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Two refinements of Frink's metrization theorem and fixed point results for Lipschitzian mappings on quasimetric spaces

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With respect and admiration to Professor Karol Baron on the occasion of his jubilee.

Abstract. Quasimetric spaces have been an object of thorough investigation since Frink's paper appeared in 1937 and various generalisations of the axioms of metric spaces are now experiencing their well-deserved renaissance. The aim of this paper is to present two improvements of Frink's metrization theorem along with some fixed point results for single-valued mappings on quasimetric spaces. Moreover, Cantor's intersection theorem for sequences of sets which are not necessarily closed is established in a quasimetric setting. This enables us to give a new proof of a quasimetric version of the Banach Contraction Principle obtained by Bakhtin. We also provide error estimates for a sequence of iterates of a mapping, which seem to be new even in a metric setting.

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1. Introduction

The theory of metric spaces initiated in 1906 by Fréchet [16] has developed into a huge branch of mathematics. The three axioms of a metric can be considered as three pillars of this theory. However, some of these conditions might be substituted by others or even omitted completely. This practice stems either from a purely theoretic curiosity-driven tendency to generalise already known concepts or is motivated by the necessity of practical applications.

Thus, several distinct concepts have been developed. For instance, if we omit the first axiom of a metric, then we obtain the so-called pseudometric,

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which is broadly used within the theory of uniform spaces. Skipping the symmetry axiom allows us to define hemimetric spaces (see, e.g., [19]). In many real-life situations it is more reasonable to use this method of measuring distance. Lastly, the generalisation which will be the focus of our attention in this paper is obtained by dropping the triangle inequality and thus obtaining the so-called semimetrics (see, e.g., [5,11,27]). Within the scope of this article, we will deal mainly with a subclass of those spaces called quasimetric spaces (see, e.g., [4,7,25,26]), which are also known under the name of b-metric spaces (see, e.g., [2,13]). Some readers might be misled by the notion of a semimetric, which differs from the one introduced in [23, Chapter 10]. However, we would like to adapt most terminology from Wilson's paper [27], who is one of the pioneers of this field of topology.

The paper is organized as follows. The first section is devoted to introducing the notions used in the article, as well as recalling a few selected results in the field of semimetric spaces. Then we dedicate the next section to recalling an important metrization technique of Frink [17] and propose two possible refinements of Frink's theorem. Each of them optimises the result obtained by Frink in some way either by dropping some restrictions on the assumptions or by improving the thesis in the original statement. The third section starts with establishing Cantor's intersection theorem for semimetric spaces satisfying one of Wilson's [27] axioms. A proof uses a metrization theorem obtained in our previous paper [11]. Next, as a simple consequence, we get a further generalization of Cantor's theorem in which we allow sets to be non-closed. This enables us to derive from that result a quasimetric version of the Banach Contraction Principle established by Bakhtin [4]. A novelty here is that we also provide quite a lot of error estimates for sequences of approximate fixed points and iterates of a mapping. Some of them seem to be new even in a metric setting.

2. Main notions

Let us start with recalling the core notion of this article, i.e., the definition of a semimetric space, which can be found in many papers, starting from [10,24,27], and ending with a recent monograph [22].

Definition 2.1. A pair (X, d) consisting of a non-empty set X and a function $d : X \times X \to [0, +\infty)$ is said to be a semimetric space if it satisfies the following conditions:

(Q1) $d(x, y) = 0 \iff x = y$;
(Q2) $d(x, y) = d(y, x)$

for all $x, y \in X$. The function d is then called a semimetric.

A myriad of possible subcategories of such a broad class of spaces is obtainable by adding some extra axioms, which can be considered as triangle-like conditions. As we have already mentioned, in the scope of this paper we will be dealing mainly with K-quasimetric spaces, which are a proper subclass of semimetric spaces.

Definition 2.2. A semimetric space (X, d) is called a quasimetric space or, more specifically, a K-quasimetric space, where $K \geqslant 1$ is fixed, if it satisfies the following condition:

(Q3) $d(x, z) \leqslant K \cdot \max\{d(x, y), d(y, z)\}$

for all $x, y, z \in X$. In this case, the function d is called a K-quasimetric.

A 1-quasimetric space is known broadly in the literature as an ultrametric space. Every 1-quasimetric space is therefore a metric space. On the other hand, every metric space is, in fact, a 2-quasimetric space. The reverse, however, does not hold even for $1 < K \leqslant 2$, i.e., there exist K-quasimetric spaces which are not metric, which is shown in the following simple

Example 2.3. Consider a set $X := \{a, b, c\}$ consisting of three distinct elements. Define a function $d : X \times X \to \mathbb{R}$ as follows: put 0 for $d(a, a)$, $d(b, b)$ and $d(c, c)$, and $d(a, b) = d(b, a) = 1,\ d(a, c) = d(c, a) = 4,\ d(c, b) = d(b, c) = 2$. One may think of such a space as a 'non-existent triangle', where the lengths of its sides are equal to 1, 2 and 4, respectively.

Actually, every semimetric space (X, d) is a quasimetric space, if the set X is finite.

An equivalent definition of the above notion can be given by replacing (Q3) by the following axiom: for a fixed $M \geqslant 1$ and all $x, y, z \in X$,

(Q3′) $d(x, z) \leqslant M \cdot (d(x, y) + d(y, z))$.

It is easily seen that conditions (Q3) and (Q3′) are equivalent. In fact, (Q3′) implies (Q3) with a constant $K = 2M$, whereas (Q3) implies (Q3′) with a constant $M = K$. In further sections (unless stated otherwise), by a K-quasimetric we understand a semimetric satisfying the condition (Q3).

A stronger concept was given by Fagin et al. [14]:

Definition 2.4. A semimetric d is said to satisfy a c-relaxed polygonal inequality (c-rpi in short) if

$$d(x, y) \leqslant c \cdot (d(x, x_1) + d(x_1, x_2) + \cdots + d(x_n, y))$$

for a fixed $c \geqslant 1$, any $n \in \mathbb{N}$ and all finite sequences $x, x_1, \ldots, x_n, y \in X$.

In the same paper, the authors proved the following useful and interesting

Theorem 2.5 (Fagin et al.). *For any semimetric space* (X, d) *and* $c \geqslant 1$, *the following conditions are equivalent:*

(i) (X, d) *satisfies c-rpi;*

(ii) there exists a metric ρ on X such that

$$\forall_{x,y\in X}\ \rho(x,y) \leqslant d(x,y) \leqslant c\cdot\rho(x,y).$$

Topology in semimetric spaces can be defined in various ways. The first and most common approach is to define a topology in the following way:

$$A \text{ is open in } (X,d) \iff \forall_{x\in A}\ \exists_{r>0}\ B(x,r)\subset A,$$

where $B(x,r) := \{y \in X : d(x,y) < r\}$ is an open ball. However, for some spaces this topology is not coherent with the descriptive words 'open ball', as examples of semimetric spaces in which open balls are not open sets have been provided in [2,3,11,25]. What makes this topology particularly useful is the fact that the convergence of a sequence $(x_n) \in X^{\mathbb{N}}$ to some point $x \in X$ with respect to this topology is equivalently described by $d(x_n, x) \to 0$. This topology will be considered as a default one throughout this paper, unless explicitly stated otherwise. A topology in which each open ball is an open set (i.e., open balls form a subbasis of the topology in (X, d)) is considered in [2,11].

