On multiplicatively-additive iteration groups

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Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. Define on the set $G := \mathbb{R}^+ \times \mathbb{R}$ the operation $(t, a) * (s, b) = (ts, tb + a)$. $(G, *)$ is a non-commutative group with the neutral element $(1, 0)$. We consider a non-commutative translation equation $F(\eta, F(\xi, x)) = F(\eta * \xi, x)$, $\eta, \xi \in G$, $x \in I$, $F(1, 0) = \text{id}$, where I is an open interval and $F : G \times I \rightarrow I$ is a continuous mapping. This equation can be written in the form: $F((t, a), F((s, b), x)) = F((ts, tb + a), x)$, $t, s \in \mathbb{R}^+$, $x \in I$. For $t = 1$ the family $\{F(t, a)\}$ defines an additive iteration group, however for $a = 0$ it defines a multiplicative iteration group. We show that if $F(t, 0)$ for some $t \neq 1$ has exactly one fixed point x_t , $(F(t, 0) - \text{id})(x_t - \text{id}) \geq 0$ and for an $a > 0$ $F(1, a) > \text{id}$, then there exists a unique homeomorphism $\varphi : I \rightarrow \mathbb{R}$ such that $F((s, b), x) = \varphi^{-1}(s\varphi(x) + b)$ for $s \in \mathbb{R}^+$ and $b \in \mathbb{R}$.

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1. Introduction

In this paper we deal with the continuous solutions of the functional equation

$$F((t, a), F((s, b), x)) = F((ts, tb + a), x), \quad t, s > 0, \quad a, b \in \mathbb{R}, \quad x \in I, \quad (1)$$

where I is an open non-empty interval.

An inspiration which led to consider the above equation were some models in economics connected to the description of a special order in a space of sequences of preference of consumption outcomes called “impatience” (see [2]). Arsen Kochov in a private correspondence presented an idea that every sequence of consumption outcomes can be represented by an increasing homeomorphism with one fixed point and the problem of the existence of impatience order can be reduced to the determination of suitable properties of semigroups G of increasing homeomorphisms of $\mathbb{R}$ onto $\mathbb{R}$ satisfying the conditions:

- (i) $f, g \in G$ have the same fixed point if and only if $f \circ g = g \circ f$;
- (ii) if f has a fixed point p_f and g has a fixed point p_g , then $p_f > p_g$ if and only if $f \circ g > g \circ f$.

The existence of the relation of impatience is related to the “affinization” of the above semigroups, that is to the problem of describing when these semigroups are conjugate to the family of affine functions.

The aim of the present paper is to show a construction of large families of semigroups of homeomorphisms possessing at most one fixed point satisfying conditions (i) and (ii) which are conjugate to the family of all affine functions. Their construction relates to the determination of two parameter families of homeomorphisms $F(t, a) : \mathbb{R} \rightarrow \mathbb{R}$ satisfying Eq. (1).

Let $\varphi : I \rightarrow \mathbb{R}$ be an increasing surjection. Define

$$\text{Realm } \varphi := \{f : I \rightarrow I, \exists t > 0 \exists a \in \mathbb{R} \varphi \circ f = t\varphi + a\}$$

and

$$\text{Realm}^* \varphi := \{f : I \rightarrow I, \exists t \in (0, 1) \exists a \in \mathbb{R} \varphi \circ f = t\varphi + a\}.$$

The parameters t and a are determined uniquely. Thus we may write a_f instead of a and t_f instead of t and define

$$\text{Ind } f := (t_f, a_f).$$

If φ is a homeomorphism then $\text{Realm } \varphi$ with the operation of composition is a group and $\text{Realm}^* \varphi$ is a semigroup.

Define the family of functions

$$l_p^s(x) := sx + p(1 - s), \quad x \in \mathbb{R},$$

where $p \in \mathbb{R}$ and $s > 0$.

Every function l_p^s has a unique fixed point, namely $x = p$, if $s \neq 1$. Moreover,

$$l_p^s \circ l_q^t - l_q^t \circ l_p^s = (p - q)(1 - s)(1 - t).$$

Hence $l_p^s \circ l_q^t = l_q^t \circ l_p^s$ if and only if $p = q$. However for $t, s \in (0, 1)$, $l_p^s \circ l_q^t > l_q^t \circ l_p^s$ if and only if $p > q$. Thus the family $\{l_p^s, s \in (0, 1), p \in \mathbb{R}\}$ is a semigroup satisfying (i) and (ii). $\text{Realm}^* \varphi$ is conjugate to the family of affine functions $\{l_p^s, s \in (0, 1), p \in \mathbb{R}\}$. Hence we get the following.

