) + s(y)), & (d) \quad & K(x, y) = k(\ell(x) + m(y)), \\ (e) \quad & M(x, y) = \ell^{-1}(p(x) + r(y)), & (f) \quad & N(x, y) = m^{-1}(q(x) + s(y)). \end{aligned}$$

Theorem 20. *Let $\mathcal{F} = \{F_\mu\}$, $\mathcal{G} = \{G_\mu\}$, $\mathcal{H} = \{H_\mu\}$, $\mathcal{K} = \{K_\mu\}$, $\mathcal{M} = \{M_\mu\}$ and $\mathcal{N} = \{N_\mu\}$ be six SU-meaningful collections of codes, and suppose that each of these six collections contains a code that is*

1. *strictly increasing in both variables;*
2. *homogeneous;*
3. *associative.*

Suppose moreover that there is a sextuple of codes $(F_\mu, G_\mu, H_\mu, K_\mu, M_\mu, N_\mu)$, with $F_\mu \in \mathcal{F}$, $G_\mu \in \mathcal{G}$, $H_\mu \in \mathcal{H}$, $K_\mu \in \mathcal{K}$, $M_\mu \in \mathcal{M}$ and $N_\mu \in \mathcal{N}$, that satisfies the 6-FT equation:

$$F_\mu(G_\mu(x, y), H_\mu(u, v)) = K_\mu(M_\mu(x, u), N_\mu(y, v)). \quad (102)$$

Then there exists a number $\theta > 0$ such that for the sextuple (F, G, H, K, M, N) of initial codes, we have

$$(x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} = F(G(x, y), H(u, v)) = K(M(x, u), N(y, v)) \quad (103)$$

$$= F_\alpha(G_\alpha(x, y), H_\alpha(u, v)) = K_\alpha(M_\alpha(x, u), N_\alpha(y, v)) \quad (104)$$

for any sextuple of codes $(F_\alpha, G_\alpha, H_\alpha, K_\alpha, M_\alpha, N_\alpha)$ with $\alpha > 0$.

FIGURE 1. A geometrical representation with $\theta = 2$ of the 6-FT equation for the sextuple of initial codes in the context of Theorem 20. This figure is known as an *orthodiagonal quadrilateral*. As its diagonals are perpendicular, each side is the hypotenuse of a right triangle. We have: $(x^2 + y^2) + (u^2 + v^2) = G(x, y)^2 + H(u, v)^2 = (x^2 + u^2) + (y^2 + v^2) = M(x, u)^2 + N(y, v)^2$

Proof. Without loss of generality, we may assume that the codes $F_\mu, G_\mu, H_\mu, K_\mu, M_\mu, N_\mu$ are the initial codes F, G, H, K, M , and N . Note that an associative code must be symmetric (cf. [8]). By the combined results of Theorem 11 and Lemma 19, there exist six positive numbers $\theta_a, \dots, \theta_f$, and nine continuous, strictly increasing functions $f, g, k, \ell, m, p, q, r, s$, satisfying the equations below:

$$\begin{aligned}
 (a) \quad & F(x, y) = (x^{\theta_a} + y^{\theta_a})^{\frac{1}{\theta_a}} = k(f(x) + g(y)), \\
 (b) \quad & G(x, y) = (x^{\theta_b} + y^{\theta_b})^{\frac{1}{\theta_b}} = f^{-1}(p(x) + q(y)), \\
 (c) \quad & H(x, y) = (x^{\theta_c} + y^{\theta_c})^{\frac{1}{\theta_c}} = g^{-1}(r(x) + s(y)), \\
 (d) \quad & K(x, y) = (x^{\theta_d} + y^{\theta_d})^{\frac{1}{\theta_d}} = k(\ell(x) + m(y)), \\
 (e) \quad & M(x, y) = (x^{\theta_e} + y^{\theta_e})^{\frac{1}{\theta_e}} = \ell^{-1}(p(x) + r(y)), \\
 (f) \quad & N(x, y) = (x^{\theta_f} + y^{\theta_f})^{\frac{1}{\theta_f}} = m^{-1}(q(x) + s(y)).
 \end{aligned}$$

