On an alternative d'Alembert's equation

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Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. Roger Cuculière [*Problem 11998*, The American Mathematical Monthly **124** no. 7 (2017)] has posed the following problem: *Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy $f(z) \leq 1$ for some nonzero real number z and*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \quad (\text{C})$$

for all real numbers x and y . We present the general Lebesgue measurable solution of (C) in the class of complex valued functions defined on the real line. Moreover, applying the invariant ideals method, we shall discuss a corresponding alternative d'Alembert equation

$$f(x+y) \neq f(x-y) \implies f(x+y) + f(x-y) = 2f(x)f(y), \quad (\text{CA})$$

stemming from Eq. (C) in the class of scalar valued functions defined on suitable groups. Equations (CA) seems to be of interest on its own.

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1. Introduction

The following question was asked by a French mathematician Roger Cuculière [*Problem 11998*, The American Mathematical Monthly **124** no. 7 (2017)]:

Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that satisfy $f(z) \leq 1$ for some nonzero real number z and

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \quad (\text{C})$$

for all real numbers x and y .

Let $(G, +)$ be a semigroup (not necessarily commutative). Equation (C) may be rewritten in the form

$$[f(x+y) - f(x)f(y)]^2 = (1 - f(x)^2)(1 - f(y)^2), \quad (1)$$

in the class of all complex valued functions f defined on G . Assuming that $(G, +)$ admits a neutral element 0 on setting $c := f(0)$ and $x = y = 0$ in (1) we infer that

$$2(c-1)^2 \left(c + \frac{1}{2} \right) = 2c^3 - 3c^2 + 1 = 0, \quad \text{i.e. } c \in \left\{ -\frac{1}{2}, 1 \right\}.$$

If we had $c = -\frac{1}{2}$ then putting $y = 0$ in (C) we would get $f(x)^2 \equiv \frac{1}{4}$ which, because of (1), forces the equality

$$[f(x+y) - f(x)f(y)]^2 = \frac{9}{16}, \quad x, y \in G.$$

Hence $S := \{x \in G : f(x) = -\frac{1}{2}\}$ yields a submonoid of G .

All these observations lead to the following two propositions.

Proposition 1. *If $(G, +; 0)$ is a monoid and $f : G \rightarrow \mathbb{C}$ satisfies Eq. (1), then $f(0) = 1$ provided that $\text{card } f(G) \geq 3$. If $f(0) \neq 1$, then either $f(x) \equiv -\frac{1}{2}$ or there exists a proper submonoid $(S, +; 0)$ of $(G, +; 0)$ such that*

$$f(x) = \begin{cases} -\frac{1}{2} & \text{for } x \in S, \\ \frac{1}{2} & \text{for } x \in G \setminus S =: S' \end{cases}$$

with $S' + S' \subset S$, $S + S' \subset S'$, and $S' + S \subset S'$; in particular, $(S, +)$ yields a subgroup of index 2 of the group $(G, +)$.

Conversely, a constant function $f = -\frac{1}{2}$ as well as any function f of the type described above yields a solution to Eq. (1).

Example. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ stand for the set of all nonnegative integers. Then the function $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} -\frac{1}{2} & \text{for } x \in 2\mathbb{N}_0, \\ \frac{1}{2} & \text{for } x \in 2\mathbb{N}_0 + 1 \end{cases}$$

satisfies Eq. (1).

Proposition 2. *If $(G, +; 0)$ is a monoid and $f : G \rightarrow \mathbb{C}$ satisfies Eq. (1) with $f(0) = 1$ and $\text{card } f(G) \leq 2$, then either $f(x) \equiv 1$ or there exists a proper submonoid $(S, +; 0)$ of $(G, +; 0)$ such that*

$$f(x) = \begin{cases} 1 & \text{for } x \in S, \\ c_0 & \text{for } x \in G \setminus S =: S' \end{cases}$$

with $c_0 \in \mathbb{C} \setminus \{1\}$ and $S' + S' \subset S$, $S + S' \subset S' \neq \emptyset$, $S' + S \subset S'$, (in particular, $(S, +)$ yields a subgroup of index 2 of the group $(G, +)$),
or

$$f(x) = \begin{cases} 1 & \text{for } x \in S, \\ -\frac{1}{2} & \text{for } x \in S' \end{cases}$$

with $S + S' \subset S' \neq \emptyset$, $S' + S \subset S'$.