Many authors have also posed questions concerning the metrizability of semimetric spaces (usually equipped with some additional conditions). In our previous paper [11] we showed that, in particular, semimetric spaces satisfying Wilson's [27] axiom $(W5)$ are uniformly metrizable. A semimetric space is said to satisfy $(W5)$ if for any three sequences $(x_n),(y_n),(z_n) \in X^{\mathbb{N}}$,

$$d(x_n,y_n)\to 0 \text{ and } d(y_n,z_n)\to 0 \text{ imply that } d(x_n,z_n)\to 0.$$

Definition 2.6. Let X be a non-empty set and d_1, d_2 be semimetrics defined on X. We say that d_1 and d_2 are uniformly equivalent if the following two conditions hold:

$$\begin{array}{llll}\forall\varepsilon>0 & \exists\delta>0 & \forall x,y\in X & (d(x,y)<\delta \implies \rho(x,y)<\varepsilon)\,;\\ \forall\varepsilon>0 & \exists\delta>0 & \forall x,y\in X & (\rho(x,y)<\delta \implies d(x,y)<\varepsilon)\,.\end{array}$$

In the next section, the topic of metrizability is revisited, focusing on metric bounds, which can be obtained for a certain class of K-quasimetric spaces.

3. Refinements of Frink's theorem

In [17] Frink provided an innovative method for constructing a metric equivalent to a 2-quasimetric. We recall his result in an equivalent form indicated by Schroeder [26].

Theorem 3.1 (Frink). *Let (X,d) be a 2-quasimetric space. Then there exists a metric ρ on X such that*

$$\forall_{x,y\in X}\ \rho(x,y)\leqslant d(x,y)\leqslant 4\rho(x,y).$$

In this theorem the metric ρ is obtained by the so-called *chain approach*, namely, ρ is defined by

$$\rho(x,y) := \inf \sum_{i=1}^{n} d(x_{i-1}, x_i), \tag{3.1}$$

where the infimum is taken over all finite sequences of points $x_0, x_1, x_2, \ldots, x_n$, where $x_0 = x$ and $x_n = y$, thus guaranteeing that the triangle inequality is satisfied. In what follows, we will denote by $\rho_{\inf}$ the function defined by (3.1).

Theorem 3.1 was extended by Schroeder [26], who obtained the following

Theorem 3.2 (Schroeder). *Let (X,d) be a K-quasimetric space with $K \leqslant 2$. Then there exists a metric ρ on X such that*

$$\forall_{x,y\in X}\ \rho(x,y) \leqslant d(x,y) \leqslant 2K\rho(x,y).$$

A natural question arises, whether one can obtain a better estimation of d than that in Theorem 3.2. We will now present two possible refinements of this theorem. The first one leaves the restriction $K \leqslant 2$ and gives an optimal bounding constant. The other relaxes the restriction and provides an optimal bound but for the pth power of the quasimetric d and not d itself.

3.1. The first refinement

In this section we present the first of the two mentioned generalisations of Theorems 3.1 and 3.2. Note that the idea to use $p := \log_K 2$ in the proof of the following theorem originates from the paper of Paluszyński and Stempak [25].

Theorem 3.3. *If (X,d) is a K-quasimetric space with $K \leqslant 2$, then there exists a metric ρ on X for which the following inequalities hold:*

$$\forall_{x,y\in X}\ \rho(x,y) \leqslant d(x,y) \leqslant K^2\rho(x,y). \tag{3.2}$$

Proof. The case where $K = 1$ yields an ultrametric space which is, in fact, a metric space. Fix $K \in (1,2]$ and set $p := \log_K 2$. Then $p \geqslant 1$ and d^p is a 2-quasimetric. Applying Theorem 3.1 to d^p, we obtain the existence of a metric ρ' such that

$$\forall_{x,y\in X}\ \rho'(x,y) \leqslant d^p(x,y) \leqslant 4\cdot\rho'(x,y).$$

To proceed further, we recall the following well-known

Lemma 3.4. *Let (X,ρ) be a metric space. If $f : [0,+\infty) \to [0,+\infty)$ is concave, continuous and $f^{-1}(\{0\}) = \{0\}$, then $f \circ \rho$ defines a metric equivalent to ρ.*

Since the function $[0, +\infty) \ni x \mapsto x^{\frac{1}{p}}$ satisfies the assumptions of Lemma 3.4, we have that $\rho := (\rho')^{\frac{1}{p}}$ is also a metric. Moreover, since $K^{2p} = 4$, ρ satisfies the condition

$$\forall_{x,y \in X}\ \rho(x,y) \leqslant d(x,y) \leqslant K^2 \cdot \rho(x,y),$$

which completes the proof. $\square$

Of course, since $K^2 \leqslant 2K$ for $K \in [1,2]$, Theorem 3.3 gives a better estimation than Theorem 3.2.

A natural question arises whether the constant K^2 is optimal in Theorem 3.3. Our next result gives a positive answer to this question.

Theorem 3.5. *Let a function $\varphi : [1,2] \to [0,\infty)$ be such that for any $K \in [1,2]$ and any K-quasimetric space (X,d), there exists a metric ρ such that*

$$\rho \leqslant d \leqslant \varphi(K)\rho.$$

Then $\varphi(K) \geqslant K^2$ for all $K \in [1,2]$.

Proof. Suppose to the contrary that there exists $K \in [1,2]$ such that $\alpha := \varphi(K) < K^2$. Consider a set $X := \{x_0, x_1, x_2, x_3\}$, where $x_i \neq x_j$ for $i \neq j$. Since $\alpha < K^2$, there exists $m \in \mathbb{N}$ such that $(1 - \frac{1}{m})K^2 > \alpha$. Then define a K-quasimetric d on X as follows:

$$\begin{aligned} d(x_0,x_1) &:= d(x_2,x_3) = \frac{1}{2m};\\ d(x_1,x_2) &:= 1 - \frac{1}{m};\\ d(x_0,x_2) &:= d(x_1,x_3) = K\left(1 - \frac{1}{m}\right);\\ d(x_0,x_3) &:= K^2\left(1 - \frac{1}{m}\right);\\ d(x_i,x_j) &:= d(x_j,x_i) \text{ if } i > j \end{aligned}$$

and $d(x_i,x_i) = 0$ for $i = 0,1,2,3$. It is not difficult to check that d is indeed a K-quasimetric. Thus by hypothesis, we can find a metric ρ satisfying $\rho \leqslant d \leqslant \varphi(K)\rho = \alpha\rho$. Hence

$$\rho(x_0,x_3) \leqslant \sum_{i=1}^{3} \rho(x_{i-1},x_i) \leqslant \sum_{i=1}^{3} d(x_{i-1},x_i) = \frac{1}{2m} + 1 - \frac{1}{m} + \frac{1}{2m} = 1.$$

Since $d(x_0,x_3) = K^2\left(1 - \frac{1}{m}\right) > \alpha$, we get that

$$\alpha < d(x_0,x_3) \leqslant \alpha\rho(x_0,x_3) \leqslant \alpha \cdot 1 = \alpha,$$

which gives a contradiction. $\square$

A natural question arises whether the constant $K = 2$ plays a special role in Theorem 3.1. More precisely, does Frink's chain approach work if we apply it to a K-quasimetric with $K > 2$? This question was answered negatively by Schroeder [26], who gave, for any $K > 2$, an interesting but complicated example of a K-quasimetric space (X, d) for which the function $\rho_{\inf}$ defined by (3.1) is not a metric. Hence, taking into account that the metric ρ in Theorem 3.1 was constructed in fact as $\rho_{\inf}$, we obtain the following

Corollary 3.6. *Let K be a real number. Then $K \leqslant 2$ if and only if for any K-quasimetric space (X, d), the function $\rho_{\inf}$ defined by (3.1) is a metric on X.*

On the other hand, Dung and Hang [13], answering a question of Kirk and Shahzad [22], constructed a Caristi [9] mapping T on a 16-quasimetric space such that T has no fixed point. For the reader's convenience we recall the definition of such mappings.