Remark 1. If φ is an increasing homeomorphism, then every semigroup $G \subset \text{Realm}^* \varphi$ satisfies conditions (i) and (ii).

The second statement is a simple consequence of the fact that the set of affine functions $\{tx + a, t \in (0, 1), a \in \mathbb{R}\}$ satisfies (i) and (ii) and is conjugate to the family $\text{Realm}^* \varphi$.

Remark 2. When proving that $\text{Realm } \varphi$ is a semigroup we get that for every $f, g \in \text{Realm } \varphi$

$$t_{f \circ g} = t_f t_g \quad \text{and} \quad a_{f \circ g} = t_f a_g + a_f.$$

These equalities lead us to the definition of the algebraic structures $\mathcal{G} := (\mathbb{R}^+ \times \mathbb{R}, \oplus)$ and $\mathcal{G}^* := ((0, 1] \times \mathbb{R}, \oplus)$, with the operation “ $\oplus$ ” defined as follows

$$(t, a) \oplus (s, b) = (ts, tb + a).$$

Note that $\mathcal{G}$ is a non-commutative group and $\mathcal{G}^*$ is a non-commutative semigroup and

$$\text{Ind } f \circ g = \text{Ind } f \oplus \text{Ind } g \quad \text{for } f, g \in \text{Realm } \varphi.$$

Note that for every homeomorphism $\varphi : I \rightarrow \mathbb{R}$ and $\hat{u} \in \mathcal{G}$ there exists a unique homeomorphism $F(\hat{u}, \cdot) \in \text{Realm } \varphi$ such that $\text{Ind } F(\hat{u}, \cdot) = \hat{u}$. Hence the last relation implies that the function $F : \mathcal{G} \times I \rightarrow I$ satisfies the non-commutative translation equation

$$F(\hat{u}, F(\hat{v}, x)) = F(\hat{u} \oplus \hat{v}, x), \quad x \in I, \quad \hat{u}, \hat{v} \in \mathcal{G}.$$

Putting $\hat{u} = (t, a)$ and $\hat{v} = (s, b)$, the last equation can be written in the following form

$$F((t, a), F((s, b), x)) = F((ts, tb + a), x).$$

For $t = 1$ the family $\{F(t, a)\}$ defines an additive iteration group of homeomorphisms, however for $a = 0$ it defines a multiplicative iteration group of homeomorphisms. In fact, putting

$$G_t(x) := F(t, 0, x) \quad \text{and} \quad H^a(x) := F(1, a, x), \quad x \in I \quad (2)$$

we get

$$G_t \circ G_s = G_{ts}, \quad t, s > 0, \quad (3)$$

$$H^a \circ H^b = H^{a+b}, \quad a, b \in \mathbb{R}. \quad (4)$$

Hence $\{G_t, t > 0\}$ is a multiplicative iteration group and $\{H^a, a \in \mathbb{R}\}$ is an additive iteration group. Thus Eq. (1) describes simultaneously multiplicative and additive iteration groups.

Definition. A family of continuous functions $\{F(t, a) : I \rightarrow I, \quad t > 0, \quad a \in \mathbb{R}\}$, where F satisfies (1) is said to be a *multiplicatively-additive iteration group*.

The main purpose of this paper is to investigate when a multiplicatively-additive iteration group is conjugate to a group of affine functions.

2. Main results

In this section we deal with multiplicatively-additive iteration groups $\{F(t, a), t > 0, a \in \mathbb{R}\}$, possessing the property that at least one function $F(t, 0)$ has a unique fixed point and at least one function $F(1, a)$ does not have a fixed point. We answer when these groups are conjugate to the family of all affine functions. More precisely, we show the following.