Conversely, the constant function $f = 1$ as well as any function f of the type described above yields a solution to Eq. (1).

In what follows we shall assume that $(G, +)$ is an Abelian group and $f : G \rightarrow \mathbb{C}$ is a solution of Eq. (1) with $\text{card } f(G) \geq 3$. Then, by Proposition 1, we have $f(0) = 1$. Moreover, the set

$$Z := \{x \in G : f(x) = 1\}$$

forms an additive subgroup of $(G, +)$. In fact, $Z \neq \emptyset$ because $0 \in Z$ and $(Z, +)$ yields a submonoid of $(G, +)$ because of (1). Setting $y = -x$ in (1) we infer that

$$1 + f(x)^2 + f(-x)^2 - 2f(x)f(-x) = 1, \quad x \in G,$$

which proves that f is even.

Consequently,

$$(Z, +) \text{ yields a subgroup of } (G, +).$$

Now, replacing y by $-y$ in (1) and subtracting the resulting equation from (1) side by side we arrive at the following alternative d'Alembert functional equation:

$$f(x+y) \neq f(x-y) \implies f(x+y) + f(x-y) = 2f(x)f(y) \quad (\text{CA})$$

valid for all $x, y \in G$.

In particular, on setting $y = x$ in (CA) we get

$$f(2x) \in \{1, 2f(x)^2 - 1\} \text{ for all } x \in G. \quad (2)$$

We shall discuss Eq. (CA) separately, in Sect. 4 below.

2. A tool: invariant ideals

In the sequel we are going to consider a restricted version of d'Alembert's functional equation (C), assuming that it is satisfied for *almost all* pairs of elements of a (possibly noncommutative) group $(X, +)$, i.e. for all except those that belong to a "small" set from a proper invariant set ideal in X^2 . Recall that a proper nonempty subfamily $\mathcal{J} \subset 2^X$ is called a *proper invariant ideal* (p.i. ideal) in X whenever it is closed under finite set-theoretical unions, hereditary with respect to descending inclusions and such that jointly with any member E of $\mathcal{J}$ it contains the family $\{x - E : x \in X\}$.

We say that a property holds $\mathcal{J}$ -almost everywhere ($\mathcal{J}$ -a.e.) on X provided that it is satisfied on the whole X except for some set $E \in \mathcal{J}$.

For a subset $N \subset X^2$ we define the vertical section of N through the point $x \in X$ by the following formula

$$N[x] := \{y \in X : (x, y) \in N\}.$$

If $\widehat{\mathcal{J}}$ is a p.i. ideal in X^2 and the following Fubini type condition is satisfied:

for all $N \in \widehat{\mathcal{J}}$ the section $N[x]$ falls into $\mathcal{J}$, $\mathcal{J}$ -a.e. on X ,

then $\widehat{\mathcal{J}}$ is termed to be *conjugate* with $\mathcal{J}$.

To justify the use of a seemingly too abstract machinery let us keep in mind the undermentioned important examples of p.i. ideals:

- Let $(X, +)$ be a group of infinite order; the family of all finite subsets of X yields a p.i. ideal.
- Let $(X, +)$ be a Baire topological group; the family of all first category subsets of X yields a p.i. ideal.
- Let $(X, +)$ be a metric topological group with invariant metric and infinite diameter; the family of all subsets of X that are metrically bounded yields a p.i. ideal.
- Let $(X, +)$ be an Abelian locally compact topological group with a Haar measure h such that $h(X)$ is infinite; the family of all measurable subsets of X having finite measure and all their subsets yields a p.i. ideal.
- Let $(X, +)$ be a locally compact topological group with a completed Haar measure h ; the family of all sets $E \subset X$ such that $h(E) = 0$ yields a p.i. ideal.
- Let $\mathcal{J}$ stand for a p.i. ideal in a group $(X, +)$. Then the families

$$\Pi(\mathcal{J}) := \{N \subset X^2 : N \subset (X \times E) \cup (E \times X) \text{ for some } E \in \mathcal{J}\}$$

and

$$\Omega(\mathcal{J}) := \{N \subset X^2 : N[x] \in \mathcal{J} \text{ for } \mathcal{J}\text{-almost all } x \in X\}$$

yield p.i. ideals in the product group $(X^2, +)$. Both of them are conjugate with $\mathcal{J}$; moreover, $\Pi(\mathcal{J})$ is the smallest (in the sense of inclusion) and $\Omega(\mathcal{J})$ is the maximal one among all p.i. ideals in $(X^2, +)$ conjugate with $\mathcal{J}$.