Definition 3.7. Let (X, d) be a K-quasimetric space. $T : X \to X$ is called a Caristi mapping if there exists a lower semicontinuous function $\phi : X \to \mathbb{R}$ which is bounded from below and satisfies $d(x, Tx) \leqslant \phi(x) - \phi(Tx)$ for all $x \in X$.

Following [8] we say that T is *asymptotically regular* if for any $x \in X$, $d(T^n x, T^{n+1}x) \to 0$.

Now we extend Corollary 3.6 by establishing a list of seven equivalent conditions including two concerning the fixed point property for Caristi mappings.

Theorem 3.8. *Let K be a real number. The following conditions are equivalent:*

(i) $K \leqslant 2$;

(ii) *for every K-quasimetric space (X, d) there exists a metric ρ such that for some $c \geqslant 1$,*

$$\forall_{x,y \in X}\ \rho(x, y) \leqslant d(x, y) \leqslant c \cdot \rho(x, y);$$

(iii) *for every K-quasimetric space (X, d), the c-relaxed polygonal inequality is satisfied for some $c \geqslant 1$;*

(iv) *for every complete K-quasimetric space (X, d), any Caristi mapping has a fixed point;*

(v) *for every complete K-quasimetric space (X, d), any continuous and asymptotically regular Caristi mapping on X has a fixed point;*

(vi) *for every K-quasimetric space (X, d), the function $\rho_{\inf}$ defined by (3.1) is a metric;*

(vii) *for every K-quasimetric space (X, d) with $|X| > 1$, the function $\rho_{\inf}$ defined by (3.1) is nonzero.*

Proof. Notice that if $K < 1$, then all the conditions (i)–(vii) are true, since X is then a singleton. So in the remaining part of the proof we assume that $K \geqslant 1$.

(i) $\implies$ *(ii)* follows from Theorem 3.1.

(ii) $\iff$ *(iii)* holds by Theorem 2.5.

(ii) $\implies$ *(iv)*: Let (X,d) be a complete K-quasimetric space and let $T : X \to X$ be a Caristi mapping with an associated lower semicontinuous, bounded from below function ϕ. Hence for any $x \in X$,

$$\rho(x,Tx) \leqslant d(x,Tx) \leqslant \phi(x) - \phi(Tx).$$

Moreover, since ρ and d are uniformly equivalent, (X,ρ) is also complete and ϕ is lower semicontinuous with respect to ρ, so by Caristi's theorem [9] T has a fixed point.

$(iv) \implies (v)$ is obvious.

(v) $\implies$ *(i)*: Suppose to the contrary that $K > 2$. Set $p := \log_2 K$. Then $p > 1$ and $K = 2^p$. Clearly, if $\eta(t) := t^p$ for $t \geqslant 0$, then $\eta(0) = \eta'(0) = 0$, so by [21, Theorem 7], there exist a complete metric space (X,ρ) and a continuous and asymptotically regular mapping $T : X \to X$ such that

$$\rho^p(x,Tx) \leqslant \phi(x) - \phi(Tx) \quad \text{for} \quad x \in X,$$

where $\phi : X \to [0,\infty)$ is continuous but T has no fixed point. Set

$$d(x,y) := \rho^p(x,y) \quad \text{for} \quad x,y \in X.$$

By [26, Remark 1.1], d is a K-quasimetric. Moreover, it is easily seen that d and ρ are uniformly equivalent, so (X,d) is complete and T is a continuous and asymptotically regular Caristi mapping with respect to d, which contradicts (v) since T has no fixed point.

(i) $\implies$ *(vi)* follows from Frink's construction used in the proof of Theorem 3.1.

(vi) $\implies$ *(vii)* is obvious.

(vii) $\implies$ *(i)*: Suppose to the contrary that $K > 2$. Set $p := \log_2 K$ and consider the set $X := \mathbb{R}$ equipped with a quasimetric d given by the following formula:

$$\forall_{x,y\in\mathbb{R}}\, d(x,y) := |x-y|^p.$$

Again, by [26, Remark 1.1], d is a K-quasimetric. Take a pair of arbitrary distinct points $x_0, y_0 \in \mathbb{R}$. We will show that $\rho_{\inf}(x_0,y_0) = 0$. Without loss of generality we may assume that $x_0 < y_0$. Consider the following sequence of *paths* leading from x_0 to y_0:

$$x_i^{(j)} := \left(1 - \frac{i}{j}\right) \cdot x_0 + \frac{i}{j} \cdot y_0,$$

for all $j \in \mathbb{N}$ and $i \in \{0,1,\ldots,j\}$. Thus in the j-th step we split the distance between x_0 and y_0 into j equal fragments, each of the length $\left(\frac{1}{j}|x_0 - y_0|\right)^p$

(described by the semi-metric d). From the definition of $\rho_{\inf}$ we obtain that

$$\begin{aligned}\rho_{\inf}(x_0,y_0) &= \sum_{i=0}^{j-1} d(x_i^{(j)},x_{i+1}^{(j)}) = \sum_{i=0}^{j-1}\left(\frac{1}{j}|x_0-y_0|\right)^p\\ &= \left(\frac{j}{j^p}|x_0-y_0|^p\right) = \frac{|x_0-y_0|^p}{j^{p-1}} \xrightarrow{j\to\infty} 0.\end{aligned}$$

Since j can be arbitrarily large, $\rho_{\inf}(x_0,y_0)$ has to be equal to 0. Due to the fact that both x_0 and y_0 were chosen arbitrarily, $\rho_{\inf}$ vanishes everywhere on X, which yields a contradiction. $\square$

3.2. The second refinement

The restrictions imposed on the constant K in both Theorem 3.3 and Frink's Theorem 3.1 can be omitted if the quasimetric d is replaced by d raised to a proper power $p > 0$. The advantage of this method over the previous one is that it allows us (by manipulating the power p) to obtain arbitrarily narrow metric bounds. This is established precisely in the following

Proposition 3.9. *Let (X,d) be a K-quasimetric space. Then for any $\varepsilon > 0$, there exist $p \in (0,1]$ and a metric ρ for which*

$$\forall_{x,y\in X}\ \rho(x,y) \leqslant d^p(x,y) \leqslant (1+\varepsilon)\rho(x,y). \tag{3.3}$$