Theorem 1. *Let $F : (0, \infty) \times \mathbb{R} \times I \rightarrow I$ be a non-constant continuous solution of (1) such that $F(1, 0) = \text{id}$. If for some $s \in (0, 1)$ there exists an $x_0 \in I$ such that $F((s, 0), x) > x$ for $x < x_0$ and $F((s, 0), x) < x$ for $x > x_0$, $F(1, c) > \text{id}$ for some $c > 0$, then there exists a unique increasing homeomorphism $\varphi : I \rightarrow \mathbb{R}$ such that*

$$F((t, a), x) = \varphi^{-1}(t\varphi(x) + a), \quad t > 0, a \in \mathbb{R}, x \in I. \quad (5)$$

In order to prove the theorem we will first show a few lemmas. In the assumptions of these lemmas the function F by default is a non-constant continuous solution of (1) such that $F(1, 0) = \text{id}$.

Put

$$F_t^a(x) := F(t, a, x), \quad x \in I, t > 0, a \in \mathbb{R}.$$

Then (1) can be written in the form

$$F_t^a \circ F_s^b = F_{ts}^{tb+a}, \quad t, s > 0, a, b \in \mathbb{R}.$$

Hence $F_t^a \circ F_{\frac{1}{t}}^{-\frac{a}{t}} = F_1^0 = \text{id}$ and $F_{\frac{1}{t}}^{-\frac{a}{t}} \circ F_t^a = F_1^0 = \text{id}$, so F_t^a are bijections and, consequently, they are homeomorphisms. Let $t > 0$ and $a \in \mathbb{R}$. Put $s := \sqrt{t}$ and $b := \frac{a}{1+\sqrt{t}}$. Then $F_t^a = F_{s^2}^{sb+b} = F_s^b \circ F_s^b$, hence the homeomorphisms F_t^a are increasing.

Lemma 1. *If for some $s \in (0, 1)$ the function $F(s, 0)$ has exactly one fixed point x_0 , then for every $t \neq 1$ $F(t, 0, x_0) = x_0$ and x_0 is the only fixed point of $F(t, 0)$.*

Proof. Let $F(s, 0, x_0) = x_0$ for an $x_0 \in I$. Put $x_1 := F(s^{\frac{1}{2}}, 0, x_0)$. We have $F(s^{\frac{1}{2}}, 0, x_1) = F(s^{\frac{1}{2}}, 0, F(s^{\frac{1}{2}}, 0, x_0)) = F(s, 0, x_0) = x_0$. Suppose $x_0 < x_1$. Since $F(s^{\frac{1}{2}}, 0, \cdot)$ is strictly increasing, we have $x_1 := F(s^{\frac{1}{2}}, 0, x_0) < F(s^{\frac{1}{2}}, 0, x_1) = x_0$. This is a contradiction. It is the same when we assume that $x_0 < x_1$. Hence $x_0 = x_1$, so $F(s^{\frac{1}{2}}, 0, x_0) = x_0$.

Further, inductively, we get that $F(s^{\frac{1}{2^k}}, 0, x_0) = x_0$ for $k \in \mathbb{N}$. Note that if $F(t, 0, x_0) = x_0$ for some $t > 0$ then $F(t^n, 0, x_0) = x_0$. Hence $F(s^{\frac{n}{2^k}}, 0, x_0) = F((s^{\frac{1}{2^k}})^n, 0, x_0) = x_0$ for $k, n \in \mathbb{N}$. Since the set $\{\frac{n}{2^k}, n, k \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$ the continuity of F with respect to t implies that $F(t, 0, x_0) = x_0$ for $t > 0$. Hence we infer that if $F(s, 0)$ has a unique fixed point, then all functions $F(t, 0)$, $t > 0$ have this property. $\square$

Lemma 2. *If $F(1, a)$ has no fixed points for some a , then for every $b \neq 0$ the function $F(1, b)$ has no fixed points either.*

Proof. Suppose that there exist $x_0 \in I$ and $b \neq 0$ such that $F((1, b), x_0) = x_0$. Similarly as in the previous proof, by induction we get that $F(1, \frac{b}{2^n}, x_0) = x_0$ and $F(1, \frac{ma}{2^n}, x_0) = x_0$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. By the density of the set $\{\frac{ma}{2^n}, m, n \in \mathbb{Z}\}$ in $\mathbb{R}$ and the continuity of F with respect to the second variable we get that $F(1, a, x_0) = x_0$ for all $a \in \mathbb{R}$, which is a contradiction. $\square$

Lemma 3. *If for some $s \in (0, 1)$ and some $x_0 \in I$, $F((s, 0), x) > x$ for $x < x_0$ and $F((s, 0), x) < x$ for $x > x_0$ then there exists an increasing homeomorphism $\varphi : I \rightarrow \mathbb{R}$ such that $\varphi(x_0) = 0$ and*

$$F((t, 0), x) = \varphi^{-1}(t\varphi(x)). \quad (6)$$

The homeomorphisms $\varphi|_{[x_0, \sup I)}$ and $\varphi|_{(\inf I, x_0]}$ are determined uniquely up to multiplicative constants.