In fact, the six positive numbers $\theta_a, \dots, \theta_f$ are equal. Indeed, observe that:

- (i) the function ℓ occurs in (d) and (e), so $\theta_e = \theta_d = \theta$;
- (ii) the function k occurs in (a) and (d); so (i) implies: $\theta_a = \theta_d = \theta$;
- (iii) the function p occurs in (b) and (e); so (i) implies: $\theta_b = \theta_e = \theta$;
- (iv) the function g occurs in (a) and (c); so (ii) implies: $\theta_c = \theta_a = \theta$;
- (v) the function s occurs in (c) and (f); so (iv) implies: $\theta_c = \theta_f = \theta$.

We thus have: $\theta_a = \dots = \theta_f = \theta$.

Rewriting the six functions F , G , H , K , M and N in terms of θ , we get successively:

$$\begin{aligned}
 F(G(x, y), H(u, v)) &= (G(x, y)^\theta + H(u, v)^\theta)^{\frac{1}{\theta}} \\
 &= ((x^\theta + y^\theta)^{\frac{1}{\theta}\theta} + H(u, v)^\theta)^{\frac{1}{\theta}} \\
 &= (x^\theta + y^\theta + (u^\theta + v^\theta)^{\frac{1}{\theta}\theta})^{\frac{1}{\theta}} \\
 &= (x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} \\
 &= ((x^\theta + u^\theta)^{\frac{1}{\theta}\theta} + (y^\theta + v^\theta)^{\frac{1}{\theta}\theta})^{\frac{1}{\theta}} \\
 &= (M(x, u)^\theta + N(y, v)^\theta)^{\frac{1}{\theta}} \\
 &= K(M(x, u), N(y, v)).
 \end{aligned}$$

This yields:

$$F(G(x, y), H(u, v)) = (x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} = K(M(x, u), N(y, v)). \quad (105)$$

Accordingly, for any $\alpha > 0$, by the SU-meaningfulness of F , G and H , we have

$$\begin{aligned}
 F_\alpha(G_\alpha(x, y), H_\alpha(u, v)) &= \alpha F\left(\frac{G_\alpha(x, y)}{\alpha}, \frac{H_\alpha(u, v)}{\alpha}\right) = \alpha F\left(G\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), H\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right)\right) \\
 &= \alpha \left(\left(\frac{x}{\alpha}\right)^\theta + \left(\frac{y}{\alpha}\right)^\theta + \left(\frac{u}{\alpha}\right)^\theta + \left(\frac{v}{\alpha}\right)^\theta \right)^{\frac{1}{\theta}} \\
 &= (x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} \\
 &= K_\alpha(M_\alpha(x, u), N_\alpha(y, v)) \quad (\text{by the Fifth Propagation Lemma}).
 \end{aligned}$$

We thus obtain

$$F_\alpha(G_\alpha(x, y), H_\alpha(u, v)) = (x^\theta + y^\theta + u^\theta + v^\theta)^{\frac{1}{\theta}} = K_\alpha(M_\alpha(x, u), N_\alpha(y, v)). \quad (106)$$

Considering (105) and (106), the proof of the theorem is complete. $\square$

Remark 21. The consequences of the meaningfulness condition illustrated by our five examples clearly depend upon the particular side conditions chosen. Other side conditions could lead to an essentially different representation. In the case of the 6-FT equation, for instance, Lemma 19 leads to the representation

$$F(x, y) = k(f(x) + g(y)) \quad (107)$$

for the function F , and to similar equations for the five other functions. Such equations could lead to a representation of the 6-FT equation quite different from that of Eqs. (103)–(104) in Theorem 20. Indeed, Lemma 3 implies that, under some background conditions, the function F represented by Eq. (107)

must be quasi-permutable (cf. Definition 2). Following the path of our treatment of the 4-FT equation, we could then use the Quasi-Permutability Theorem 13. Assuming meaningfulness, this would lead for the function F to the representation

$$F(y, r) = \phi y r^\gamma \quad (108)$$

for some positive constants ϕ and γ , and to similar representations for the five other functions of the 6-FT equation. Eventually, we would end up representing the 6-FT equation by a monomial, instead of Eqs. (103)–(104). In general, the choice of particular side conditions will depend on the application context of meaningful representations.