Proof. The case where $K = 1$ is obvious, so we assume further that $K > 1$. Put $q := \min\{1, \log_K 2\}$. Then, $q \in (0,1]$ and d^q is a K'-quasimetric with $K' \leqslant 2$. By Frink's Theorem 3.1, there exists a metric D on X, for which

$$\forall_{x,y\in X}\ D(x,y) \leqslant d^q(x,y) \leqslant 4D(x,y). \tag{3.4}$$

Fix $\varepsilon > 0$. Since $4^{\frac{1}{n}} \to 1$, there exists $k \in \mathbb{N}$ such that $4^{\frac{1}{k}} \leqslant 1+\varepsilon$. Define a function f by $f(x) := x^{\frac{1}{k}}$ for $x \geqslant 0$. Applying f to both sides of (3.4), we obtain that for any $x,y \in X$,

$$D^{\frac{1}{k}}(x,y) \leqslant d^{\frac{q}{k}}(x,y) \leqslant 4^{\frac{1}{k}} D^{\frac{1}{k}}(x,y) \leqslant (1+\varepsilon)D^{\frac{1}{k}}(x,y).$$

By Lemma 3.4, if $\rho := D^{\frac{1}{k}}$, then ρ is a metric and putting $p := \frac{q}{k}$ we get that $\rho \leqslant d^p \leqslant (1+\varepsilon)\rho$. Clearly, $p \in (0,1]$. $\square$

Checking whether a given function fulfills the first two axioms of a metric is, in general, not problematic. The difficulty is usually to determine if the condition (Q3) is satisfied. Sometimes it is easier to verify another condition, which is equivalent to either (Q3) or (Q3′). The next theorem provides such conditions. However, it is worth noting here that a similar result appeared in [3] (see [3, Proposition 4.1]), where the authors proved the implication (i) $\implies$ (ii) formulated below. Here we expand the list with two additional conditions. We say that a function $\varphi : [0, \infty) \to [0, \infty)$ satisfies condition Δ_2 with a constant C if

$$\varphi(2t) \leqslant C\varphi(t) \quad \text{for all} \quad t \geqslant 0.$$

Theorem 3.10. *Let (X, d) be a semimetric space. The following conditions are equivalent:*

(i) d is a K-quasimetric for some $K \geqslant 1$;
(ii) for every non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ fulfilling Δ_2 and such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi \circ d$ is a quasimetric;
(iii) for every $\varepsilon > 0$ there exists $p \in (0, 1]$ and a metric ρ on X such that $\rho(x, y) \leqslant d^p(x, y) \leqslant (1 + \varepsilon)\rho(x, y)$ for all $x, y \in X$;
(iv) there exists $p \in (0, 1]$ such that d^p fulfills the c-relaxed polygonal inequality.

Proof. *(i) ⇒ (ii)*: Assume that d is a K-quasimetric. Let φ be as in (ii), where condition Δ_2 is satisfied with a constant C. Of course the condition $\varphi^{-1}(\{0\}) = \{0\}$ guarantees that $\varphi \circ d$ fulfills $(Q1)$ and it is obvious that $\varphi \circ d$ is symmetric. Let $x, y, z \in X$. Let $n_0 \in \mathbb{N}$ be such that $\frac{K}{2^{n_0}} \leqslant 1$. Then:

$$\begin{aligned}\varphi(d(x,z)) &\leqslant \varphi\left(K \max\{d(x,y), d(y,z)\}\right) \leqslant C\varphi\left(\frac{K}{2} \max\{d(x,y), d(y,z)\}\right) \\ &\leqslant \cdots \leqslant C^{n_0}\varphi\left(\frac{K}{2^{n_0}} \max\{d(x,y), d(y,z)\}\right) \\ &\leqslant C^{n_0}\varphi\left(\max\{d(x,y), d(y,z)\}\right) = C^{n_0}\max\{\varphi\left(d(x,y)\right), \varphi\left(d(y,z)\right)\},\end{aligned}$$

which means that $\varphi \circ d$ is a C^{n_0}-quasimetric.

(ii) ⇒ (iii): First note that $\varphi = id$ satisfies the conditions in (ii), so d is a quasimetric. Fix $\varepsilon > 0$. By Proposition 3.9, there exist $p \in (0, 1)$ and a metric ρ such that $\rho(x, y) \leqslant d^p(x, y) \leqslant (1 + \varepsilon)\rho(x, y)$ for all $x, y \in X$.

(iii) ⇒ (iv): This implication is due to Fagin et al. [14,15].

(iv) ⇒ (i): Assume that d^p fulfills the c-relaxed polygonal inequality. Clearly, in particular, d^p is then a quasimetric. Hence, d is a quasimetric as well, because $\varphi(t) = t^{\frac{1}{p}}$ fulfills the conditions from (ii), so we may refer to (i) $\implies$ (ii) with d replaced by d^p. □

4. Theorems of Cantor and Banach in a quasimetric setting

In this section we generalise the Cantor intersection theorem onto quasimetric spaces, presenting also its more general version. Moreover, we derive from it the Banach Fixed-Point Theorem for quasimetric spaces, which was first proved by Bakhtin [4]. Then in [1] the authors showed that the quasimetric version can be obtained from its metric counterpart using remetrization techniques. We prove that a more general version of the Banach Contraction Principle is also true for quasimetric spaces and present a proof of it using the generalised version of Cantor's theorem.

First, let us start with a technical lemma which will be of use in the proof of the Cantor intersection theorem for quasimetric spaces.

Lemma 4.1. *Let d_1 and d_2 be two semimetrics defined on a nonempty set X. The following conditions are equivalent:*

(i) *d_1 and d_2 are uniformly equivalent;*
(ii) *for every sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of X, the following equivalence holds:*

$$\operatorname{diam}_{d_1} A_n \to 0 \iff \operatorname{diam}_{d_2} A_n \to 0,$$

where $\operatorname{diam}_{d_i} A := \sup_{x,y\in A} d_i(x,y)$ *is the diameter of a set $A \subset X$ with respect to the semimetric d_i for $i = 1, 2$.*

Proof. *(i)* $\implies$ *(ii)*: Assume that d_1 is uniformly equivalent to d_2 and let (A_n) be a sequence of subsets of X such that $\operatorname{diam}_{d_1} A_n \to 0$. We will show that $\operatorname{diam}_{d_2} A_n \to 0$. Fix $\varepsilon > 0$. Then there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for any $x, y \in X$, if $d_1(x,y) < \delta$, then $d_2(x,y) < \varepsilon$ and $\operatorname{diam}_{d_1} A_n < \delta$ for each $n \geqslant n_0$. Hence, for $n \geqslant n_0$ and $x, y \in A_n$, we have that $d_1(x,y) \leqslant \operatorname{diam}_{d_1} A_n < \delta$, so $d_2(x,y) < \varepsilon$, which implies that $\operatorname{diam}_{d_2} A_n \leqslant \varepsilon$ for $n \geqslant n_0$. This shows that $\operatorname{diam}_{d_2} A_n \to 0$. Now, by interchanging the roles of d_1 and d_2, we obtain that the stated equivalence holds.

(ii) $\implies$ *(i)*: Suppose to the contrary that d_1 and d_2 are not uniformly equivalent. Without loss of generality we may assume that the first condition from Definition 2.6 does not hold, i.e., there exists $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there exist points x_n, y_n in X such that

$$d_1(x_n, y_n) < \frac{1}{n} \text{ and } d_2(x_n, y_n) \geqslant \varepsilon_0.$$

If we set $A_n := \{x_n, y_n\}$ for $n \in \mathbb{N}$, then $\operatorname{diam}_{d_1} A_n \to 0$, but $\operatorname{diam}_{d_2} A_n \geqslant \varepsilon_0$ which yields a contradiction. □

Now with the help of Lemma 4.1 and remetrization techniques, we will prove the Cantor intersection theorem for semimetric spaces satisfying $(W5)$.