Put

$$G_t^+(x) := F((t, 0), x), \quad x \in [x_0, \sup I), \quad t > 0$$

and

$$G_t^-(x) := F((t, 0), x), \quad x \in (\inf I, x_0], \quad t > 0.$$

By Lemma 1 $\{G_t^+, t > 0\}$ and $\{G_t^-, t > 0\}$ are continuous multiplicative iteration groups, respectively on $[x_0, \sup I)$ and $(\inf I, x_0]$, with one fixed point x_0 .

Note that the family of functions $\{g_t^+, t \in \mathbb{R}\}$, where $g_t^+ := G_{\exp t}^+$ for $t \in \mathbb{R}$ is an additive iteration group such that $H_{\log s}^+(x) < x$ for $x \in (x_0, \sup I)$ and $g_{\log s}^+(x_0) = x_0$. It is known (see e.g. [5, Th.7.1], [6, Th.7], [1, Ch.6.1]) that then there exists a homeomorphism $h : (-\infty, \infty) \rightarrow (x_0, \sup I)$ such that $g_t^+(x) = h(t + h^{-1}(x))$. Putting $\alpha(t) := h \circ \log t$ for $t > 0$ we have $\alpha(t\alpha^{-1}(x)) = h(\log(t \exp h^{-1}(x))) = h(\log t + h^{-1}(x)) = g_{\log t}^+(x) = G_t^+(x)$ for $t > 0$ $x \in (x_0, \sup I)$. Thus we infer that there exists a homeomorphism $\alpha : [0, \infty) \rightarrow [x_0, \sup I)$ such that

$$G_t^+(x) = \alpha(t\alpha^{-1}(x)), \quad x \in [x_0, \sup I), \quad t > 0.$$

Similarly, we show that there exists a homeomorphism $\beta : [0, \infty) \rightarrow (\inf I, x_0]$ such that

$$G_t^-(x) = \beta(t\beta^{-1}(x)), \quad x \in (\inf I, x_0], \quad t > 0.$$

Let $x_1 < x_0 < x_2$. By the assumption $G_s^+(x_2) < x_2$ and $G_s^-(x_1) > x_1$ for given $0 < s < 1$, we have $\alpha(s\alpha^{-1}(x_2)) < x_2$ and $\beta(s\beta^{-1}(x_1)) > x_1$, so $\alpha(st_2) < \alpha(t_2)$ for $t_2 = \alpha^{-1}(x_2)$ and $\beta(st_1) > \beta(t_1)$ for $t_1 = \beta^{-1}(x_1)$. Thus α is an increasing homeomorphism and β is decreasing. Moreover, $\alpha(0) = x_0 = \beta(0)$.

Define

$$\xi(t) := \begin{cases} \alpha(t), & t \geq 0 \\ \beta(-t), & t \leq 0. \end{cases}$$

Note that ξ is an increasing homeomorphism and

$$\xi^{-1}(x) := \begin{cases} \alpha^{-1}(x), & x \geq x_0 \\ -\beta^{-1}(x), & x \leq x_0. \end{cases}$$

It is easy to verify that $G_t^+(x) = \xi(t\xi^{-1}(x))$, $x \in [x_0, \sup I)$, $t > 0$ and $G_t^-(x) = \xi(t\xi^{-1}(x))$, $x \in (\inf I, x_0]$, $t > 0$. Hence putting $\varphi = \xi^{-1}$ we get (6).