9. Toward a Catalogue of Possible Meaningful Laws

The examples described here and in several other publications (see e.g. [2–4]), suggest that, in a long term, and assuming that many other examples of meaningful laws could be obtained, it might be possible to create a more or less comprehensive catalogue of possible laws. Such a catalogue might be useful to a scientist searching for a mathematical formula describing a phenomenon that is investigated.

However, this would require one additional, critical step for each of the laws. In our introductory comments, we wrote that the meaningfulness condition “*may be a powerful tool in a search for the exact form of a scientific or geometric law because it entails the analysis of the relevant phenomenon in terms of some critical features which are either empirically intuitive (such as continuity, monotonicity, or symmetry) or can be independently gauged or tested.*”

In this section, we make a first pass at what this additional step might be in the example discussed in Sect. 4, which dealt with the phenomenon of the passing of radiant energy through an absorbing medium and its relation with the transformation equation

$$F(F(x, y), z) = F(x, K(y, z)).$$

In this case, we have to connect this functional equation, and in particular the empirical role of the function K , to this phenomenon, and to the assumptions of Theorem 12. How can we empirically justify, a priori, the assumptions that some code is left-homogeneous, and the function K_ν is homogeneous, associative and solvable?

The phenomenon under study is the passing of a wave form through an absorbing medium: This is pictured by Fig. 2. Figure 3 pictures the left-homogeneity property.

FIGURE 2. A picture of the phenomenon: x is the intensity of the incoming wave; y is the power of the absorbing medium; $F(x,y)$ is the intensity of the outgoing wave

FIGURE 3. LEFT HOMOGENEITY. The intensity of the incoming wave is multiplied by a . The result is that the intensity of the outgoing wave is also multiplied by a . So we get $F(ax,y) = aF(x,y)$

The next idea is to iterate the process. What if the outgoing wave is fed through a similar absorbing medium, with a possibly higher absorbing power? What would be the effect of that? Analyzing this situation might help our understanding of the process and lead us to a correct formalization. The situation is pictured by Fig. 4.

FIGURE 4. ITERATION OF THE PROCESS. With possibly different absorbing powers y and z

Looking at Fig. 4 suggests that the two absorbing media of powers y and z could be combined into one absorbing medium, with the combined power $K(y,z)$. Our problem would then be to find an equation for the function K formalizing the combination of the two absorbing powers (Fig. 5).

FIGURE 5. THE CODE F IS K -TRANSFORMABLE. With an absorbing medium of combined powers $K(y, z)$, leading to the functional equation $F(F(x, y), z) = F(x, K(y, z))$

We now have to assess possible conditions for the function K combining the powers of the two absorbing media. Condition (d) of Theorem 12 states that the code K_ν must be strictly increasing in both variables, which makes intuitive sense. The solvability assumption of Condition (c) is also intuitively plausible.

The homogeneity condition in Condition (c) is also credible, as suggested by Fig. 6: multiplying each component absorbing power by the same value a amounts to multiplying the combining function K by a .

FIGURE 6. THE FUNCTION K IS HOMOGENEOUS. If we multiply the absorbing powers y and z of the two media by the same constant a , the absorbing power of the combined media should be multiplied by the same constant. So we should have $K(ax, ay) = aK(x, y)$

Even the associativity assumption can be given a credible representation: See Fig. 7.

FIGURE 7. A REPRESENTATION OF THE ASSOCIATIVITY CONDITION

In any event, one of the consequences of associativity and the other conditions of Theorem 11 is the symmetry condition, which is very plausible, as suggested by Fig. 8.