- Let $(X, +)$ be a group. Given a nonempty collection $\mathcal{R}$ of subsets of X , the family $\mathcal{J}(\mathcal{R})$ of all finite unions of the sets

$$x + [A \cup (-A)] + y, \quad A \in \mathcal{R}, x, y \in X,$$

and all their subsets yields the smallest invariant ideal of subsets of X among those containing the family $\mathcal{R}$. An ideal $\mathcal{J}(\mathcal{R})$ becomes a p.i. ideal provided that no finite union of sets of that form coincides with X . If that is the case, this p.i. ideal is termed to be *generated* by the family $\mathcal{R}$.

3. The main results

The following theorem offers a solution of an abstract version of Cuculière's problem for complex valued functions defined on suitable groups. Jointly with

the succeeding corollaries it also gives a solution of the original Cuculière's problem (cf. the Introduction).

Theorem 1. *Let $(G, +)$ be an Abelian uniquely 2-divisible group and let $\mathcal{J}_1, \mathcal{J}_2$ be two conjugate p.i. ideals in G and G^2 , respectively, enjoying the following properties:*

- $U \in \mathcal{J}_1$ implies $2U, \frac{1}{2}U \in \mathcal{J}_1$,
- $N \in \mathcal{J}_2$ implies $\varphi(N) \in \mathcal{J}_2$, where $\varphi(x, y) := (x+y, x-y), (x, y) \in G^2$.

Let further $f : G \longrightarrow \mathbb{C}$ be a solution to equation

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1, \quad (x, y) \in G^2, \quad [\text{C}]$$

such that $\text{card } f(G) \geq 3$ and

$$\{(x, y) \in G^2 : f(x+y) = f(x-y)\} \in \mathcal{J}_2.$$

Then there exists a complex valued exponential function m on G such that

$$f(x) = \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in G; \quad (3)$$

m is unique except that it can be replaced by $m \circ (-\text{id})$.

Conversely, each function $f : G \longrightarrow \mathbb{R}$ defined by formula (3) yields a solution to Eq. (C).

Proof. (Necessity.) We already know that f yields a solution to (CA). Applying Theorem 1 from Adamaszek's paper [2] we infer that f has to be $\mathcal{J}_1$ -almost everywhere equal to a solution of d'Alembert's equation (A), i.e. there exist a member E of the ideal $\mathcal{J}_1$ and a function $g : G \longrightarrow \mathbb{C}$ such that

$$\begin{aligned} g(x+y) + g(x-y) \\ = 2g(x)g(y) \quad \text{for all } x, y \in G \quad \text{and} \quad f(x) = g(x) \quad \text{for all } x \in G \setminus E. \end{aligned}$$

By a simple change of variables equations (CA) and (A) may equivalently be written in the form

$$f(s) \neq f(t) \implies f(s) + f(t) = 2f\left(\frac{s+t}{2}\right)f\left(\frac{s-t}{2}\right), \quad s, t \in G, \quad (*)$$

and

$$g(s) + g(t) = 2g\left(\frac{s+t}{2}\right)g\left(\frac{s-t}{2}\right), \quad s, t \in G, \quad (**)$$

respectively. □

Fix arbitrarily an $s \in G$. By choosing a t off the set $E_s := E \cup (2E - s) \cup (s - 2E) \in \mathcal{J}_1$ we get

$$f(t) = g(t), \quad f\left(\frac{s+t}{2}\right) = g\left(\frac{s+t}{2}\right), \quad \text{and} \quad f\left(\frac{s-t}{2}\right) = g\left(\frac{s-t}{2}\right),$$

whence, on account of (*) and (**), we infer that

$$f(s) \neq g(t) \implies f(s) + g(t) = 2g\left(\frac{s+t}{2}\right)g\left(\frac{s-t}{2}\right) = g(s) + g(t), \quad t \in G \setminus E_s,$$

i.e.