Let $\psi : I \rightarrow \mathbb{R}$ be an increasing homeomorphism such that $\varphi^{-1}(t\varphi(x)) = \psi^{-1}(t\psi(x))$ for $x \in I$ and $t > 0$. Then $\gamma(tr) = t\gamma(r)$ for $t > 0$ and $r \in \mathbb{R}$, where $\gamma = \psi \circ \varphi^{-1}$. Putting $r = 1$ and $r = -1$ we get $\gamma(t) = t\gamma(1)$ and $\gamma(-t) = t\gamma(-1)$ for $t > 0$. Note that $\gamma(1) > 0$ and $\gamma(-1) < 0$, since $\varphi(x_0) = \psi(x_0) = 0$ and φ, ψ are strictly increasing. Thus

$$\gamma(t) = \begin{cases} pt, & t \geq 0 \\ qt, & t \leq 0 \end{cases}$$

for $p = \gamma(1)$ and $q = -\gamma(-1) > 0$. Hence $\psi(x) = p\varphi(x)$ for $x \geq x_0$ and $\psi(x) = q\varphi(x)$ for $x \leq x_0$. After assuming that ψ is a decreasing homeomorphism we get the same relation but with negative p and q . Thus φ is uniquely determined up to two positive multiplicative constants but both have the same sign.

As we mentioned earlier, the family $\{H^a, a \in \mathbb{R}\}$, where $H^a = F(1, a)$, is an additive continuous iteration group. Hence we have the following statement (see e.g. [5, Th.7.1], [6, Th.7], [1, Ch.6.1]).

Lemma 4. *If $F(1, b) > id$ for some $b > 0$, then there exists an increasing homeomorphism $\psi : I \rightarrow \mathbb{R}$ such that*

$$F((1, a), x) = \psi^{-1}(\psi(x) + a), \quad x \in I, \quad a \in \mathbb{R}. \quad (7)$$

ψ is determined uniquely up to an additive constant.

Now we prove Theorem 1.

Putting in (1) $b = 0$ and $t = 1$ we get

$$F((1, a), F((s, 0), x), x) = F((s, a), x), \quad s > 0.$$

Thus, by Lemmas 3 and 4

$$F((s, a), x) = \psi^{-1}[\psi(\varphi^{-1}(s\varphi(x))) + a], \quad s > 0, \quad a \in \mathbb{R}. \quad (8)$$

Now, put in (1) $a = 0$ and $s = 1$. Then

$$F((t, 0), F((1, b), x)) = F((t, tb), x), \quad t > 0, \quad b \in \mathbb{R},$$

which gives

$$F((t, a), x) = \varphi^{-1} \left[t\varphi \left(\psi^{-1} \left(\psi(x) + \frac{a}{t} \right) \right) \right], \quad t > 0, \quad a \in \mathbb{R}. \quad (9)$$

Comparing (8) with (9) and putting $\gamma := \varphi \circ \psi^{-1}$ we get

$$\gamma(\gamma^{-1}(sb) + a) = s\gamma\left(\gamma^{-1}(b) + \frac{a}{s}\right), \quad s > 0, \quad a, b \in \mathbb{R}. \quad (10)$$

We may assume that $\psi(x_0) = 0$, since ψ is uniquely determined up to an additive constant. Since $\varphi(x_0) = 0$ we get $\gamma(0) = 0$. Putting $b = 0$ in (10) we obtain

$$\gamma(a) = s\gamma\left(\frac{a}{s}\right), \quad s > 0, \quad a \in \mathbb{R}.$$

Putting successively $a = 1$ and $a = -1$ we get $\gamma(t) = t\gamma(1)$ for $t > 0$ and $\gamma(t) = -t\gamma(-1)$ for $t < 0$. Note that $\psi^{-1}(-1) < x_0 < \psi^{-1}(1)$. By Lemma 3 we may choose homeomorphisms $\varphi|_{[x_0, \sup I)}$ and $\varphi|_{(\inf I, x_0]}$ such that $\varphi(\psi^{-1}(1)) = 1$ and $\varphi(\psi^{-1}(-1)) = -1$. Hence $\gamma(t) = t$ for $t \in \mathbb{R}$, which gives that $\varphi = \psi$. Hence, by (8), we get (5).

Let a bijection $\bar{\varphi} : I \rightarrow \mathbb{R}$ satisfy (5). It is easy to see that the function $\xi := \bar{\varphi} \circ \varphi^{-1}$ satisfies

$$\xi(tb + a) = t\xi(b) + a, \quad t > 0, \quad a, b \in \mathbb{R}. \quad (11)$$

Putting $a = 0$ and $b = 1$ we get $\xi(t) = tc$, where $c = \xi(1)$. Substituting this function in (11) we get $c(tb + a) = ctb + a$, so $c = 1$ and, consequently, $\bar{\varphi} = \varphi$. $\square$

Remark 3. The assumption of continuity of F in Theorem 1 can be replaced by the condition that F is Lebesgue measurable with respect to the first and second variables and continuous with respect to the third one.

