



Vector valued Banach limits and generalizations applied to the inhomogeneous Cauchy equation

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Dedicated to Karol Baron on the occasion of his 70th birthday.

Abstract. In Prager and Schwaiger (Grazer Math Ber 363:171–178, 2015) the classical notion of Banach limits was used to solve the inhomogeneous Cauchy equation $f(x + y) - f(x) - f(y) = \phi(x, y)$ for real functions of one real variable. Here these methods are generalized to more general target spaces, namely Banach spaces which admit vector valued Banach limits.

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1. Introduction

The main topic of this paper is the investigation of the inhomogeneous Cauchy equation

$$f(x + y) - f(x) - f(y) = \phi(x, y), \quad (1)$$

where $\phi : V \times V \rightarrow W$ is given and $f : V \rightarrow W$ is to be determined. The emphasis lies on giving explicit formulas for solutions when certain hypotheses on the inhomogeneity are fulfilled. It is well known (see [7]) that the solvability of this equation is equivalent to the properties

$$\phi(\xi, \eta) = \phi(\eta, \xi) \quad (2)$$

$$\phi(\xi, \eta) + \phi(\xi + \eta, \zeta) = \phi(\eta, \zeta) + \phi(\xi, \eta + \zeta) \quad (3)$$

of the inhomogeneity ϕ . Solving the equation in general is based on the application of Zorn's Lemma. In our case, if the domain and co-domain are rational vector spaces, assuming the existence of a basis for such vector spaces is enough to give *explicit* solutions. In the generic case the existence of such a basis is granted only by applying also Zorn's Lemma. Nevertheless situations may

appear where a basis of a vector space is known a priori. Thus this construction may be interesting in such cases. In [10] solutions of the inhomogeneous Cauchy equation were constructed for $\phi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by using the existence of *Banach limits* on \mathbb{R} ([3]). Just recently this notion has been extended to certain Banach spaces. Here we try to generalize some results from [10] to that case. Since, as in the original case, these generalizations are applicable only if (1) is solvable, the construction of solutions fits perfectly into this framework.

2. Solution of the inhomogeneous Cauchy-equation

We start with a restatement of the solvability conditions (2), (3), appropriate for our purposes.

Lemma 1. *The function $\phi: V \times V \rightarrow W$ satisfies (2) and (3) for all $\xi, \eta, \zeta \in V$ if and only if ϕ satisfies*

$$\phi(0, \alpha) = \phi(\alpha, 0) = \phi(0, 0) \quad (4)$$

for all $\alpha \in V$ and

$$\phi(\alpha + \beta, \gamma + \delta) - \phi(\alpha, \gamma) - \phi(\beta, \delta) = \phi(\alpha + \gamma, \beta + \delta) - \phi(\alpha, \beta) - \phi(\gamma, \delta) \quad (5)$$

for all $\alpha, \beta, \gamma, \delta \in V$.

Proof. Suppose ϕ satisfies (2) and (3). Taking $\xi = \eta = 0, \zeta = \alpha$ in (3), we have $\phi(0, 0) = \phi(0, \alpha)$, which equals $\phi(\alpha, 0)$ by (2). Taking in (3) $\xi = \alpha, \eta = \gamma, \zeta = \beta + \delta$ and $\xi = \alpha, \eta = \beta, \zeta = \gamma + \delta$, respectively, we have

$$\begin{aligned} \phi(\alpha, \gamma) + \phi(\alpha + \gamma, \beta + \delta) &= \phi(\gamma, \beta + \delta) + \phi(\alpha, \beta + \gamma + \delta), \\ \phi(\alpha, \beta) + \phi(\alpha + \beta, \gamma + \delta) &= \phi(\beta, \gamma + \delta) + \phi(\alpha, \beta + \gamma + \delta), \end{aligned}$$

hence

$$\phi(\alpha + \beta, \gamma + \delta) - \phi(\alpha, \gamma) + \phi(\gamma, \beta + \delta) = \phi(\alpha + \gamma, \beta + \delta) - \phi(\alpha, \beta) + \phi(\beta, \gamma + \delta).$$

Moreover, from (3) we get

$$\begin{aligned} \phi(\gamma, \beta + \delta) &= \phi(\gamma, \beta) + \phi(\gamma + \beta, \delta) - \phi(\beta, \delta), \\ \phi(\beta, \gamma + \delta) &= \phi(\beta, \gamma) + \phi(\beta + \gamma, \delta) - \phi(\gamma, \delta). \end{aligned}$$

Observing (2) this renders (5). Suppose ϕ satisfies (4) and (5). Taking $\alpha = \xi, \beta = \eta, \gamma = 0, \delta = \zeta$, we get from (5)

$$\phi(\xi + \eta, \zeta) - \phi(\xi, 0) - \phi(\eta, \zeta) = \phi(\xi, \eta + \zeta) - \phi(\xi, \eta) - \phi(0, \zeta),$$

which implies (3) since $\phi(\xi, 0) = \phi(0, 0) = \phi(0, \zeta)$ by (1). Moreover, taking $\alpha = \delta = 0, \beta = \xi, \gamma = \eta$ in (5), we get (2) since $\phi(0, \eta) = \phi(\eta, 0) = \phi(\xi, 0) = \phi(0, \xi)$ by (1). \square

In what follows, given a (Hamel) basis of V over \mathbb{Q} , all solutions of (1) will be given. The ideas for the definition of the solution f arise from some necessary conditions on f in (1).

