Aequat. Math. 93 (2019), 239–246

0001-9054/19/010239-8
published online November 17, 2018
https://doi.org/10.1007/s00010-018-0616-x

Aequationes Mathematicae

Simultaneous difference equations on a restricted domain

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Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. Given a set $T \subset (0, +\infty)$, a function $c\colon T \to \mathbb{R}$ and a real number p we study continuous solutions φ of the simultaneous equations

$$\varphi(tx) = \varphi(x) + c(t)x^p, \qquad t \in T.$$

Here φ is defined on an interval $I \subset (0, +\infty)$, so the equations are postulated on a restricted domain: for any fixed $t \in T$ we assume that $x \in I$ is such that $tx \in I$. In the case when T is large in a sense, we determine the form of φ on a non-trivial subinterval of I. The research is a continuation of that of "non-restricted", where $I = (0, +\infty)$, made in Jarczyk (Ann Univ Sci Budapest Sect Comp 40:353–362, 2013).

Mathematics Subject Classification. Primary 39A13, 39B72; Secondary 39B12, 39B22.

Keywords. Simultaneous equations, Difference equations, Equations on restricted domains, Form of continuous solutions.

Introduction

Let $T \subset (0, +\infty)$ be any non-empty set. Given a function $c\colon T \to \mathbb{R}$ and a number $p \in \mathbb{R}$ consider the simultaneous difference equations

$$\varphi(tx) = \varphi(x) + c(t)x^p, \qquad t \in T. \tag{1}$$

Systems of such equations appear naturally while studying weak generalized stabilities of random variables in [1]. Namely, the main problem of that paper has been reduced to determining Lebesgue (or Baire) measurable solutions $f : (0, +\infty) \to (0, +\infty)$ of the functional equation

$$\begin{aligned}&(f\left(t(x+y)\right) - f(tx))\left(f(x+y) - f(y)\right)\\ &\quad = \left(f(t(x+y)) - f(ty)\right)\left(f(x+y) - f(x)\right).\end{aligned}$$

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Solving this equation one can show that either $\log f$ satisfies the system of equations

$$\varphi(nx) = \varphi(x) + c(n), \qquad n \in \mathbb{N},$$

where $c(n) = \log \frac{f(n)}{f(1)}$, or f is a solution of the system

$$\varphi(nx) = \varphi(x) + c(n)x^p, \qquad n \in \mathbb{N},$$

with a $p \in \mathbb{R}$ and some sequence c.

In [2, Theorem 2.2] the first present author, assuming that the multiplicative group $<T>$ generated by T is dense in $(0, +\infty)$, found, among others, all the continuous solutions $\varphi\colon (0, +\infty) \to \mathbb{R}$ of Eq. (1). Of course, at this point also the form of possible functions c were determined.

Here we study a more general situation of Eq. (1) on a restricted domain where we assume that their solution φ is defined on a given interval $I \subset (0, +\infty)$ only. Clearly, this requires that given a $t \in T$ we have $tx \in I$ whenever $x \in I$.

In what follows I is a fixed non-empty interval contained in $(0, +\infty)$. Observe that if $t \in T$ and we can find an $x \in I$ such that $tx \in I$, then $x \in I \cap t^{-1}I$ and, consequently, $I \cap t^{-1}I \neq \emptyset$. Otherwise, if $I \cap t^{-1}I = \emptyset$, then there is no point to considering the individual equation

$$\varphi(tx) = \varphi(x) + c(t)x^p. \tag{2}$$

Consequently, without loss of generality, we may (and should!) assume the condition

$$t \in T \text{ implies } I_t \neq \emptyset, \qquad t \in (0, +\infty), \tag{3}$$

where $I_t := I \cap t^{-1}I$, ignoring those $t \in T$ for which $I_t = \emptyset$. In other words, for any $\varphi\colon I \to \mathbb{R}$ the phrase "φ is a solution of Eq. (1)" means "φ satisfies equalities (2) for all $t \in T$ and $x \in I_t$".

1. Continuous solutions of (1)

Assume condition (3). Then, since

$$I_{1/t} = I \cap tI = t\left(t^{-1}I \cap I\right) = tI_t,$$

also $I_{1/t} \neq \emptyset$ for all $t \in T$, that is $I_t \neq \emptyset$ for all $t \in T^{-1}$, where $T^{-1} := \left\{t^{-1}\colon\ t \in T\right\}$. Putting

$$T^* := T \cup T^{-1} \cup \{1\}$$

we have

$$I_t \neq \emptyset, \qquad t \in T^*.$$

If $\inf I = \sup I$ then I is a singleton which implies $T = \{1\}$. In that case the problem of solutions $\varphi\colon I \to \mathbb{R}$ of (1) has a trivial answer: φ is arbitrary (and c is the zero function). So further we may assume that $\inf I < \sup I$.