Let G_t , $t > 0$ and H^a , $a \in \mathbb{R}$ be the functions defined by (2). The weakened assumption on F says that all these functions are continuous and for every $x \in I$ the functions $t \rightarrow G_t(x)$ and $a \rightarrow H^a(x)$ are Lebesgue measurable. Hence, it follows by (3) and (4), that $\{G_t, t > 0\}$ and $\{H^a, a \in \mathbb{R}\}$ are measurable iteration groups. It is known that measurable iteration groups are continuous, that is the functions $(t, x) \rightarrow G_t(x)$ and $(a, x) \rightarrow H^a(x)$ are continuous (see [5, Th.1.1], [7]). It follows, by (1), the following equality $F((t, a), x) = H^a(G_t(x))$ holds for $t > 0$, $a \in \mathbb{R}$ and $x \in I$. Thus F is continuous. The remaining part of the proof is the same.

Corollary 1. *If F satisfies the assumptions of Theorem 1 for $I = \mathbb{R}$, then the family of functions $\{F(t, a), t \in (0, 1), a \in \mathbb{R}\}$ is a semigroup of homeomorphisms possessing at least one fixed point satisfying conditions (i) and (ii).*

By Theorem 1 there exists a homeomorphism φ such that $\{F(t, a), 0 < t < 1, a \in \mathbb{R}\} = \text{Realm } \varphi$. By Remark 1 we get our assumption.

Conjecture. For every semigroup G of homeomorphisms with at most one fixed point satisfying (i) and (ii) there exists an increasing surjection φ such that $G \subset \text{Realm}^* \varphi$.

Let F satisfy (1). Define

$$G_p^t(x) := F((t, p(1-t)), x), \quad p \in \mathbb{R}, \quad t > 0.$$

It is easy to verify that for every $p \in \mathbb{R}$ the family of functions $\{G_p^t, t > 0\}$ is a multiplicative iteration group.

Remark 4. If F is given by (5) then for every $p \in \mathbb{R}$ G_p^t for $t > 0, t \neq 1$ has a unique fixed point $x_p = \varphi^{-1}(p)$ and

$$G_p^t(x) = \varphi^{-1}(t\varphi(x) + (1-t)\varphi(x_p)).$$

Remark 5. If F satisfies (1) then we have the following decomposition

$$\{F(t, a), \quad t > 0, \quad a \in \mathbb{R}\} = \bigcup_{p \in \mathbb{R}} \{G_p^t, \quad t > 0\} \cup \{F(1, a), \quad a \in \mathbb{R} \setminus \{0\}\}.$$

In fact, if $t > 0$ and $a \in \mathbb{R}$ then $F(t, a) = G_p^t$ for $p = \frac{a}{1-t}$.

Theorem 2. *If F satisfies the assumptions of Theorem 1, then all iteration groups $\{G_p^t\}$ and $\{G_q^t\}$ are conjugate by a one parameter family of bijections γ_c . These bijections are continuous. If $q \neq 1$, then $\gamma_c(x) = F((\frac{p-c}{q}, c), x)$ for $c < p$. If $q = 1$, then $\gamma_c(x) = F((c, p), x)$ for $c > 0$.*

Proof. Suppose that there exists a bijection $\alpha : I \rightarrow I$ such that $\alpha \circ G_q^t = G_p^t \circ \alpha$. By Theorem 1 there exists a homeomorphism φ such that $G_p^t(x) = \varphi^{-1}(t\varphi(x) + p(1-t))$ and $G_q^t(x) = \varphi^{-1}(t\varphi(x) + q(1-t))$. Putting $\beta := \varphi \circ \alpha \circ \varphi^{-1}$ we get

$$\beta(ty + q(1-t)) = t\beta(y) + p(1-t), \quad y \in \mathbb{R}, \quad t > 0. \quad (12)$$

Let $q \neq 0$. Note that the function $\bar{\beta}(y) = \frac{p}{q}y$ satisfies (12). Since β satisfies (12) $\rho = \beta - \bar{\beta}$ satisfies the equation

$$\rho(ty + q(1-t)) = t\rho(y), \quad t > 0, y \in \mathbb{R}.$$

Hence putting $y = 0$ and $\rho(0) =: c$ we get $\rho(y) = c\frac{q-y}{q}$, so $\beta(y) = \frac{p}{q}y + c\frac{q-y}{q} = \frac{p-c}{q}y + c$. Thus $\alpha(x) = \varphi^{-1}(\frac{p-c}{q}\varphi(x) + c) = F((\frac{p-c}{q}, c), x)$, if $c < 0$.