Rewriting this equation as $f(x + y) = f(x) + f(y) + \phi(x, y)$ we get by induction that

$$f\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n f(x_i) + \sum_{i=1}^{n-1} \phi\left(x_i, \sum_{j=i+1}^n x_j\right),$$

which for $x_1 = x_2 = \dots = x_n = x$ means

$$f(nx) = nf(x) + \sum_{i=1}^{n-1} \phi(x, (n-i)x) = nf(x) + \sum_{i=1}^{n-1} \phi(x, ix).$$

Since $f(0) = f(-x) + f(x) + \phi(x, -x)$ and $f(x) = nf(\frac{1}{n}x) + \sum_{i=1}^{n-1} \phi(\frac{1}{n}x, \frac{i}{n}x)$ all values $f(\sum_{b \in B} \lambda_b b)$ may be expressed in terms of $f(b)$ and certain values of ϕ , where B is a finite subset of V and all $\lambda_b \in \mathbb{Q}$.

Now we gather some auxiliary results needed for the construction.

Lemma 2. *Suppose ϕ satisfies (2) and (3). Then ϕ satisfies*

$$\sum_{i=1}^{n-1} \phi\left(a_i, \sum_{j=i+1}^n a_j\right) = \sum_{i=1}^{n-1} \phi\left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)}\right) \tag{6}$$

for all $n \in \mathbb{N}$, $a_1, \dots, a_n \in V$, and for all permutations $\pi \in \text{Sym}_n$.

Proof. The proof is by induction on n . For $n = 1$, (6) is trivially true. Let be $n \in \mathbb{N}$ and as the induction hypothesis suppose (6) is true for $n - 1$. Let $\pi \in \text{Sym}_n$ be any permutation and consider

$$\sum_{i=1}^{n-1} \phi\left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)}\right) = \phi\left(a_{\pi(1)}, \sum_{j=2}^n a_{\pi(j)}\right) + \sum_{i=2}^{n-1} \phi\left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)}\right). \tag{7}$$

In case $\pi(1) = 1$ applying the induction hypothesis by permuting the arguments $a_{\pi(2)}, \dots, a_{\pi(n)}$ with $\pi^{-1}|_{\{2,3,\dots,n\}} \in \text{Sym}_{n-1}$ we get

$$\begin{aligned} \sum_{i=2}^{n-1} \phi\left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)}\right) &= \sum_{i=1}^{(n-1)-1} \phi\left(a_{\pi(i+1)}, \sum_{j=i+1}^{n-1} a_{\pi(j+1)}\right) \\ &= \sum_{i=2}^{n-1} \phi\left(a_i, \sum_{j=i+1}^n a_j\right), \end{aligned}$$

which proves the statement for this case. In case $\pi(1) \neq 1$ we introduce the transposition $\sigma := (\pi(2) 1) \in \text{Sym}_n$ and, observing that $\sigma(\pi(1)) = \pi(1)$, apply

to the right hand side of (7) the induction hypothesis with the permutation $\sigma|_{\{\pi(2),\pi(3),\dots,\pi(n)\}} \in \text{Sym}_{n-1}$ to obtain

$$\begin{aligned} & \sum_{i=1}^{n-1} \phi \left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)} \right) \\ &= \phi \left(a_{\pi(1)}, \sum_{j=2}^n a_{\sigma \circ \pi(j)} \right) + \sum_{i=2}^{n-1} \phi \left(a_{\sigma \circ \pi(i)}, \sum_{j=i+1}^n a_{\sigma \circ \pi(j)} \right) \\ &= \phi \left(a_{\pi(1)}, a_{\sigma \circ \pi(2)} + \sum_{j=3}^n a_{\sigma \circ \pi(j)} \right) + \phi \left(a_{\sigma \circ \pi(2)}, \sum_{j=3}^n a_{\sigma \circ \pi(j)} \right) \\ & \quad + \sum_{i=3}^{n-1} \phi \left(a_{\sigma \circ \pi(i)}, \sum_{j=i+1}^n a_{\sigma \circ \pi(j)} \right). \end{aligned}$$