Take any numbers $\gamma_- \in (0,1)$ and $\gamma_+ \in (1,+\infty)$. For each $n \in \mathbb{N}$ we define

$$T_{\gamma_-,\gamma_+,n} := \{t \in (0,+\infty) : \text{ there exist } t_1,\dots,t_n \in T^* \text{ such that } t = t_1 \cdot \ldots \cdot t_n \text{ and } \gamma_- \le t_1 \cdot \ldots \cdot t_k \le \gamma_+ \text{ for all } k = 1,\dots,n\}.$$

Moreover, put

$$T_{\gamma_-,\gamma_+} := \bigcup_{n=1}^{\infty} T_{\gamma_-,\gamma_+,n}.$$

Proposition 1. *Let $c\colon T \to \mathbb{R}$ and fix $\gamma_- \in (0,1)$ and $\gamma_+ \in (1,+\infty)$ such that $\inf I/\gamma_- < \sup I/\gamma_+$. Then, for every $n \in \mathbb{N}$,*

(i) *$T_{\gamma_-,\gamma_+,n} \subset T_{\gamma_-,\gamma_+,n+1}$;*

(ii) *there exists a unique function $c_n : T_{\gamma_-,\gamma_+,n} \to \mathbb{R}$ such that for every solution $\varphi\colon I \to \mathbb{R}$ of Eq. (1) the equality*

$$\varphi(tx) = \varphi(x) + c_n(t)x^p$$

holds for all $t \in T_{\gamma_-,\gamma_+,n}$ and $x \in (\inf I/\gamma-, \sup I/\gamma+)$.

Proof. To get (i) it is enough to observe that

$$T_{\gamma_-,\gamma_+,n} = T_{\gamma_-,\gamma_+,n} \cdot 1 \subset T_{\gamma_-,\gamma_+,n+1}$$

for all $n \in \mathbb{N}$.

Now we uniquely extend c to the set T^* in such a way that equality (2) holds for all $t \in T^*$ and $x \in I_t$. If $1 \in T$ then (2) forces that $c(1) = 0$. If $1 \notin T$ then define $c(1) := 0$. Moreover, put

$$c(t) := -t^p c\left(t^{-1}\right)$$

for all $t \in T^{-1}$. (One can check that for $t \in T \cap T^{-1}$ the above equality follows immediately from (1).) Now, if $t \in T^{-1}$ and $x \in I_t$ then $t^{-1} \in T$ and $tx \in I_{1/t}$, hence

$$\varphi\left(t^{-1}tx\right) = \varphi(tx) + c\left(t^{-1}\right)(tx)^p,$$

that is

$$\varphi(tx) = \varphi(x) - t^p c\left(t^{-1}\right)x^p = \varphi(x) + c(t)x^p,$$

which is (2).

The sequence $(c_n)_{n\in\mathbb{N}}$ will be defined inductively. Notice that $T_{\gamma_-,\gamma_+,1} = T^* \cap [\gamma_-,\gamma_+]$ and define

$$c_1 := c|_{T^*\cap[\gamma_-,\gamma_+]}.$$

For every $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$ we have $x \in I$ because of the inequalities $\gamma_- < 1 < \gamma_+$. If, in addition, $t \in T_{\gamma_-,\gamma_+,1}$ then $\gamma_- \le t \le \gamma_+$, so $\inf I < \gamma_- x \le$

$tx \le \gamma_+ x < \sup I$, and thus $x \in I \cap t^{-1}I = I_t$. Consequently, it follows from (2) that

$$\varphi(tx) = \varphi(x) + c(t)x^p = \varphi(x) + c_1(t)x^p.$$

Now fix an integer $n \ge 2$ and assume that we have defined a unique $c_{n-1}\colon T_{\gamma_-,\gamma_+,n-1} \to \mathbb{R}$ such that for every solution $\varphi\colon I \to \mathbb{R}$ of Eq. (1) we have

$$\varphi(tx) = \varphi(x) + c_{n-1}(t)x^p$$

for all $t \in T_{\gamma_-,\gamma_+,n-1}$ and $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$. Fix any $t \in T_{\gamma_-,\gamma_+,n}$ and $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$. Choose $t_1, \ldots, t_n \in T^*$ such that $t = t_1 \cdot \ldots \cdot t_n$ and $\gamma_- \le t_1 \cdot \ldots \cdot t_k \le \gamma_+$ for all $k \in \{1, \ldots, n\}$. Then