Theorem 4.2. *Let (X, d) be a complete semimetric space in which (W5) holds. Let $(A_n)_{n\in\mathbb{N}}$ be a descending sequence of closed nonempty subsets of X such that* $\operatorname{diam} A_n \to 0$. *Then $\bigcap_{n\in\mathbb{N}} A_n = \{x_*\}$ for some $x_* \in X$.*

Proof. Due to [11, Theorem 3.2], there exists a metric ρ which is uniformly equivalent to d on X. Clearly, (X, ρ) is then complete. By Lemma 4.1, since $\operatorname{diam}_d A_n \to 0$, we get that $\operatorname{diam}_\rho A_n \to 0$, so the result follows from the classic Cantor's intersection theorem. □

It is well known that the classic Cantor's intersection theorem yields the Banach Contraction Principle. This observation is due to Boyd and Wong [6] (see also [18, p. 8]), who proved that if T is a Banach contraction on a metric space (X, ρ), then sets A_n defined by

$$A_n := \left\{x \in X : \rho(x, Tx) \leqslant \frac{1}{n}\right\} \text{ for } n \in \mathbb{N},$$

satisfy the assumptions of Cantor's theorem. Thus, a natural question arises whether Theorem 4.2 could be used to prove the following quasimetric version of the Banach Contraction Principle established by Bakhtin [4]. We denote by $L(T)$ the Lipschitz constant of a mapping T.

Theorem 4.3 (Bakhtin). *Let (X, d) be a complete quasimetric space, $T : X \to X$ be Lipschitzian with $L(T) \in [0, 1)$. Then T has a unique fixed point $x_* \in X$ and for any $x \in X$, $T^n x \to x_*$.*

However, it turns out that, in general, Bakhtin's theorem cannot be proved via Cantor's intersection theorem for quasimetric spaces. This is caused by the fact that, as shown in Example 4.4 given below, there exists a complete quasimetric space (X, d) (even satisfying a c-relaxed polygonal inequality) and a Banach contraction $T : X \to X$ such that for any $\varepsilon > 0$, the set

$$\operatorname{Fix}_\varepsilon T := \{x \in X : d(x, Tx) \leqslant \varepsilon\}$$

is not closed. Consequently, there does not exist a sequence (α_n) such that $\alpha_n \searrow 0$ and the sets $\operatorname{Fix}_{\alpha_n} T$ satisfy the assumptions of Theorem 4.2, so the Boyd-Wong [6] trick does not work in this case.

Example 4.4. Let $X = \mathbb{R}_+ := [0, \infty)$, $\mathbb{Q}_+ := \mathbb{Q} \cap [0, \infty)$, $Tx := \frac{x}{\sqrt{5}}$ for $x \in X$ and for $x, y \in X$, let

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in \mathbb{Q} \text{ or } x, y \notin \mathbb{Q}; \\ 2|x - y|, & \text{elsewhere.} \end{cases}$$

Note that $|x - y| \leqslant d(x, y) \leqslant 2|x - y|$. Hence, (X, d) satisfies the 2-rpi and if $x, y \in X$, then

$$d(Tx, Ty) \leqslant 2|Tx - Ty| = \frac{2}{\sqrt{5}}|x - y| \leqslant \frac{2}{\sqrt{5}} d(x, y),$$

so $L(T) \leqslant \frac{2}{\sqrt{5}} < 1$. Let $x \in X$ and consider the following cases:

(a) if $x \in \mathbb{Q}$, or $x \notin \mathbb{Q}$ and $\frac{x}{\sqrt{5}} \in \mathbb{Q}$, then $d(x, Tx) = 2x\left(1 - \frac{1}{\sqrt{5}}\right)$;

(b) if $x \notin \mathbb{Q}$ and $\frac{x}{\sqrt{5}} \notin \mathbb{Q}$, then $d(x, Tx) = x\left(1 - \frac{1}{\sqrt{5}}\right)$.

Let $\varepsilon > 0$. In case (a), $d(x, Tx) \leqslant \varepsilon$ if and only if $\frac{2x}{\sqrt{5}}(\sqrt{5} - 1) \leqslant \varepsilon$, i.e., $x \leqslant \frac{5+\sqrt{5}}{8}\varepsilon$. In case (b), $d(x, Tx) \leqslant \varepsilon$ if and only if $\frac{x}{\sqrt{5}}(\sqrt{5} - 1) \leqslant \varepsilon$, i.e.,

$x \leqslant \frac{5+\sqrt{5}}{4}\varepsilon$. Hence we may infer that

$$\mathrm{Fix}_\varepsilon T := \left[0, \frac{5+\sqrt{5}}{8}\varepsilon\right] \cup \left(\left(\frac{5+\sqrt{5}}{8}\varepsilon, \frac{5+\sqrt{5}}{4}\varepsilon\right] \cap (\mathbb{R}\setminus\mathbb{Q}) \cap (\mathbb{R}\setminus A)\right),$$

where $A := \{\frac{m}{n}\sqrt{5} : m, n \in \mathbb{N}\}$. It is easily seen that $(\mathbb{R}\setminus\mathbb{Q}) \cap (\mathbb{R}\setminus A)$ is dense in $\mathbb{R}_+$. This implies that for any open set $U \subset \mathbb{R}_+$, $\overline{U \cap (\mathbb{R}\setminus\mathbb{Q}) \cap (\mathbb{R}\setminus A)} = \overline{U}$. Hence we easily obtain that $\overline{\mathrm{Fix}_\varepsilon T} = \left[0, \frac{5+\sqrt{5}}{4}\varepsilon\right]$, so $\mathrm{Fix}_\varepsilon T$ is not closed.

Now we will generalise Theorem 4.2 to get a version of Cantor's theorem which yields the Banach Contraction Principle for mappings on quasimetric spaces.

Theorem 4.5. (Generalized Cantor's intersection theorem for semimetric spaces) *Let (X, d) be a complete semimetric space satisfying (W5), $(A_n)_{n\in\mathbb{N}}$ be a descending sequence of nonempty subsets of X such that* $\mathrm{diam}\, A_n \to 0$ *and there exists a subsequence (A_{k_n}) such that for any $n \in \mathbb{N}$, $\overline{A}_{k_n} \subset A_n$. Then $\bigcap_{n\in\mathbb{N}} A_n = \{x_*\}$ for some $x_* \in X$.*

Proof. We will apply Theorem 4.2 to the sequence $(\overline{A}_{k_n})$. Of course, $(\overline{A}_{k_n})$ is a descending sequence of closed sets with diameters tending to 0, since $\mathrm{diam}\, \overline{A}_{k_n} \leqslant \mathrm{diam}\, A_n$. Moreover,

$$\bigcap_{n\in\mathbb{N}} A_n \subset \bigcap_{n\in\mathbb{N}} A_{k_n} \subset \bigcap_{n\in\mathbb{N}} \overline{A}_{k_n} \subset \bigcap_{n\in\mathbb{N}} A_n,$$

which implies that $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \overline{A}_{k_n} = \{x_*\}$ for some $x_* \in X$. $\square$

We will now present a proof of the quasimetric version of the Banach Fixed Point Theorem using Theorem 4.5. For this purpose let us recall some basic notions. For a semimetric space (X, d) and a mapping $T : X \to X$, by $\mathrm{Fix}\, T$ we understand the set of all fixed points of T. Moreover, recall that if $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $\alpha_n \searrow 0$, then we denote

$$A_n := \{x \in X : d(x, Tx) \leqslant \alpha_n\} = \mathrm{Fix}_{\alpha_n} T.$$

The following result explains when the sequence (A_n) satisfies the assumptions of Theorem 4.5. For metric spaces, the condition (2) of Lemma 4.6 was established in [21].

