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On a functional equation characterizing linear similarities

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Abstract. The aim of this paper is to give an answer to a question posed by Alsina, Sikorska and Tomás. Namely, we show that, under suitable assumptions, a function $f\colon X\to Y$ from a normed space X into a normed space Y, satisfying the functional equation

$$f\left(y-\frac{\rho'_+(x,y)}{\|x\|^2}x\right)=f(y)-\frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x),\quad x,y\in X$$

has to be a linear similarity (scalar multiple of a linear isometry).

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1. Introduction

Let $(X,\|\cdot\|)$ be a real normed space. We define *norm derivatives* $\rho'_\pm\colon X\times X\to\mathbb{R}$ by $\rho'_\pm(x,y):=\|x\|\cdot\lim_{t\to 0^\pm}\frac{\|x+ty\|-\|x\|}{t}$. The convexity of the norm yields that ρ'_+ and ρ'_- are well-defined. Now we define ρ_+*-orthogonality*: $x\perp_{\rho_+}y\ :\Leftrightarrow \rho'_+(x,y)=0$. The following properties can be found, e.g., in [1,2].

(nd1) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{R}}\quad \rho'_\pm(x,\alpha x+y)=\alpha\|x\|^2+\rho'_\pm(x,y)$;
(nd2) $\forall_{x,y\in X}\ \forall_{\alpha\geqslant 0}\quad \rho'_\pm(\alpha x,y)=\alpha\rho'_\pm(x,y)=\rho'_\pm(x,\alpha y)$;
(nd3) $\forall_{x,y\in X}\ \forall_{\alpha<0}\quad \rho'_\pm(\alpha x,y)=\alpha\rho'_\mp(x,y)=\rho'_\pm(x,\alpha y)$;
(nd4) $\forall_{x,y\in X}\quad |\rho'_\pm(x,y)|\leqslant\|x\|\cdot\|y\|,\ \rho'_\pm(x,x)=\|x\|^2,\ \rho'_-(x,y)\leqslant\rho'_+(x,y)$;
(nd5) $\forall_{x,y,z\in X}\quad \rho'_+(x,y+z)\leqslant\rho'_+(x,y)+\rho'_+(x,z)$.

A normed space X is said to be *smooth* if for every $x\in X\setminus\{0\}$ there is a unique *supporting functional at* x, i.e., a unique functional $x^*\in X^*$ such that $\|x^*\|=1$ and $x^*(x)=\|x\|$. Moreover, we may state this definition in an equivalent form, namely: X is smooth $\Leftrightarrow\rho'_+=\rho'_-\Leftrightarrow\forall_{x\in X}\ \rho'_+(x,\cdot)$ is linear. If X is smooth, then the following condition holds (see [1]):

(nd6) for any two-dimensional subspace P of X and for every $x \in P$, $\lambda \in (0,+\infty)$, there exists a $y \in P$ such that $x \perp_{\rho_+} y$ and $x+y \perp_{\rho_+} \lambda x - y$.

In a real inner product space $(X, \langle\cdot|\cdot\rangle)$, given the triangle determined by two linearly independent vectors x, y and the zero vector (i.e., $\triangle\{x,y,0\}$), one can compute the height vector from y to the side x and orthogonal to x using the formula $h(x,y) := y - \frac{\langle x|y\rangle}{\|x\|^2}x$. Then $x \perp h(x,y)$. The same might be done for normed spaces using the function ρ'_+ as a generalization of an inner product. In this case we consider the height function $h(x,y) := y - \frac{\rho'_+(x,y)}{\|x\|^2}x$.

Alsina et al. [1] investigated functions $f\colon X \to X$ that transform the height of the triangle with sides x, y, $x-y$ into the corresponding height of the triangle determined by sides $f(x)$, $f(y)$, $f(x)-f(y)$, i.e. $f(\triangle\{x,y,0\}) = \triangle\{f(x), f(y), 0\}$. Namely, they studied the condition $f(h(x,y)) = h(f(x), f(y))$, which leads to the functional equation $f\left(y - \frac{\rho'_+(x,y)}{\|x\|^2}x\right) = f(y) - \frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x)$. In particular, Alsina et al. [1] obtained the following result.

Theorem 1. [1, p. 102, Theorem 3.7.2] *If X is a real normed linear space and $f\colon X \to X$ is a continuous function, then f is a solution of*

$$f\left(y - \frac{\rho'_+(x,y)}{\|x\|^2}x\right) = f(y) - \frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x), \quad x,y \in X$$

and vanishes only at zero if and only if, f is a linear similarity.

At the end of their book [1, p. 178, Open problem 6] Alsina, Sikorska and Tomás put the following problem.

Open problem Solve the functional equation

$$f\left(y - \frac{\rho'_+(x,y)}{\|x\|^2}x\right) = f(y) - \frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x), \quad x,y \in X, \tag{1}$$

where $f\colon X \to X$ is injective and $f(x) \neq 0$ whenever $x \neq 0$.