$$\inf I < \gamma_- x \le t_1 \cdot \ldots \cdot t_k x \le \gamma_+ x < \sup I,$$

and thus $t_1 \cdot \ldots \cdot t_k x \in I$ whenever $k \in \{1, \ldots, n\}$. Observe that $t_n \in T^*$ and $t_1 \cdot \ldots \cdot t_{n-1} \in T_{\gamma_-,\gamma_+,n-1}$, and thus

$$\begin{aligned}
\varphi(tx) &= \varphi(t_1 \cdot \ldots \cdot t_n x) = \varphi(t_n t_1 \cdot \ldots \cdot t_{n-1} x) \\
&= \varphi(t_1 \cdot \ldots \cdot t_{n-1} x) + c(t_n)(t_1 \cdot \ldots \cdot t_{n-1} x)^p \\
&= \varphi(x) + c_{n-1}(t_1 \cdot \ldots \cdot t_{n-1})x^p + c(t_n)(t_1 \cdot \ldots \cdot t_{n-1})^p x^p \\
&= \varphi(x) + \left[c_{n-1}(t_1 \cdot \ldots \cdot t_{n-1}) + c(t_n)(t_1 \cdot \ldots \cdot t_{n-1})^p\right] x^p.
\end{aligned}$$

If, in addition, $t = s_1 \cdot \ldots \cdot s_n$ with some $s_1, \ldots, s_n \in T^*$ satisfying $\gamma_- \le s_1 \cdot \ldots \cdot s_k \le \gamma_+$ for all $k \in \{1, \ldots, n\}$, then an analogous argument gives

$$\begin{aligned}
\varphi(tx) &= \varphi(s_1 \cdot \ldots \cdot s_n x) \\
&= \varphi(x) + \left[c_{n-1}(s_1 \cdot \ldots \cdot s_{n-1}) + c(s_n)(s_1 \cdot \ldots \cdot s_{n-1})^p\right] x^p.
\end{aligned}$$

Therefore, the value

$$c_n(t) := c_{n-1}(t_1 \cdot \ldots \cdot t_{n-1}) + c(t_n)(t_1 \cdot \ldots \cdot t_{n-1})^p$$

does not depend on the representation $t = t_1 \cdot \ldots \cdot t_n$ with $t_1, \ldots, t_n \in T^*$ such that $\gamma_- \le t_1 \cdot \ldots \cdot t_k \le \gamma_+$ whenever $k \in \{1, \ldots, n\}$ and defines a function c_n on $(\inf I/\gamma_-, \sup I/\gamma_+)$ satisfying the equality

$$\varphi(tx) = \varphi(x) + c_n(t)x^p$$

for all $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$. The uniqueness of c_n follows from its definition and the uniqueness of the functions c_{n-1} and c. $\square$

As an almost immediate consequence of Proposition 1 we obtain the following result,

Corollary. *Let* $c\colon T \to \mathbb{R}$ *and fix* $\gamma_- \in (0, 1)$ *and* $\gamma_+ \in (1, +\infty)$ *such that* $\inf I/\gamma_- < \sup I/\gamma_+$. *Then there exists a unique function* $c_{\gamma_-,\gamma_+}\colon T_{\gamma_-,\gamma_+} \to \mathbb{R}$ *such that for every solution* $\varphi\colon I \to \mathbb{R}$ *of Eq.* (1) *the equality*

$$\varphi(tx) = \varphi(x) + c_{\gamma_-,\gamma_+}(t)x^p$$

holds for all $t \in T_{\gamma_-,\gamma_+}$ *and* $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$.

Proof. It is enough to observe that the properties of the sequence $(c_n)_{n\in\mathbb{N}}$, established in Proposition 1, imply the equalities

$$c_{n+1}|_{T_{\gamma_-,\gamma_+,n}} = c_n$$

for all $n \in \mathbb{N}$ and to define the function c_{γ_-,γ_+} on T_{γ_-,γ_+} by

$$c_{\gamma_-,\gamma_+}(t) = c_n(t)$$

if $t \in T_{c_{\gamma_-,\gamma_+,n}}$ for an $n \in \mathbb{N}$. □

Now we are in a position to formulate and prove the main result.