It is easy to verify that for every $c < p$ $F(\frac{p-c}{q}, c) \circ G_q^t = G_p^t \circ F(\frac{p-c}{q}, c)$.

Now, assume that $q = 0$. Then $\beta_0(y) := y + p$ satisfies (12) and $\omega := \beta - \beta_0$ satisfies the equation $\omega(ty) = t\omega(y)$ for $t > 0$ and $y \in \mathbb{R}$, so $\omega(t) = t\omega(1)$ and $\beta(t) = ct$ for $c = \omega(1)$. Thus $\alpha(x) = \varphi(c\varphi^{-1}(x) + p) = F((c, p), x)$. $\square$

3. The embedding problem

Let $f : I \rightarrow I$ be a homeomorphism with one fixed point x_0 such that $f(x) > x$ for $x < x_0$ and $f(x) < x$ for $x > x_0$. It is easy to show that

there exists a function F satisfying the assumptions of Theorem 1 such that $f \in \{F(t, a), \ t > 0, \ a \in \mathbb{R}\}$. In fact, the equation

$$\varphi(f(x)) = t\varphi(x) + a \quad (13)$$

for some $0 < t < 1$ has a continuous strictly increasing solution. Its general form is $\varphi = \psi + \frac{a}{1-t}$, where ψ is a continuous and strictly increasing solution of the equation

$$\psi(f(x)) = t\psi(x). \quad (14)$$

The above solutions depend on an arbitrary function (see e.g. [3, Th. 2.2], [4, Th.3.1.1]).

If we assume a regularity condition on f which guarantees that (13) has in a suitable class of functions a unique solution, then f has a unique regular embedding in this class. As an example of the application of this method we get

Theorem 3. *Let f be a diffeomorphism of class C^2 with one fixed point x_0 , $f'(x_0) =: t > 0$, $t \neq 1$. Then for every $a \in \mathbb{R}$ there exists a unique multiplicatively-additive iteration group of diffeomorphisms such that*

$$F((t, a), x) = f(x), \quad x \in I.$$

This is a simple consequence of the fact that in this case equation (14) has a unique diffeomorphic solution (see e.g. [3, Th. 6.1], [4, Th.3.5.1]).

Now, let f and g be increasing homeomorphisms possessing at most one fixed point. We want to discuss when f and g belong to the same multiplicatively-additive continuous iteration group. That is, when can we find a continuous function F satisfying (1) and $s, t > 0$, $a, b \in \mathbb{R}$ so that

$$f(x) = F((t, a), x) \quad \text{and} \quad g(x) = F((s, b), x)?$$

This problem is equivalent to the existence of homeomorphic solutions of the system of equations

$$\varphi(f(x)) = t\varphi(x) + a, \quad \varphi(g(x)) = s\varphi(x) + b. \quad (15)$$

Assume that f and g commute and Eq. (15) has a homeomorphic solution. If f has exactly one fixed point, say x_0 , then g also has x_0 as a unique fixed point. Moreover, $a = \varphi(x_0)(1 - t)$ and $b = \varphi(x_0)(1 - s)$. Putting $\phi(x) = \varphi(x) - \varphi(x_0)$ we get a system of Schröder equations

$$\phi(f(x)) = t\phi(x), \quad \phi(g(x)) = s\phi(x).$$

It is easy to see that if f has no fixed points, then in Eq. (13) $t = 1$. Thus if both f and g have no fixed points we get a system of Abel equations

$$\varphi(f(x)) = \varphi(x) + a, \quad \varphi(g(x)) = \varphi(x) + b.$$

In these cases the problem is completely solved in [8] (see also [4, Ch. 9.4–6]). Then weak very natural assumptions, imply the uniqueness of continuous solutions up to a multiplicative or additive constant.

Assume that f and g have different fixed points, then they do not commute and the problem of continuous solutions of system (15) is still open.

If f has one fixed point and g has no fixed point, then we get the system consisting of one Schröder and one Abel equation

$$\varphi(f(x)) = t\varphi(x), \quad \varphi(g(x)) = \varphi(x) + b.$$

The existence of homeomorphic solutions in this case is also an open problem.

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