FIGURE 8. THE FUNCTION K IS SYMMETRICAL. It makes sense to assume that the order in which the incoming wave goes through the two components of the combined absorbing media is inconsequential

To sum up, Theorem 12 says that, in the context of meaningfulness, these conditions imply that, for some positive constants θ and κ , we have:

$$F(x,y) = x e^{\frac{y^\theta}{\kappa}},$$
$$K(x,y) = (x^\theta + y^\theta)^{\frac{1}{\theta}}.$$

Appendix

What follows is the statement and proof of Falmagne and Doble [3, Theorem 7.2.1, page 88].

Theorem 22. *Let $\mathcal{F} = \{F_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{R}_{++}\}$ be a ST-meaningful collection, with $F_{\alpha,\beta} : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{\text{onto}} \mathbb{R}_{++}$ for all $\alpha, \beta \in \mathbb{R}_{++}$. Moreover, suppose that each code is permutable with respect to the initial code $F = F_{1,1}$ and that one of the codes is strictly increasing in both variables. Then there are constants $\phi, \gamma > 0$ such that Case A holds:*

$$\text{Case A.} \quad F(y, r) = \phi y r^\gamma, \quad (109)$$

$$F_{\alpha,\beta}(y, r) = \phi y \left(\frac{r}{\beta}\right)^\gamma \quad \text{for all } F_{\alpha,\beta} \in \mathcal{F}. \quad (110)$$

If $F_{\alpha,\beta} = F_{\alpha,\alpha} = F_\alpha$ for each code in $\mathcal{F}$, so that $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_{++}\}$, with all of the other hypotheses the same as above, then for some positive constants ϕ, γ, θ and λ and some constant η , either Case B or Case C holds:

$$\text{Case B.} \quad F(y, r) = \phi y r^\gamma, \quad (111)$$

$$F_\alpha(y, r) = \phi y \left(\frac{r}{\alpha}\right)^\gamma \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (112)$$

If some code in $\mathcal{F}$ is symmetric, then $\gamma = 1$, and if in addition some code is homogeneous, then

$$F_\alpha(y, r) = F(y, r) = \theta y r \quad \text{for all } F_\alpha \in \mathcal{F}, \quad (113)$$

$$\text{Case C.} \quad F(y, r) = (y^\theta + \lambda r^\theta + \eta)^{\frac{1}{\theta}}, \quad (114)$$

$$F_\alpha(y, r) = (y^\theta + \lambda r^\theta + \alpha^\theta \eta)^{\frac{1}{\theta}} \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (115)$$

If some code in $\mathcal{F}$ is symmetric, then $\lambda = 1$, and if in addition some code is homogeneous, then

$$F_\alpha(y, r) = F(y, r) = (y^\theta + r^\theta)^{\frac{1}{\theta}} \quad \text{for all } F_\alpha \in \mathcal{F}. \quad (116)$$

Proof. CASE A. Since each code in the family is permutable with respect to the initial code F , the code F is permutable. (This is true by Lemma 4.3.2, page 49 in [3]). Using Lemma 3(ii) and the fact that $\mathcal{F}$ is a ST-meaningful collection, we get for all $F_{\alpha,\beta}$ in $\mathcal{F}$:

$$F_{\alpha,\beta}(y, r) = \alpha F\left(\frac{y}{\alpha}, \frac{r}{\beta}\right) = \alpha f^{-1}\left(f\left(\frac{y}{\alpha}\right) + g\left(\frac{r}{\beta}\right)\right), \quad (117)$$

for some continuous, strictly increasing functions f and g , with in particular

$$F(y, r) = f^{-1}(f(y) + g(r)). \quad (118)$$

Using Lemma 3(ii) (twice) and meaningfulness, we get successively

$$\begin{aligned} F_{\alpha,\beta}(F(y, r), z) \\ = F_{\alpha,\beta}(f^{-1}(f(y) + g(r)), z) \end{aligned} \quad (119)$$

$$= \alpha F\left(\frac{1}{\alpha}f^{-1}(f(y) + g(r)), \frac{z}{\beta}\right) \quad (120)$$

$$= \alpha f^{-1}\left(f\left(\frac{1}{\alpha}f^{-1}(f(y) + g(r))\right) + g\left(\frac{z}{\beta}\right)\right) \quad (121)$$

$$= \alpha f^{-1}\left(f\left(\frac{1}{\alpha}f^{-1}(f(y) + g(z))\right) + g\left(\frac{r}{\beta}\right)\right) \quad (122)$$

by quasi-permutability.