Lemma 4.6. *Let (X, d) be a semimetric space and let T, (α_n), (A_n) be as defined above. Then we have:*

1. *there exists a subsequence (A_{k_n}) such that $\overline{A_{k_n}} \subset A_n$ for each $n \in \mathbb{N}$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{\mathrm{Fix}_\delta T} \subset \mathrm{Fix}_\varepsilon T$;*
2. $\mathrm{diam}\, A_n \to 0$ *if and only if for any sequences $(x_n), (y_n)$ of elements of X, $d(x_n, Tx_n) \to 0$ and $d(y_n, Ty_n) \to 0$ imply that $d(x_n, y_n) \to 0$.*

Proof. Ad 1. '$\implies$': Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\alpha_n < \varepsilon$ and put $\delta := \alpha_{k_n}$. Then

$$\overline{\mathrm{Fix}_\delta T} = \overline{A}_{k_n} \subset A_n \subset \mathrm{Fix}_\varepsilon T.$$

'$\impliedby$': Let $n \in \mathbb{N}$ and $\varepsilon = \alpha_n$. From the assumption, there exists $\delta_n > 0$ such that $\overline{\mathrm{Fix}_{\delta_n} T} \subset \mathrm{Fix}_{\alpha_n} T = A_n$. Choose $m_n \in \mathbb{N}$ such that for all $j \geqslant m_n$, $\alpha_j < \delta_n$. We define the sequence (k_n) recursively. Put $k_1 := m_1$. Having defined k_n, put $k_{n+1} := \max\{k_n + 1, m_{n+1}\}$. Then $k_{n+1} > k_n \geqslant m_n$, so $\alpha_{k_n} < \delta_n$ and hence

$$\overline{A}_{k_n} \subset \overline{\mathrm{Fix}_{\delta_n} T} \subset A_n.$$

Ad 2. '$\implies$': Assume that $\operatorname{diam} A_n \to 0$. Let (x_n) and (y_n) be such that $d(x_n, Tx_n) \to 0$ and $d(y_n, Ty_n) \to 0$. Fix $\varepsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\operatorname{diam} A_k < \varepsilon$. Since $\alpha_k > 0$, there is $p \in \mathbb{N}$ such that for each $n \geqslant p$, $d(x_n, Tx_n) \leqslant \alpha_k$ and $d(y_n, Ty_n) \leqslant \alpha_k$, i.e., $x_n, y_n \in A_k$. Hence

$$d(x_n, y_n) \leqslant \operatorname{diam} A_k < \varepsilon \quad \text{for all} \quad n \geqslant p,$$

so $d(x_n, y_n) \to 0$.

'$\impliedby$': Since $A_{n+1} \subset A_n$, we have that $\operatorname{diam} A_{n+1} \leqslant \operatorname{diam} A_n$ for each $n \in \mathbb{N}$. Hence $\operatorname{diam} A_n \searrow r$ for some $r \in [0, \infty]$. Suppose to the contrary that $r > 0$. Choose any $s \in (0, r)$. Then $s < \infty$ and $\operatorname{diam} A_n > s$ for any $n \in \mathbb{N}$, so there exist x_n and y_n in A_n such that $d(x_n, y_n) > s$. On the other hand,

$$d(x_n, Tx_n) \leqslant \alpha_n \quad \text{and} \quad d(y_n, Ty_n) \leqslant \alpha_n,$$

so $d(x_n, Tx_n) \to 0$ and $d(y_n, Ty_n) \to 0$. By hypothesis, $d(x_n, y_n) \to 0$. Hence, since $d(x_n, y_n) > s$, we obtain, letting n tend to ∞, that $0 \geqslant s > 0$, which yields a contradiction. □

We will also need the following two lemmas.

Lemma 4.7. *Let (X, d) be a quasimetric space satisfying (Q3′) with a constant $M \geqslant 1$, and let $T : X \to X$ be a Lipschitzian mapping with the Lipschitz constant α. Then for any $x, y \in X$ and $k \in \mathbb{N}$, the following inequalities hold:*

(1) $d(x, y)(1 - \alpha M^2) \leqslant M d(x, Tx) + M^2 d(y, Ty)$;
(2) *if* $x_* = Tx_*$, *then* $d(x, x_*)(1 - \alpha M) \leqslant M d(x, Tx)$;
(3) $d(x, T^k x) \leqslant d(x, Tx)\left(M \frac{1-\alpha^{k-1}M^{k-1}}{1-\alpha M} + \alpha^{k-1}M^{k-1}\right)$, *if* $\alpha \neq \frac{1}{M}$;
(4) $d(x, T^k x) \leqslant d(x, Tx)\,(M(k-1) + 1)$ *if* $\alpha = \frac{1}{M}$;
(5) $d(x, Tx) \leqslant d(x, y) M\,(1 + \alpha M) + M^2 d(y, Ty)$.

Moreover, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{\mathrm{Fix}_\delta T} \subset \mathrm{Fix}_\varepsilon T$.

Proof. By (Q3′) used twice, we obtain

$$\begin{aligned} d(x, y) &\leqslant M\,(d(x, Tx) + M\,(d(Tx, Ty) + d(Ty, y))) \\ &\leqslant \alpha M^2 d(x, y) + M d(x, Tx) + M^2 d(y, Ty), \end{aligned}$$

which yields (1). Similarly,

$$d(x,x_*) \leqslant M\left(d(x,Tx)+d(Tx,Tx_*)\right) \leqslant \alpha M d(x,x_*) + M d(x,Tx),$$

so (2) holds. We show that (3) and (4) are satisfied. We have that

$$\begin{aligned} d(x,T^k x) &\leqslant M\left(d(x,Tx)+\alpha d(x,T^{k-1}x)\right) \\ &\leqslant M\left(d(x,Tx)+\alpha M\left(d(x,Tx)+\alpha d(x,T^{k-2}x)\right)\right). \end{aligned}$$

Continuing in this fashion we obtain that

$$d(x,T^k x) \leqslant d(x,Tx)\left(\sum_{j=1}^{k-1}\alpha^{j-1}M^j + \alpha^{k-1}M^{k-1}\right),$$

which yields both (3) and (4). To show (5) we again use (Q3′) twice to get that

$$d(x,Tx) \leqslant M\left(d(x,y)+M\left(d(y,Ty)+\alpha d(x,y)\right)\right),$$

so (5) holds.

Finally, fix $\varepsilon > 0$ and set $\delta := \frac{\varepsilon}{M^2}$. Let (x_n) be such that $d(x_n,Tx_n) \leqslant \delta$ and $x_n \to x$. Then, by (5),

$$\begin{aligned} d(x,Tx) &\leqslant d(x,x_n)M\left(1+\alpha M\right) + M^2 d(x_n,Tx_n) \\ &\leqslant d(x,x_n)M(1+\alpha M) + M^2\delta. \end{aligned}$$

Letting n tend to infinity, we obtain that

$$d(x,Tx) \leqslant M^2\delta = \varepsilon.$$

□

The following result is well known for selfmaps of metric spaces. We omit the proof, since it does not differ from its metric version.