The aim of this paper is to present a partial solution of the above open problem. In particular, we will prove that the assumption of the continuity of f is redundant in some circumstances. Moreover, it is not necessary to assume that f is injective.

2. Results

Throughout this section we will work with real normed spaces of dimensions not less than 2. We will consider the norm derivatives in various spaces (X and Y); however, we will use one common symbol ρ'_+ for them. We will prove that $f\colon X \to Y$ is a solution of (1) if and only if it is a linear similarity (scalar multiple of a linear isometry). This assertion, however, can be obtained under the assumption of the smoothness of X. But, unlike Theorem 1, it will not be assumed that a function f is continuous.

Lemma 2. *Let X, Y be normed spaces, let $f: X \to Y$ satisfy (1). Then $f(0)=0$.*

Proof. By (1) we get $f(0) = f\left(y - \frac{\rho'_+(y,y)}{\|y\|^2} y\right) = f(y) - \frac{\rho'_+(f(y),f(y))}{\|f(y)\|^2} f(y) = f(y) - f(y) = 0$. □

Now we prove the first main result of this paper.

Theorem 3. *Let X, Y be normed spaces and let $f: X \to Y$ satisfy (1). Suppose that $z \neq 0 \Rightarrow f(z) \neq 0$. Then f is additive.*

Proof. First we will prove that f preserves the linear independence of two vectors. Suppose that $f(y)=\alpha f(x)$ and $x \neq 0$. Then

$$f\left(y - \frac{\rho'_+(x,y)}{\|x\|^2} x\right) \stackrel{(1)}{=} \alpha f(x) - \frac{\rho'_+(f(x), \alpha f(x))}{\|f(x)\|^2} f(x) \stackrel{(nd1)}{=}$$
$$\stackrel{(nd1)}{=} \alpha f(x) - \alpha \frac{\rho'_+(f(x), f(x))}{\|f(x)\|^2} f(x) \stackrel{(nd4)}{=} 0.$$

From the assumption (i.e. $f(z) = 0 \Rightarrow z = 0$) we have that $y - \frac{\rho'_+(x,y)}{\|x\|^2} x = 0$, hence the vectors x, y are linearly dependent. So, we have proved that f preserves the linear independence of two vectors.

Fix two linearly independent vectors $a, b \in X$. Then we have

$$f(b) - \frac{\rho'_+(f(a), f(b))}{\|f(a)\|^2} f(a) \stackrel{(1)}{=} f\left(b - \frac{\rho'_+(a,b)}{\|a\|^2} a\right) \stackrel{(nd1)}{=}$$
$$\stackrel{(nd1)}{=} f\left(a+b - \frac{\rho'_+(a, a+b)}{\|a\|^2} a\right) \stackrel{(1)}{=}$$
$$\stackrel{(1)}{=} f(a+b) - \frac{\rho'_+(f(a), f(a+b))}{\|f(a)\|^2} f(a).$$

It follows from the above equalities that

$$f(a+b) = f(b) + \left(\frac{\rho'_+(f(a), f(a+b))}{\|f(a)\|^2} - \frac{\rho'_+(f(a), f(b))}{\|f(a)\|^2}\right) f(a). \tag{2}$$

Putting b, a in place of a, b, respectively, in the above equality we get

$$f(a+b) = f(a) + \left(\frac{\rho'_+(f(b), f(b+a))}{\|f(b)\|^2} - \frac{\rho'_+(f(b), f(a))}{\|f(b)\|^2}\right) f(b). \tag{3}$$

We know that $f(a), f(b)$ are linearly independent. Thus, combining (2) and (3), we immediately get $\frac{\rho'_+(f(a),f(a+b))}{\|f(a)\|^2} - \frac{\rho'_+(f(a),f(b))}{\|f(a)\|^2} = 1$. Now equality (2) becomes $f(a+b) = f(b) + 1 \cdot f(a)$. To sum up, it has been shown that

$$a, b \text{ are linearly independent} \Rightarrow f(a+b) = f(a) + f(b). \tag{4}$$

Now let x and y be linearly dependent. We may assume that $x \neq 0 \neq y$. We consider two cases. Assume first that $y = \gamma x$ for some $\gamma \in \mathbb{R} \setminus \{-1\}$. There are linearly independent vectors $a, b \in X$ such that $a + b = x$. Then

$f(x+y) = f(x+\gamma x) = f(a+b+\gamma x) \overset{(4)}{=} f(a)+f(b+\gamma x) \overset{(4)}{=} f(a)+f(b)+f(\gamma x) \overset{(4)}{=} f(a+b)+f(\gamma x) = f(x)+f(\gamma x)$. To sum up, it has been shown that

$$x \in X \setminus \{0\}, \gamma \in \mathbb{R} \setminus \{-1\} \;\Rightarrow\; f(x+\gamma x) = f(x)+f(\gamma x). \tag{5}$$

