Another refinement of the right-hand side of the Hermite–Hadamard inequality for simplices

MONIKA NOWICKA AND ALFRED WITKOWSKI

Abstract. We establish a new refinement of the right-hand side of the Hermite–Hadamard inequality for convex functions of several variables defined on simplices.

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The classical Hermite–Hadamard inequality states that if $f : I \rightarrow \mathbb{R}$ is a convex function then for all $a < b \in I$ the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}$$

is valid. This powerful tool has found numerous applications and has been generalized in many directions (see e.g. [1, 2]). One of those directions is its multivariate version:

Theorem 1. ([1]) *Let $f : U \rightarrow \mathbb{R}$ be a convex function defined on a convex set $U \subset \mathbb{R}^n$ and $\Delta \subset U$ be an n -dimensional simplex with vertices $x_0, x_1, \dots, x_n$. Then*

$$f(b_\Delta) \leq \frac{1}{\text{Vol } \Delta} \int_\Delta f(x) dx \leq \frac{f(x_0) + \dots + f(x_n)}{n+1}, \quad (1)$$

where $b_\Delta = \frac{x_0 + \dots + x_n}{n+1}$ is the barycenter of Δ and the integration is with respect to the n -dimensional Lebesgue measure.

The aim of this note is to prove a refinement of the right-hand side of (1) stated in Theorem 2.

Let us start with a set of definitions.

A function $f : I \rightarrow \mathbb{R}$ defined on an interval I is called *convex* if for any $x, y \in I$ and $t \in (0, 1)$ the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds.

If U is a convex subset of $\mathbb{R}^n$, then a function $f : U \rightarrow \mathbb{R}$ is convex if its restriction to every line segment in U is convex.

For $n+1$ points $x_0, \dots, x_n \in \mathbb{R}^n$ in general positions the set $\Delta = \text{conv}\{x_0, \dots, x_n\}$ is called an n -dimensional *simplex*. If K is a nonempty subset of the set $N = \{0, \dots, n\}$ of cardinality k , the set $\Delta_K = \text{conv}\{x_i : i \in K\}$ is called a *face* (or a $k-1$ -face) of Δ . The point $b_K = \frac{1}{k} \sum_{i \in K} x_i$ is called a *barycenter* of Δ_K . The barycenter of Δ will be denoted by b . By $\text{card } K$ we shall denote the cardinality of the set K .

For each $k-1$ -face Δ_K we calculate the average value of f over Δ_K using the formula

$$\text{Avg}(f, \Delta_K) = \frac{1}{\text{Vol}(\Delta_K)} \int_{\Delta_K} f(x) dx,$$

where the integration is with respect to the $k-1$ -dimensional Lebesgue measure (in case $k=1$ this is the counting measure).

For $k=1, 2, \dots, n+1$ we define

$$\mathcal{A}(k) = \frac{1}{\binom{n+1}{k}} \sum_{\substack{K \subset N \\ \text{card } K=k}} \text{Avg}(f, \Delta_K).$$

Note that the right-hand side of the inequality (1) can be rewritten as $\mathcal{A}(n+1) \leq \mathcal{A}(1)$. It turns out, that

Theorem 2. *The following chain of inequalities holds:*

$$\mathcal{A}(n+1) \leq \mathcal{A}(n) \leq \dots \leq \mathcal{A}(2) \leq \mathcal{A}(1).$$

In the proof we shall use the following

Lemma 1. ([3, Theorem 4.1]) *If $K_i = N \setminus \{i\}$ and b is the barycenter of Δ , then*

$$\text{Avg}(f, \Delta) \leq \frac{1}{n+1} f(b) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}).$$

Proof of Theorem 2. We shall prove first the inequality $\mathcal{A}(n+1) \leq \mathcal{A}(n)$. Let us use the notation from Lemma 1. For $i=0, 1, \dots, n$ we have

$$b_{K_i} = \frac{1}{n} \sum_{\substack{j=0 \\ j \neq i}}^n x_j = \frac{1}{n} \left(\sum_{j=0}^n x_j - x_i \right) = \frac{1}{n} ((n+1)b - x_i). \quad (2)$$

Summing (2) we obtain

$$b = \frac{1}{n+1} \sum_{j=0}^n b_{K_j}. \quad (3)$$

Now using Lemma 1 and the convexity of f applied to (3) we get

$$\begin{aligned} \text{Avg}(f, \Delta) &\leq \frac{1}{n+1} f(b) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}) \\ &\leq \frac{1}{n+1} \frac{1}{n+1} \sum_{i=0}^n f(b_{K_i}) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}), \end{aligned}$$

thus, by the left-hand side of (1)

$$\begin{aligned} &\leq \frac{1}{n+1} \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}) \\ &= \frac{1}{n+1} \sum_{i=0}^n \text{Avg}(f, \Delta_{K_i}). \end{aligned}$$

This shows the inequality $\mathcal{A}(n+1) \leq \mathcal{A}(n)$.

Let $K \subset N$ be a set of cardinality $k > 1$. Applying the above reasoning to Δ_K we obtain

$$\text{Avg}(f, \Delta_K) \leq \frac{1}{k} \sum_{\substack{K' \subset K \\ \text{card } K' = k-1}} \text{Avg}(f, \Delta_{K'}).$$

Summing the above for all k -element subsets we get

$$\begin{aligned} \sum_{\substack{K \subset N \\ \text{card } K = k}} \text{Avg}(f, \Delta_K) &\leq \frac{1}{k} \sum_{\substack{K \subset N \\ \text{card } K = k}} \sum_{\substack{K' \subset K \\ \text{card } K' = k-1}} \text{Avg}(f, \Delta_{K'}) \\ &= \frac{n-k+2}{k} \sum_{\substack{K' \subset K \\ \text{card } K' = k-1}} \text{Avg}(f, \Delta_{K'}). \end{aligned}$$

The equality follows from the fact that every $k-2$ -face belongs to $n-k+2$ distinct $k-1$ -faces so every term $\text{Avg}(f, \Delta_{K'})$ appears in the sum exactly $n-k+2$ times. Dividing both sides by $\binom{n+1}{k}$ we get $\mathcal{A}(k) \leq \mathcal{A}(k-1)$, which completes the proof. $\square$

Just for completeness note that a similar refinement of the left-hand side of (1) can be found in [4, Corollary 2.6]. It reads as follows:

Theorem 3. *For a nonempty subset K of N define the simplex Σ_K as follows: let A_K be the affine span of Δ_K and A'_K be the affine space of the same dimension, parallel to A_K and passing through the barycenter of Δ . Then $\Sigma_K = \Delta \cap A'_K$.*

For $k = 1, 2, \dots, n+1$ let

$$\mathcal{B}(k) = \frac{1}{\binom{n+1}{k}} \sum_{\substack{K \subseteq N \\ \text{card } K = k}} \text{Avg}(f, \Sigma_K).$$

Then

$$f(b) = \mathcal{B}(1) \leq \mathcal{B}(2) \leq \dots \leq \mathcal{B}(n+1) = \text{Avg}(f, \Delta).$$

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Monika Nowicka and Alfred Witkowski
 Institute of Mathematics and Physics
 UTP University of Science and Technology
 Al. Prof. Kaliskiego 7
 85-796 Bydgoszcz
 Poland
 e-mail: monika.nowicka@utp.edu.pl

Alfred Witkowski
 e-mail: alfred.witkowski@utp.edu.pl

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