Theorem. *Let $c\colon T \to \mathbb{R}$ and let $\varphi\colon I \to \mathbb{R}$ be a continuous solution of Eq. (1). Assume that $\gamma_- \in (0,1)$ and $\gamma_+ \in (1,+\infty)$ are such that $\inf I/\gamma_- < \sup I/\gamma_+$ and T_{γ_-,γ_+} is a dense subset of the interval (γ_-,γ_+).*

(i) *If $p \neq 0$ then there exist $a, b \in \mathbb{R}$ such that*

$$\varphi(x) = ax^p + b, \qquad x \in (\inf I/\gamma_-, \sup I/\gamma_+)\,.$$

(ii) *If $p = 0$ then there exist $a, b \in \mathbb{R}$ such that*

$$\varphi(x) = a\log x + b, \qquad x \in (\inf I/\gamma_-, \sup I/\gamma_+)\,.$$

Proof. Since φ is continuous it follows from the Corollary that for every $x \in (\inf I/\gamma_-, \sup I/\gamma_+)$ the function

$$(\gamma_-,\gamma_+) \ni t \longmapsto \frac{\varphi(tx)-\varphi(x)}{x^p}$$

is a continuous extension of c_{γ_-,γ_+}. As the domain T_{γ_-,γ_+} of c_{γ_-,γ_+} is a dense subset of $(\inf I/\gamma_-, \sup I/\gamma_+)$ such an extension is unique. This means that the function $c_\infty\colon (\gamma_-,\gamma_+) \to \mathbb{R}$, given by

$$c_\infty(t) = \frac{\varphi(tx)-\varphi(x)}{x^p},$$

does not depend on x. In other words, $c_\infty|_{T_{\gamma_-,\gamma_+}} = c_{\gamma_-,\gamma_+}$ and

$$\varphi(tx) = \varphi(x) + c_\infty(t)x^p, \quad t \in (\gamma_-,\gamma_+)\,,\ x \in (\inf I/\gamma_-, \sup I/\gamma_+)\,. \tag{4}$$

Fix any $\vartheta \in (1,+\infty)$ such that $\left(\vartheta^{-1},\vartheta\right) \subset (\gamma_-,\gamma_+)$ and $\vartheta \inf I/\gamma_- < \vartheta^{-1}\sup I/\gamma_+$. Take any $s,t \in \left(\vartheta^{-1},\vartheta\right)$ and $x \in \left(\vartheta \inf I/\gamma_-, \vartheta^{-1}\sup I/\gamma_+\right)$. Then $sx, tx \in (\inf I/\gamma_-, \sup I/\gamma_+)$.

To prove (i) assume that $p \neq 0$. Then, by (4),

$$\varphi(stx) = \varphi(tx) + c_\infty(s)(tx)^p = \varphi(x) + \left(c_\infty(t) + c_\infty(s)t^p\right)x^p$$

and

$$\varphi(tsx) = \varphi(sx) + c_\infty(t)(sx)^p = \varphi(x) + \left(c_\infty(s) + c_\infty(t)s^p\right)x^p,$$

hence

$$c_\infty(t) + c_\infty(s)t^p = c_\infty(s) + c_\infty(t)s^p,$$

that is

$$c_\infty(s)\left(t^p-1\right)=c_\infty(t)\left(s^p-1\right).$$

Therefore, there exists an $a\in\mathbb{R}$ such that

$$c_\infty(t)=a\left(t^p-1\right),\qquad t\in\left(\vartheta^{-1},\vartheta\right).$$

Taking any $x\in(\inf I/\gamma_-,\sup I/\gamma_+)$ and $t\in\left(\vartheta^{-1},\vartheta\right)$, and using (4) we get

$$\frac{\varphi(tx)-\varphi(x)}{tx-x}=\frac{c_\infty(t)x^p}{(t-1)x}=a\frac{t^p-1}{t-1}x^{p-1}\underset{t\to1}{\longrightarrow}apx^{p-1}.$$

Thus φ is differentiable at x and $\varphi'(x)=apx^{p-1}$; consequently,

$$\varphi(x)=ax^p+b,\qquad x\in(\inf I/\gamma_-,\sup I/\gamma_+),$$

with some $b\in\mathbb{R}$.