$$f(s) \neq g(t) \implies f(s) = g(s), \quad t \in G \setminus E_s. \quad (4)$$

Put $c := f(s)$ and $T := \{t \in G : g(t) \neq c\}$. If we had $T \in \mathcal{J}_1$, then for every $x \in G$ and $t \in G$ taken off the set

$$T_x := T \cup (2T - x) \cup (x - 2T) \in \mathcal{J}_1$$

we would have

$$g(t) = g\left(\frac{x+t}{2}\right) = g\left(\frac{x-t}{2}\right) = c.$$

Thus, in view of (**), $g(x) + c = 2c^2$ whence

$$g(x) \equiv 2c^2 - c = \text{const} =: \alpha.$$

This gives $f(x) = \alpha$ for all $x \in G \setminus E$. Fix arbitrarily a $u \in G$ and choose a v off the set $E \cup (2E - u) \cup (u - 2E) \in \mathcal{J}_1$ to obtain

$$v \notin E, \quad \frac{u+v}{2} \notin E, \quad \text{and} \quad \frac{u-v}{2} \notin E.$$

Now, an appeal to (*) yields

$$f(u) \neq \alpha \implies f(u) + \alpha = 2\alpha^2,$$

whence

$$f(u) \in \{\alpha, 2\alpha^2 - \alpha\} \quad \text{for every } u \in G.$$

This leads to a contradiction:

$$3 \leq \text{card} f(G) \leq 2.$$

Therefore $T \notin \mathcal{J}_1$, which guarantees the existence of a $t \in T \setminus E_s$. Consequently, relation (4) implies the equality $f(s) = g(s)$ whence $f = g$ because of the unrestricted choice of an element s from G . Now, an appeal to Pl. Kannappan's Theorem 2 from his paper [5], proves that there exists a unique function $m : G \rightarrow \mathbb{C}$ [except that it can be replaced by $m \circ (-id)$] such that

$$\begin{aligned} m(x+y) &= m(x)m(y) \text{ for all } x, y \in G \quad \text{and} \quad f(x) = g(x) \\ &= \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in G, \end{aligned}$$

as claimed.

This finishes the proof because the detailed verification of the sufficiency reduces to a mechanical calculation.

In what follows the symbol ℓ_k denotes the k -dimensional Lebesgue measure, $k \in \{1, 2\}$, whereas $\mathcal{J}_k$ will stand for the p.i. ideal of all Lebesgue nullsets in

$\mathbb{R}^k$, $k \in \{1, 2\}$. Plainly, the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate and satisfy the conditions occurring in the statement of Theorem 1.

Theorem 2. *A Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$ yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \quad (\text{C})$$

for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or

$$f(x) = \begin{cases} -\frac{1}{2} & \text{for } x \in S, \\ \frac{1}{2} & \text{for } x \in G \setminus S =: S' \end{cases}$$

where $(S, +)$ is a Lebesgue measurable subgroup of the group $(\mathbb{R}, +)$ with $\ell_1(S) = 0$, or

$$f(x) = \cos bx \cosh ax + i \sin bx \sinh ax, \quad x \in \mathbb{R}, \quad (5)$$

where $a, b \in \mathbb{R}$ stand for arbitrary constants.

(Note that taking $a = b = 0$ we get another constant solution $f(x) = 1$, $x \in \mathbb{R}$.)

Proof. (Sufficiency.) Obviously, a constant function $f = -\frac{1}{2}$ yields a solution to (C). A straightforward verification shows that the two-valued function f defined in the statement of Theorem 2 satisfies Eq. (C) as well. Recall that so does each function of the form (3) with an exponential mapping m ; in our present case, the function f given by formula (5) has the form (3) with $m(x) = e^{(a+bi)x}$, $x \in \mathbb{R}$.

(Necessity.) Assume that a Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$ yields a solution of Eq. (C). Let us distinguish two cases:

$$(a) \quad \text{card} f(\mathbb{R}) \leq 2 \quad \text{and} \quad (b) \quad \text{non}(a).$$

Assume (a). If $f(0) \neq 1$ then, in view of the fact that the group $(\mathbb{R}, +)$ does not admit subgroups of index 2, Proposition 1 implies that $f(x) \equiv -\frac{1}{2}$. If $f(0) = 1$, then from Proposition 2 we infer that either $f(x) \equiv 1$ or

$$f(x) = \begin{cases} 1 & \text{for } x \in S, \\ -\frac{1}{2} & \text{for } x \in S' \end{cases}$$

where $S := f^{-1}(\{1\})$ is a Lebesgue measurable proper subgroup of the group $(\mathbb{R}, +)$. If the Lebesgue measure of $S = S - S$ were positive, then by the Theorem of Steinhaus (see [6]) S would contain a neighbourhood of zero, forcing the equality $S = \mathbb{R}$ and contradicting the fact that $(S, +)$ is proper.