Now assume $y=-x$. We have $f(x)=f\left(2x+\left(-\frac{1}{2}\right)2x\right) \overset{(5)}{=} f(2x)+f\left(\left(-\frac{1}{2}\right)2x\right) = f(2x)+f(-x) = f(x+x)+f(-x) \overset{(5)}{=} f(x)+f(x)+f(-x)$. It follows from the above equalities that $0 = f(x)+f(-x)$. By Lemma 2 we already know that $f(0) = 0$. Therefore $f(x+y) = f(x+(-x)) = f(0) = 0 = f(x)+f(-x) = f(x)+f(y)$. So, we have the additivity of f on the whole space X. $\square$

Lemma 4. *Let X, Y be normed spaces and let $f\colon X \to Y$ satisfy (1). Then f preserves ρ_+-orthogonality.*

Proof. Assume that $x \perp_{\rho_+} y$, i.e., $\rho'_+(x,y) = 0$. We assume that $f(x) \neq 0$ (if $f(x)=0$, then $f(x) \perp_{\rho_+} f(y)$). Notice that $f(y) = f\left(y - \frac{\rho'_+(x,y)}{\|x\|^2}x\right) \overset{(1)}{=} f(y) - \frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x)$, hence $\frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2}f(x) = 0$. This gives $\rho'_+(f(x),f(y))=0$. Hence $f(x) \perp_{\rho_+} f(y)$. Thus, in fact, f preserves ρ_+-orthogonality. $\square$

Now we prove the second main result of this paper.

Theorem 5. *Let X, Y, f be as in Theorem 3. Suppose that X is smooth. Then f is homogeneous.*

Proof. Fix y in $X \setminus \{0\}$. We know that $\dim \operatorname{span}\{y\} = 1$, so, it is best to think of $f|_{\operatorname{span}\{y\}}\colon \operatorname{span}\{y\} \to Y$ as a function $f\colon \mathbb{R} \to Y$.

Now we can prove that $f|_{\operatorname{span}\{y\}}$ is homogeneous. Since we already know that f is additive, it suffices to show that $\left\|f|_{\operatorname{span}\{y\}}(\cdot)\right\|$ is bounded below on the segment $\{\gamma y : \gamma \in [1,2]\}$. Let $\beta \in (0,1]$. Applying (nd6), there exists a $w \in X \setminus \{0\}$ such that $y \perp_{\rho_+} w$ and $y+w \perp_{\rho_+} \beta y - w$. It follows from Lemma 4 that $f(y) \perp_{\rho_+} f(w)$. Therefore,

$$\begin{aligned}\|f(y)\|^2 &= \rho'_+(f(y),f(y))+0 = \rho'_+(f(y),f(y))+\rho'_+(f(y),f(w)) \overset{(nd1)}{=} \\ &\overset{(nd1)}{=} \rho'_+(f(y),f(y)+f(w)) \overset{(nd4)}{\leqslant} \|f(y)\|\cdot\|f(y)+f(w)\|,\end{aligned}$$

and dividing by $\|f(y)\|$, we obtain $\|f(y)\| \leqslant \|f(y)+f(w)\|$. But since also $y+w \perp_{\rho_+} \beta y - w$, we conclude that $f(y+w) \perp_{\rho_+} f(\beta y - w)$, and by the additivity of f we have $f(y)+f(w) \perp_{\rho_+} f(\beta y) - f(w)$. In the same manner we can prove

$$\|f(y)+f(w)\| \leqslant \|f(y)+f(w) + f(\beta y) - f(w)\|.$$

Therefore $\|f(y)+f(w)\| \leqslant \|f(y)+f(\beta y)\|$. From this we deduce that

$$\|f(y)\| \leqslant \|f(y)+f(w)\| \leqslant \|f(y)+f(\beta y)\| = \|f(y+\beta y)\|.$$

Thus we have proved:

$$\beta \in (0,1] \Rightarrow \|f(y)\| \leqslant \|f((1+\beta)y)\|.$$

Observe that the above condition implies that $\left\|f|_{\operatorname{span}\{y\}}(\cdot)\right\|$ is bounded below on the segment $\{\gamma y : \gamma \in [1,2]\}$. The proof of Theorem 5 is complete. □

We can combine the results of Theorems 3 and 5 and Lemma 4 to obtain the third main result. Finally, we can solve (1) completely.

Theorem 6. *Let X, Y be real normed spaces. Suppose that X is smooth. Assume that $f : X \to Y$ is nonzero, suppose that $z \neq 0 \Rightarrow f(z) \neq 0$. Then, the following conditions are equivalent:*

(a) *f satisfies* (1),
(b) *f is linear and* $\exists_{\gamma>0}\ \forall_{x\in X}\ \|f(x)\| = \gamma\|x\|$.

Proof. We prove (a)⇒(b). It follows from Theorems 3,5 that f is linear. According to Lemma 4, f preserves ρ_+-orthogonality. The class of linear mappings preserving ρ_+-orthogonality coincides with the class of linear similarities (cf. [3, Theorem 5]). The proof of this implication is complete. The converse implication has a trivial verification. □

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