Similarly we prove (ii). Assume that $p=0$. Then (4) implies

$$\varphi(stx)=\varphi(tx)+c_\infty(s)=\varphi(x)+c_\infty(t)+c_\infty(s)$$

and

$$\varphi(stx)=\varphi(x)+c_\infty(st).$$

This means that

$$c_\infty(st)=c_\infty(s)+c_\infty(t),\qquad s,t\in\left(\vartheta^{-1},\vartheta\right),$$

and thus (see [3]) there exists an $a\in\mathbb{R}$ such that

$$c_\infty(t)=a\log t,\qquad t\in\left(\vartheta^{-1},\vartheta\right).$$

Now, if $x\in(\inf I/\gamma_-,\sup I/\gamma_+)$ and $t\in\left(\vartheta^{-1},\vartheta\right)$ then, by (4),

$$\frac{\varphi(tx)-\varphi(x)}{tx-x}=\frac{c_\infty(t)}{(t-1)x}=a\frac{\log t}{t-1}\frac{1}{x}\underset{t\to1}{\longrightarrow}a\frac{1}{x}.$$

Therefore

$$\varphi(x)=a\log x+b,\qquad x\in(\inf I/\gamma_-,\sup I/\gamma_+),$$

for some $b\in\mathbb{R}$. □

2. Density of T_{γ_-,γ_+} in (γ_-,γ_+)

The following question, about the possible realization of the assumptions of the Theorem, arises naturally: are there simple non-trivial examples of sets $T\subset(0,+\infty)$ and numbers $\gamma_-\in(0,1),\gamma_+\in(1,+\infty)$ such that T_{γ_-,γ_+} is a dense subset of the interval (γ_-,γ_+)?

At first it is quite easy to prove that if $T\subset(0,+\infty)$ is any set having 1 as an accumulation point, then we are done (see [4, Lemma 5.1] for details). A somewhat opposite situation is when T contains two elements only. The proposition below provides a pretty large class of examples giving a positive answer to the question in that case.

Proposition 2. *Let* $0 < \gamma_- \le a < 1 < b \le \gamma_+$ *with* $\log a/\log b \notin \mathbb{Q}$ *and let* $T = \{a, b\}$*. Then* T_{γ_-,γ_+} *is a dense subset of the interval* (γ_-, γ_+)*.*

Proof. Due to the logarithmic incommensurability of a, b the set

$$C := \{a^l b^m : l \in \mathbb{Z}, m \in \mathbb{N}_0\} \cap [\gamma_-, \gamma_+]$$

is dense in $[\gamma_-, \gamma_+]$, and thus to prove the proposition it is enough to show that $C \subset T_{\gamma_-,\gamma_+}$. Clearly $C = \bigcup_{m=0}^{\infty} C_m$ where

$$C_m := \{a^l b^m : l \in \mathbb{Z}\} \cap [\gamma_-, \gamma_+], \qquad m \in \mathbb{N}_0.$$

Since $\gamma_- \le a < 1 \le \gamma_+$, it follows that all the sets C_m are non-empty. If $t \in C_m \cap T_{\gamma_-,\gamma_+}$ for some $m \in \mathbb{N}_0$, then, as $a, a^{-1} \in T^*$ and each element of C_m can be obtained as a product of t by a non-negative power of a or a^{-1}, we see that $C_m \subset T_{\gamma_-,\gamma_+}$. Therefore

$$\text{either } C_m \cap T_{\gamma_-,\gamma_+} = \emptyset, \quad \text{or } C_m \subset T_{\gamma_-,\gamma_+}, \qquad m \in \mathbb{N}_0.$$

Using induction we prove that $C_m \subset T_{\gamma_-,\gamma_+}$ for all $n \in \mathbb{N}_0$. Clearly $1 \in C_0 \cap T_{\gamma_-,\gamma_+}$. Assume that $C_m \subset T_{\gamma_-,\gamma_+}$ for some $m \in \mathbb{N}_0$ and let $c_m = \inf C_m$. Since $a \in (0, 1)$ we see that $ac_m \notin C_m$, hence $ac_m < \gamma_- \le a$ and, consequently, $c_m < 1$. Thus

$$bc_m \in (c_m, b) \subset [\gamma_-, \gamma_+].$$

This implies two facts. First of all, as $c_m \in C_m$, we get $bc_m \in C_{m+1}$. Secondly, since $b \in T^*$ and $c_m \in T_{\gamma_-,\gamma_+}$, by the induction hypothesis, we have $bc_m \in T_{\gamma_-,\gamma_+}$. This shows the relation $C_{m+1} \cap T_{\gamma_-,\gamma_+} \ne \emptyset$, and thus $C_{m+1} \subset T_{\gamma_-,\gamma_+}$. This completes the proof. □

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Received: April 1, 2018
Revised: August 29, 2018