Lemma 4.8. *Let T be a selfmap of a semimetric space such that for some $k \in \mathbb{N}$, T^k has a unique fixed point x_* and for any $x_0 \in X$,*

$$\lim_{n\to\infty} d\left(T^{kn}x_0, x_*\right) = 0.$$

Then $\operatorname{Fix} T = \{x_*\}$ *and for any $x_0 \in X$,*

$$\lim_{n\to\infty} d\left(T^{n}x_0, x_*\right) = 0.$$

We can now proceed to the aforementioned alternative proof of the Banach Theorem 4.3. It will be more convenient for us to work with condition (Q3′). In fact, we will prove the following more general result.

Theorem 4.9. *Let (X,d) be a complete quasimetric space satisfying (Q3′) with a constant $M \geqslant 1$, i.e., for any $x,y,z \in X$,*

$$d(x,y) \leqslant M\left(d(x,z)+d(z,y)\right).$$

Let a mapping $T : X \to X$ be Lipschitzian with the Lipschitz constant α. Assume that for some $p \in \mathbb{N}$, $\beta := L(T^p) < 1$ and let $q \in \mathbb{N}$ be such that $\beta^q < \frac{1}{M^2}$. Then T has a unique fixed point $x_* \in X$ and for every $x_0 \in X$, $T^n x_0 \to x_*$. Moreover, for any sequence (x_n), if $d(x_n, Tx_n) \to 0$, then $x_n \to x_*$ and

(a) *if $\alpha \neq \frac{1}{M}$ and $\beta \neq \frac{1}{M}$, then*

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\beta^q M} \left(M \frac{1-\beta^{q-1}M^{q-1}}{1-\beta M} + \beta^{q-1}M^{q-1} \right) \cdot \\ \cdot \left(M \frac{1-\alpha^{p-1}M^{p-1}}{1-\alpha M} + \alpha^{p-1}M^{p-1} \right);$$

(b) *if $\alpha = \frac{1}{M}$ and $\beta \neq \frac{1}{M}$, then*

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\beta^q M} \left(M \frac{1-\beta^{q-1}M^{q-1}}{1-\beta M} + \beta^{q-1}M^{q-1} \right) \cdot \\ \cdot (M(p-1)+1);$$

(c) *if $\alpha \neq \frac{1}{M}$ and $\beta = \frac{1}{M}$, then*

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\beta^{q-1}} (M(q-1)+1) \cdot \\ \cdot \left(M \frac{1-\alpha^{p-1}M^{p-1}}{1-\alpha M} + \alpha^{p-1}M^{p-1} \right);$$

(d) *if $\alpha = \beta = \frac{1}{M}$, then*

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\alpha^{q-1}} (M(q-1)+1).$$

Proof. We divide the proof into three steps.

Step 1 Assume that $\alpha < \frac{1}{M^2}$. Set

$$A_n := \left\{ x \in X \ : \ d(x, Tx) \leqslant \frac{1}{n} \right\} \quad \text{for} \quad n \in \mathbb{N}.$$

By Lemmas 4.6 and 4.7, there exists a subsequence (A_{k_n}) such that $\overline{A_{k_n}} \subset A_n$ for each $n \in \mathbb{N}$. Since $\alpha < 1$ and for any $x \in X$, $d(T^n, T^{n+1}x) \leqslant \alpha^n d(x, Tx)$, we may infer that each A_n is nonempty. We show that $\operatorname{diam} A_n \to 0$. If $x, y \in A_n$, then by Lemma 4.7 (1), taking into account that $1 - \alpha M^2 > 0$, we get that

$$d(x, y) \leqslant \frac{M}{1-\alpha M^2} (d(x, Tx) + M d(y, Ty)) \leqslant \frac{M(M+1)}{1-\alpha M^2} \cdot \frac{1}{n},$$

which yields that $\operatorname{diam} A_n \to 0$. By Theorem 4.5, $\operatorname{Fix} T = \bigcap_{n\in\mathbb{N}} A_n = \{x_*\}$ for some $x_* \in X$. Now let (x_n) be such that $d(x_n, Tx_n) \to 0$. Since $\alpha < \frac{1}{M^2} \leqslant \frac{1}{M}$, we obtain by Lemma 4.7 (2) that

$$d(x_n, x_*) \leqslant \frac{M}{1-\alpha M} d(x_n, Tx_n), \tag{4.1}$$

so $x_n \to x_*$. In particular, if $x_0 \in X$ and $x_n := T^n x_0$, then $d(x_n, Tx_n) \to 0$, so by (4.1) we infer that

$$d(T^n x_0, x_*) \leqslant \frac{M\alpha^n}{1-\alpha M} d(x_0, Tx_0)$$

and hence $T^n x_0 \to x_*$.

Step 2 Now assume that $\alpha < 1$. Then there exists $k \in \mathbb{N}$ such that $\alpha^k < \frac{1}{M^2}$. By Step 1, substituting T for T^k, we obtain that $\operatorname{Fix} T^k = \{x_*\}$ and $\lim\limits_{n\to\infty} T^{kn} x_0 = x_*$ for any $x_0 \in X$. By Lemma 4.8, $\operatorname{Fix} T = \{x_*\}$ and $T^n x_0 \to x_*$ for any $x_0 \in X$. Moreover, it follows from Step 1 that if (x_n) is such that $d(x_n, T^k x_n) \to 0$, then $x_n \to x_*$ and since $L(T^k) \leqslant \alpha^k$ we may infer from (4.1) that

$$d(x_n, x_*) \leqslant \frac{M}{1-\alpha^k M} d(x_n, T^k x_n). \tag{4.2}$$

Now assume that (x_n) is such that $d(x_n, Tx_n) \to 0$. By (3), we get that

$$d(x_n, T^k x_n) \leqslant d(x_n, Tx_n) \left(M \frac{1-\alpha^{k-1}M^{k-1}}{1-\alpha M} + \alpha^{k-1}M^{k-1} \right), \tag{4.3}$$

if $\alpha \neq \frac{1}{M}$, and

$$d(x_n, T^k x_n) \leqslant d(x_n, Tx_n)(M(k-1)+1), \tag{4.4}$$

if $\alpha = \frac{1}{M}$. In both cases $d(x_n, T^k x_n) \to 0$, so $x_n \to x_*$ and by (4.2), (4.3) and (4.4), we get that if $\alpha \neq \frac{1}{M}$, then

$$d(x_n, x_*) \leqslant \frac{M}{1-\alpha^k M} \left(M \frac{1-\alpha^{k-1}M^{k-1}}{1-\alpha M} + \alpha^{k-1}M^{k-1} \right) d(x_n, Tx_n), \tag{4.5}$$

and if $\alpha = \frac{1}{M}$, then since $\alpha^k M = \alpha^{k-1}$, we get that

$$d(x_n, x_*) \leqslant \frac{M}{1-\alpha^{k-1}} \left(M(k-1)+1 \right) d(x_n, Tx_n). \tag{4.6}$$

(Let us notice that in the latter case, $k > 1$, so $1 - \alpha^{k-1} > 0$; otherwise $\frac{1}{M} = \alpha^1 < \frac{1}{M^2} \leqslant \frac{1}{M}$, a contradiction.)