Equating the last two right-hand sides, canceling the α 's, and applying the function f on both sides, we get

$$\begin{aligned} f\left(\frac{1}{\alpha}f^{-1}(f(y) + g(r))\right) + g\left(\frac{z}{\beta}\right) \\ = f\left(\frac{1}{\alpha}f^{-1}(f(y) + g(z))\right) + g\left(\frac{r}{\beta}\right). \end{aligned} \quad (123)$$

Setting $s = f(y)$, $t = g(r)$, fixing $z = 1$, and temporarily assuming that $\frac{1}{\alpha} = \frac{1}{\beta} = \nu$, Eq. (123) becomes

$$f(\nu f^{-1}(s + t)) + g(\nu) = f(\nu f^{-1}(s + g(1))) + g(\nu g^{-1}(t)). \quad (124)$$

Defining the functions

$$h_\nu = f \circ \nu f^{-1}, \quad m_\nu : s \mapsto f(\nu f^{-1}(s + g(1))), \quad k_\nu : t \mapsto g(\nu g^{-1}(t)) - g(\nu),$$

Equation (124) becomes

$$h_\nu(s + t) = m_\nu(s) + k_\nu(t),$$

a Pexider equation. (See, e.g., Section 3.1 of [1]). In view of the conditions on the functions, the solution is

$$h_\nu(s) = p(\nu)s + q(\nu) + w(\nu) \quad (125)$$

$$m_\nu(s) = p(\nu)s + q(\nu)$$

$$k_\nu(t) = p(\nu)t + w(\nu), \quad (126)$$

for some constants $p(\nu)$, $q(\nu)$ and $w(\nu)$ possibly varying with ν . Rewriting (125) and (126) in terms of the functions f and g , we obtain, with $\zeta(\nu) = q(\nu) + w(\nu)$,

$$\begin{aligned} h_\nu(s) &= (f \circ \nu f^{-1})(s) = p(\nu)s + \zeta(\nu), \\ k_\nu(t) &= g(\nu g^{-1}(t)) - g(\nu) = p(\nu)t + w(\nu) \end{aligned} \quad (127)$$

yielding, with $\gamma(\nu) = w(\nu) + g(\nu)$,

$$f(\nu y) = p(\nu)f(y) + \zeta(\nu), \quad (128)$$

$$g(\nu r) = p(\nu)g(r) + \gamma(\nu). \quad (129)$$

These are Vincze equations (cf. [9]). In principle, for each of Eqs. (128) and (129), we have two solutions for the functions f and g depending on whether or not $p(\nu)$ is a constant function. But only the case below is consistent with the hypotheses.

Suppose that p is a constant function. We have then, for some positive constants b and d and some constants a and c ,

$$f(y) = b \ln y + a, \quad \text{and so} \quad f^{-1}(z) = e^{\frac{z-a}{b}} \quad (130)$$

$$g(r) = d \ln r + c. \quad (131)$$

Rewriting F in terms of the solutions (130) and (131) for the functions f and g yields

$$\begin{aligned} F(y, r) &= f^{-1}(f(y) + g(r)) = e^{\frac{f(y)+g(r)-a}{b}} \\ &= e^{\frac{b \ln y + a + d \ln r + c - a}{b}} = ye^{\frac{d \ln r + c}{b}} = e^{\frac{c}{b}} yr^{\frac{d}{b}}, \end{aligned}$$

and with $\phi = e^{\frac{c}{b}}$ and $\gamma = \frac{d}{b}$,

$$F(y, r) = \phi yr^\gamma. \quad (132)$$

With $F_{\alpha, \beta}(y, r) = \alpha F\left(\frac{y}{\alpha}, \frac{r}{\beta}\right)$, we get

$$F_{\alpha, \beta}(y, r) = \phi y \left(\frac{r}{\beta}\right)^\gamma. \quad (133)$$