Assume (b). Then we have $f(0) = 1$, f is even, and Eq. (CA) is satisfied.

Due to the measurability of f the set $N_0 := \{(s, t) \in \mathbb{R}^2 : f(s) = f(t)\}$ is

Lebesgue measurable in $\mathbb{R}^2$. Note that the set $N := \{(x, y) \in \mathbb{R}^2 : f(x+y) = f(x-y)\}$ coincides with the image of N_0 through a diffeomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by the formula

$$\Phi(s, t) = \left(\frac{s+t}{2}, \frac{s-t}{2} \right), \quad (s, t) \in \mathbb{R}^2.$$

Therefore N is ℓ_2 -measurable as well and $\ell_2(N) = \frac{1}{2}\ell_2(N_0)$. We are going to show that

$$N = \{(x, y) \in \mathbb{R}^2 : f(x+y) = f(x-y)\} \in \mathcal{J}_2. \quad (6)$$

Indeed, otherwise, we would have $0 < \ell_2(N) = \frac{1}{2}\ell_2(N_0)$ and, by Fubini's Theorem, there exists an $s_0 \in \mathbb{R}$ such that the vertical section $N(s_0) := \{t \in \mathbb{R} : f(t) = f(s_0)\}$ of the set N_0 passing through s_0 is of positive ℓ_1 -measure. Denote by α the number $f(s_0)$ and fix arbitrarily x and y from $N(s_0)$. An appeal to Eq. (C) jointly with the evenness of f gives now the equality

$$2\alpha^2 + f(x-y)^2 - 2\alpha^2 f(x-y) = 1,$$

which implies that $f(x-y) \in \{1, 2\alpha^2 - 1\}$. In other words, $f(D) \subset \{1, 2\alpha^2 - 1\}$ where $D := N(s_0) - N(s_0)$. Recalling that $N(s_0)$ has positive measure we read from Steinhaus' theorem (see [6]) that $D \supset (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Note that due to the measurability of f the group $Z = \{x \in \mathbb{R} : f(x) = 1\}$ is measurable. It has measure zero; indeed, otherwise, applying the Steinhaus Theorem again, $Z = \mathbb{R}$ contradicting the fact that f admits at least three different values. Consequently, we get

$$f(x) = 2\alpha^2 - 1 \text{ for all } x \in (-\varepsilon, \varepsilon) \setminus Z \text{ (i.e. for } \ell_1\text{-almost all } x \in (-\varepsilon, \varepsilon)).$$

With the aid of the duplication formula (2), on taking an

$$x \in \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right) \setminus \left(Z \cup \frac{1}{2}Z\right) = \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right) \setminus \frac{1}{2}Z,$$

we obtain the (in)equalities

$$1 \neq f(2x) = 2\alpha^2 - 1 \quad \text{and} \quad 2\alpha^2 - 1 = 2(2\alpha^2 - 1)^2 - 1,$$

whence $2\alpha^2 - 1 = -\frac{1}{2}$. Thus

$$f(x) = -\frac{1}{2} \text{ for all } x \in (-\varepsilon, \varepsilon) \setminus Z$$

and, consequently, for every $t \in (-2\varepsilon, 2\varepsilon) \setminus Z$ one has $\frac{1}{2}t \in (-\varepsilon, \varepsilon) \setminus \frac{1}{2}Z \subset (-\varepsilon, \varepsilon) \setminus Z$, which implies that

$$f(t) = 2f\left(\frac{1}{2}t\right)^2 - 1 = -\frac{1}{2} \text{ for all } t \in (-2\varepsilon, 2\varepsilon) \setminus Z.$$

Hence, by induction, for every positive integer n we obtain

$$f(t) = -\frac{1}{2} \text{ for all } t \in (-2^n\varepsilon, 2^n\varepsilon) \setminus Z.$$

This says nothing else but the equality

$$f(x) = \begin{cases} -\frac{1}{2} & \text{for } x \in \mathbb{R} \setminus Z \\ 1 & \text{for } x \in Z, \end{cases}$$

contradicting the fact that $\text{card } f(\mathbb{R}) \geq 3$.