Step 3 Finally, let α be an arbitrary nonnegative real. By hypothesis, $\beta = L(T^p) < 1$ and $\beta^q < \frac{1}{M^2}$. By Step 2, replacing T, α and k by T^p, β and q, respectively, we obtain with the help of Lemma 4.8 that $\operatorname{Fix} T = \{x_*\}$ and $T^n x_0 \to x_*$ for any $x_0 \in X$. Moreover, (4.5) and (4.6) imply that if $\beta \neq \frac{1}{M}$, then

$$d(x_n, x_*) \leqslant \frac{M}{1-\beta^q M} \left(M \frac{1-\alpha^{q-1}M^{q-1}}{1-\alpha M} + \alpha^{q-1}M^{q-1} \right) d(x_n, T^p x_n), \tag{4.7}$$

and if $\beta = \frac{1}{M}$, then

$$d(x_n, x_*) \leqslant \frac{M}{1-\beta^{q-1}} \left(M(q-1)+1 \right) d(x_n, T^p x_n). \tag{4.8}$$

Now using Lemma 4.7 ((3) and (4)) with $x := x_n$ and $k := p$, we easily obtain from (4.7) and (4.8) all the four inequalities given in Theorem 4.9. □

Remark 4.10. Under the assumptions of Theorem 4.9 we have that

$$d(T^n x_0, T^{n+1} x_0) \to 0 \text{ for any } x_0 \in X.$$

Thus, the four inequalities established in Theorem 4.9 enable us to give the a posteriori error estimates for a sequence of iterates of T. For example, if $\alpha \neq \frac{1}{M}$ and $\beta \neq \frac{1}{M}$, then we have that

$$d(T^n x_0, x_*) \leqslant d(T^n x_0, T^{n+1} x_0) \frac{M}{1-\beta^q M}\Big(M\frac{1-\beta^{q-1}M^{q-1}}{1-\beta M} + \\ +\beta^{q-1}M^{q-1}\Big) \cdot \left(M\frac{1-\alpha^{p-1}M^{p-1}}{1-\alpha M} + \alpha^{p-1}M^{p-1}\right).$$

Remark 4.11. Following De Blasi and Myjak [12] we say that the fixed point problem (FPP in short) for a mapping T is *well-posed* if T has a unique fixed point x_* and for any sequence (x_n), $d(x_n, Tx_n) \to 0$ implies that $x_n \to x_*$. Thus, by Theorem 4.9, the FPP for a Lipschitzian mapping having a contractive iterate is well-posed.

Putting $p = 1$ in Theorem 4.9 and taking into account that in this case $\alpha = \beta$, we obtain the following result, which is still an extension of Bakhtin's Theorem 4.3.

Corollary 4.12. *Let (X, d) be a complete quasimetric space, satisfying (Q3′) with a constant $M \geqslant 1$, i.e., for any $x, y, z \in X$,*

$$d(x,y) \leqslant M\left(d(x,z) + d(z,y)\right).$$

Let a mapping $T : X \to X$ be Lipschitzian with $\alpha := L(T) < 1$, and let $q \in \mathbb{N}$ be such that $\alpha^q < \frac{1}{M^2}$. Then T has a unique fixed point $x_* \in X$. Moreover, for any sequence (x_n), if $d(x_n, Tx_n) \to 0$, then $x_n \to x_*$ and

(a) if $\alpha \neq \frac{1}{M}$, then

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\alpha^q M}\left(M\frac{1-\alpha^{q-1}M^{q-1}}{1-\alpha M} + \alpha^{q-1}M^{q-1}\right);$$

(b) if $\alpha = \frac{1}{M}$, then

$$d(x_n, x_*) \leqslant d(x_n, Tx_n) \frac{M}{1-\alpha^{q-1}}\left(M(q-1)+1\right).$$

In particular, for any $x_0 \in X$, $T^n x_0 \to x_*$ and

(a) if $\alpha \neq \frac{1}{M}$, then

$$d(T^n x_0, x_*) \leqslant d(x_0, Tx_0) \frac{\alpha^n M}{1-\alpha^q M}\left(M\frac{1-\alpha^{q-1}M^{q-1}}{1-\alpha M} + \alpha^{q-1}M^{q-1}\right);$$

(b) if $\alpha = \frac{1}{M}$*, then*

$$d(T^n x_0, x_*) \leqslant d(x_0, Tx_0) \frac{\alpha^{n-1}}{1-\alpha^{q-1}} \left(M(q-1)+1\right).$$

On the other hand, in the case where $M = 1$, we obtain the following metric version of Theorem 4.9. Let us note that we may then put $q = 1$.

Corollary 4.13. *Let* (X, ρ) *be a complete metric space and let a mapping* $T : X \to X$ *be Lipschitzian with the Lipschitz constant* α*. Assume that for some* $p \in \mathbb{N}$, $\beta := L(T^p) < 1$*. Then* T *has a unique fixed point* $x_* \in X$ *and for any* $x_0 \in X$, $T^n x_0 \to x_*$*. Moreover, for any sequence* (x_n)*, if* $\rho(x_n, Tx_n) \to 0$*, then* $x_n \to x_*$ *and*

(a) if $\alpha \neq 1$*, then*

$$\rho(x_n, x_*) \leqslant \rho(x_n, Tx_n) \frac{1}{1-\beta} \frac{\alpha^p - 1}{\alpha - 1};$$

(b) if $\alpha = 1$*, then*

$$\rho(x_n, x_*) \leqslant \rho(x_n, Tx_n) \frac{p}{1-\beta}.$$

In particular, for any $x_0 \in X$*, we get the following a posteriori error estimates:*

(a) if $\alpha \neq 1$*, then*

$$\rho(T^n x_0, x_*) \leqslant \rho(T^n x_0, T^{n+1} x_0) \frac{1}{1-\beta} \frac{\alpha^p - 1}{\alpha - 1};$$

(b) if $\alpha = 1$*, then*

$$\rho(T^n x_0, x_*) \leqslant \rho(T^n x_0, T^{n+1} x_0) \frac{p}{1-\beta}.$$

It is worth pointing out that the first part of Corollary 4.13 concerning the existence of a fixed point is often used in applications. In particular, in the proof of the classical Picard-Lindelöf theorem presented in [18, pp. 15–16] it is shown that the integral operator F corresponding to the Cauchy initial value problem is Lipschitzian, but not necessarily contractive. Nevertheless, there exists $p \in \mathbb{N}$ such that $L\left(F^p\right) < 1$, so Corollary 4.13 is applicable. However, we have not found in the literature any information on error estimates in such a case, so the second part of Corollary 4.13 may be new.

Remark 4.14. Step 1 of the proof of Theorem 4.9 shows that if $A_n := \left\{x \in X : d(x, Tx) \leqslant \frac{1}{n}\right\}$ for $n \in \mathbb{N}$ and $L(T)$ is sufficiently small, then the sequence (A_n) satisfies the assumptions of Theorem 4.3. However, since by Theorem 4.9 the fixed point problem for T is well posed, we may infer with the help of point 2 of Lemma 4.6 that in fact (A_n) satisfies the assumptions of Theorem 4.3 without any restrictions on $L(T)$.

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