It is easily verified that (132) and (133) imply permutability with respect to F :

$$\begin{aligned} F_{\alpha, \beta}(F(y, r), t) &= \phi F(y, r) \left(\frac{t}{\beta}\right)^\gamma = \phi^2 yr^\gamma \left(\frac{t}{\beta}\right)^\gamma = \phi^2 y \left(\frac{1}{\beta}\right)^\gamma r^\gamma t^\gamma \\ &= F_{\alpha, \beta}(F(y, t), r). \end{aligned}$$

Observation. If p takes at least two distinct values, then the form obtained for the functions $F_{\alpha, \beta}$ results in a collection $\mathcal{F}$ that is not permutable with respect to F . The argument goes as follows. From (128) and (129) we get, for some $\theta > 0$:

$$f(y) = by^\theta + a \quad \text{for some constants } b > 0 \text{ and } a, \quad (134)$$

$$g(r) = dr^\theta + c \quad \text{for some constants } d > 0 \text{ and } c. \quad (135)$$

From (134), we obtain

$$f^{-1}(t) = \left(\frac{t-a}{b} \right)^{\frac{1}{\theta}}. \quad (136)$$

Computing $F(y, r)$, we obtain from (134), (136) and (135), successively

$$\begin{aligned} F(y, r) &= f^{-1}(f(y) + g(r)) = \left(\frac{f(y) + g(r) - a}{b} \right)^{\frac{1}{\theta}} \\ &= \left(\frac{by^\theta + a + dr^\theta + c - a}{b} \right)^{\frac{1}{\theta}} = \left(y^\theta + \frac{d}{b}r^\theta + \frac{c}{b} \right)^{\frac{1}{\theta}} \end{aligned}$$

and with $\lambda = \frac{d}{b}$ and $\eta = \frac{c}{b}$, finally

$$F(y, r) = (y^\theta + \lambda r^\theta + \eta)^{\frac{1}{\theta}}.$$

With meaningfulness, this gives for any $\alpha > 0$ and $\beta > 0$

$$F_{\alpha, \beta}(y, r) = \alpha \left(\left(\frac{y}{\alpha} \right)^\theta + \lambda \left(\frac{r}{\beta} \right)^\theta + \eta \right)^{\frac{1}{\theta}} = \left(y^\theta + \lambda \left(\frac{\alpha r}{\beta} \right)^\theta + \alpha^\theta \eta \right)^{\frac{1}{\theta}}.$$

Since we have:

$$F_{\alpha, \beta}(F(y, r), t) = \left(y^\theta + \lambda r^\theta + \eta + \lambda \left(\frac{\alpha t}{\beta} \right)^\theta + \alpha^\theta \eta \right)^{\frac{1}{\theta}},$$

assuming that $F_{\alpha, \beta}(F(y, r), t) = F_{\alpha, \beta}(F(y, t), r)$ leads to

$$\left(y^\theta + \lambda r^\theta + \eta + \lambda \left(\frac{\alpha t}{\beta} \right)^\theta + \alpha^\theta \eta \right)^{\frac{1}{\theta}} = \left(y^\theta + \lambda t^\theta + \eta + \lambda \left(\frac{\alpha r}{\beta} \right)^\theta + \alpha^\theta \eta \right)^{\frac{1}{\theta}},$$

and after simplification we get

$$r^\theta + \left(\frac{\alpha t}{\beta} \right)^\theta = t^\theta + \left(\frac{\alpha r}{\beta} \right)^\theta. \quad (137)$$

For (137) to hold for all $\alpha, \beta > 0$, it must be that $r^\theta = t^\theta$.

This completes our proof of Case A.