So, under the assumptions we deal with, condition (6) holds true. Thus, Theorem 1 may be applied to get the equality

$$f(x) = \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in \mathbb{R}, \quad (7)$$

where $1 \neq m \neq 0$ is a complex valued exponential function on $\mathbb{R}$. Then m and $m \circ (-id)$ are different exponential functions and an appeal to Theorem 3.18 (f) from H. Stetkær's monograph [7] shows that the Lebesgue measurability of m is inherited from that of f . On account of Theorem 4 in Section 5.1 of the monograph by Aczél and Dhombres [1] gives now a representation of m in the form

$$m(x) = e^{cx}, \quad x \in \mathbb{R}, \quad \text{where } c = a + bi \text{ is a complex number.}$$

Consequently,

$$f(x) = \frac{e^{ax} + e^{-ax}}{2} \cos bx + i(\sin bx) \frac{e^{ax} - e^{-ax}}{2} \quad \text{for all } x \in \mathbb{R}.$$

This is the desired conclusion. $\square$

Corollary 1. *A Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

satisfied for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or

$$f(x) = \begin{cases} 1 & \text{for } x \in S, \\ -\frac{1}{2} & \text{for } x \in S' \end{cases}$$

where $(S, +)$ is a Lebesgue measurable subgroup of the group $(\mathbb{R}, +)$ with $\ell_1(S) = 0$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ are arbitrary constants.

Proof. Theorem 2 implies that in the case where $\text{card } f(\mathbb{R}) \geq 3$ one has

$$f(x) = \cos bx \cosh ax + i \sin bx \sinh ax, \quad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ stand for arbitrary constants. Since now f is a real function we conclude that either a or b has to vanish proving that either $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, respectively. $\square$

Corollary 2. (a solution of Cuculière's problem). *A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(z) \leq 1$ for some nonzero real number z yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, where $b \in \mathbb{R}$ is an arbitrary constant.

Proof. Any continuous selfmapping of $\mathbb{R}$ is Lebesgue measurable and it suffices to apply Corollary 1: either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ are arbitrary constants, because the range of a continuous real function on $\mathbb{R}$ is connected. However, the cosh function is eliminated by the assumption that $f(z) \leq 1$ for some nonzero real number z . $\square$

4. An alternative equation

Without any regularity condition whatsoever, the facts established in the Introduction allow us to state the following

Remark. Given a group $(G, +)$ (not necessarily commutative), a function $f : G \rightarrow \mathbb{C}$ such that $f(0) = 1$ yields a solution of the equation

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \quad (\text{C})$$

for all $x, y \in G$, then f satisfies an alternative d'Alembert functional equation

$$f(x+y) \neq f(x-y) \implies f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G. \quad (\text{CA})$$

So, in a natural way, we have entered the world of alternative functional equations like, for instance:

- Mikusiński's equation

$$f(x+y) \neq 0 \implies f(x+y) = f(x) + f(y)$$

- the functional equation of Dhombres

$$f(x) + f(y) \neq 0 \implies f(x+y) = f(x) + f(y)$$

- the Mikusiński–Pexider functional equation

$$f(x+y) \neq 0 \implies g(x+y) = h(x) + h(y)$$

- numerous so called conditional equations

and so on (see e.g. Dhombres and Ger [3] or [4] for a more detailed review).

In that context, Eq. (CA) [stemming from Eq. (C) occurring in Roger Cuculière's problem] seems to be of interest on its own. It will be discussed in another paper.

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References

- [1] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Encyclopedia of Mathematics and its Applications, vol. 31. Cambridge University Press, Cambridge (1989)
- [2] Adamaszek, I.: Almost trigonometric functions. *Glasnik Mat.* **19**(39), 83–104 (1984)
- [3] Dhombres, J., Ger, R.: Conditional Cauchy equations. *Glasnik Mat.* **13**(33), 39–62 (1978)
- [4] Ger, R.: Solving alternative functional equations: What for? *Ann. Univ. Sci. Budapest. Sect. Comp.* **47**, 273–283 (2018)
- [5] Kannappan, P.I.: The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups. *Proc. Am. Math. Soc.* **19**, 69–74 (1968)
- [6] Steinhaus, H.: Sur les distances des points dans les ensembles de mesure positive. *Fund. Math.* **1**, 93–104 (1920)
- [7] Stetkær, H.: Functional Equations on Groups. World Scientific, Singapore (2013)

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