NOTES ON CASES B AND C. We now assume that $F_{\alpha, \beta} = F_{\alpha, \alpha} = F_\alpha$ for each code in $\mathcal{F}$, so that $\mathcal{F} = \{F_\alpha \mid \alpha \in \mathbb{R}_{++}\}$. Applying the same derivation as in Case A, except this time assuming $\alpha = \beta$, we again obtain Eqs. (128) and (129):

$$\begin{aligned} f(\nu y) &= p(\nu)f(y) + \zeta(\nu), \\ g(\nu r) &= p(\nu)g(r) + \gamma(\nu). \end{aligned}$$

(Note that there is no assumption that α and β vary independently in obtaining Eqs. (128) and (129). For example, in going from (123) to (124), it is assumed that $\frac{1}{\alpha} = \frac{1}{\beta} = \nu$.) Again we have two cases, determined by whether or not $p(\nu)$ is constant.

Suppose that p is a constant function. (This gives Case B of the theorem.) We again obtain Eq. (132),

$$F(y, r) = \phi y r^\gamma,$$

and meaningfulness then gives, for all $\alpha > 0$,

$$F_\alpha(y, r) = \alpha F\left(\frac{y}{\alpha}, \frac{r}{\alpha}\right) = \phi y \left(\frac{r}{\alpha}\right)^\gamma. \quad (138)$$

If some code in $\mathcal{F}$ is symmetric, then by (iii) of the First Propagation Lemma, all the codes in $\mathcal{F}$ are symmetric. Then necessarily $\gamma = 1$, that is,

$$F(y, r) = \phi y r. \quad (139)$$

If in addition some code in $\mathcal{F}$ is homogeneous, then by (iv) of the First Propagation Lemma, all the codes are homogeneous, and furthermore $F_\alpha(y, r) = F(y, r)$ for all $\alpha > 0$. This gives

$$F_\alpha(y, r) = \phi y r, \quad (140)$$

which completes the proof of Case B.

Now suppose that p takes at least two distinct values. (Case C.)

Proceeding as above, we again get Eqs. (134)–(136), which result in

$$F(y, r) = (y^\theta + \lambda r^\theta + \eta)^{\frac{1}{\theta}} \quad (141)$$

for some positive constants θ and λ , and for some constant η .

From meaningfulness, this gives for all $\alpha > 0$,

$$F_\alpha(y, r) = \alpha \left(\left(\frac{y}{\alpha}\right)^\theta + \lambda \left(\frac{r}{\alpha}\right)^\theta + \eta \right)^{\frac{1}{\theta}} \quad (142)$$

$$= (y^\theta + \lambda r^\theta + \alpha^\theta \eta)^{\frac{1}{\theta}}. \quad (143)$$

The functions F and F_α in Eqs. (141) and (143) satisfy $F_\alpha(F(y, r), t) = F_\alpha(F(y, t), r)$ for all $\alpha, y, r, t > 0$. Now, if some code in $\mathcal{F}$ is symmetric, then all the codes in $\mathcal{F}$ are symmetric, and so $\lambda = 1$, that is,

$$F(y, r) = (y^\theta + r^\theta + \eta)^{\frac{1}{\theta}} \quad (144)$$

and

$$F_\alpha(y, r) = (y^\theta + r^\theta + \alpha^\theta \eta)^{\frac{1}{\theta}} \quad \text{for all } \alpha > 0. \quad (145)$$

If in addition some code in $\mathcal{F}$ is homogeneous, then all the codes are. The “Moreover” statement of the First Propagation Lemma gives that $F_\alpha(y, r) = F(y, r)$ for all $\alpha > 0$. This means that $(y^\theta + r^\theta + \alpha^\theta \eta)^{\frac{1}{\theta}} = (y^\theta + r^\theta + \eta)^{\frac{1}{\theta}}$ for all $\alpha > 0$, which implies $\eta = 0$. We thus have $F_\alpha(y, r) = F(y, r) = (y^\theta + r^\theta)^{\frac{1}{\theta}}$. This completes the proof of Case C. $\square$

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