From (2) and (3) it follows that ϕ satisfies $\phi(\xi, \zeta + \eta) + \phi(\zeta, \eta) = \phi(\zeta, \xi + \eta) + \phi(\xi, \eta)$ for all ξ, η, ζ and taking $\xi = a_{\pi(1)}, \eta = \sum_{j=3}^n a_{\sigma \circ \pi(j)}, \zeta = a_{\sigma \circ \pi(2)}$ we obtain further

$$\begin{aligned} & \sum_{i=1}^{n-1} \phi \left(a_{\pi(i)}, \sum_{j=i+1}^n a_{\pi(j)} \right) \\ &= \phi \left(a_{\sigma \circ \pi(2)}, a_{\pi(1)} + \sum_{j=3}^n a_{\sigma \circ \pi(j)} \right) + \phi \left(a_{\pi(1)}, \sum_{j=3}^n a_{\sigma \circ \pi(j)} \right) \\ & \quad + \sum_{i=3}^{n-1} \phi \left(a_{\sigma \circ \pi(i)}, \sum_{j=i+1}^n a_{\sigma \circ \pi(j)} \right) \\ &= \phi \left(a_1, \sum_{j=2}^n a_j \right) + \sum_{i=2}^{n-1} \phi \left(a_{\rho \circ \sigma \circ \pi(i)}, \sum_{j=i+1}^n a_{\rho \circ \sigma \circ \pi(j)} \right) \\ &= \sum_{i=1}^{n-1} \phi \left(a_i, \sum_{j=i+1}^n a_j \right), \end{aligned}$$

where we have used in the last line $\sigma \circ \pi(2) = 1$ and then applied the induction hypothesis with the permutation $(\rho \circ \sigma \circ \pi)^{-1}|_{\{2,3,\dots,n\}} \in \text{Sym}_{n-1}$, where $\rho \in \text{Sym}_n$ is the transposition $\rho := (\pi(1) \ 1)$. □

Lemma 3. *Suppose ϕ satisfies (2) and (3). Then*

$$\sum_{k=0}^{l p-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) = \sum_{k=0}^{p-1} \phi(k a, a) + p \sum_{k=1}^{l-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) \tag{8}$$

for all $a \in V$, $p \in \mathbb{N}_0$, $l \in \mathbb{N}$.

Proof. We fix $a \in V$ and $l \in \mathbb{N}$ arbitrarily and perform induction on p . For $p = 0$ the validity of (8) is evident. Suppose (8) is true for a $p \in \mathbb{N}_0$. Then

$$\begin{aligned} \sum_{k=0}^{l(p+1)-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) &= \sum_{k=0}^{l p-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) + \sum_{k=0}^{l-1} \phi\left(\frac{(k+l p)}{l} a, \frac{1}{l} a\right) \\ &= \sum_{k=0}^{p-1} \phi(k a, a) + p \sum_{k=1}^{l-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) \\ &\quad + \sum_{k=0}^{l-1} \left(\phi\left(p a, \frac{(k+1)}{l} a\right) - \phi\left(p a, \frac{k}{l} a\right) + \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) \right) \\ &= \sum_{k=0}^{p-1} \phi(k a, a) + p \sum_{k=1}^{l-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) - \phi(p a, 0) + \phi(p a, a) + \sum_{k=0}^{l-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right) \\ &= \sum_{k=0}^{(p+1)-1} \phi(k a, a) + (p+1) \sum_{k=1}^{l-1} \phi\left(\frac{k}{l} a, \frac{1}{l} a\right), \end{aligned}$$

where we have used (3) with $\xi = p a$, $\eta = k \frac{a}{l}$, $\zeta = \frac{a}{l}$ and $\phi(0, \frac{a}{l}) = \phi(p a, 0)$ by Lemma 1. □

Let $\mathcal{B} \subset V$ be a Hamel basis of V over \mathbb{Q} . Then for each $x \in V$ there is a unique finite subset $B = B_x \subseteq \mathcal{B}$ and a unique family $(\lambda_b)_{b \in B} \in (\mathbb{Q} \setminus \{0\})^B$ such that $x = \sum_{b \in B} \lambda_b b$. (For $x = 0$ the set B_x is the empty set and the corresponding family of λ_b is the empty family.) This implies that for any $C \subseteq \mathcal{B}$ and any family $(\mu_c)_{c \in C} \in \mathbb{Q}^C$ such that $x = \sum_{c \in C} \mu_c c$ it follows that $B_x \subseteq C$, $\mu_b = \lambda_b$ for $b \in B_x$, and $\mu_c = 0$ for $c \in C \setminus B_x$.

Lemma 4. *Let $B \subset V$ be finite, $n := \#B$ the cardinality of B , let $(\lambda_b)_{b \in B} \in \mathbb{Q}^B$ and $\alpha: \{1, 2, \dots, n\} \rightarrow B$ be bijective. Then the expression*

$$F^*(B, (\lambda_b)_{b \in B}, \alpha) := \sum_{i=1}^{n-1} \phi\left(\lambda_{b_{\alpha(i)}} b_{\alpha(i)}, \sum_{j=i+1}^n \lambda_{b_{\alpha(j)}} b_{\alpha(j)}\right)$$

is independent of α .

Proof. This follows immediately from Lemma 3. □

According to this lemma the function F ,

$$F(B, (\lambda_b)_{b \in B}) := F^*(B, (\lambda_b)_{b \in B}, \alpha) - \#B \cdot \phi(0, 0)$$

for some bijection $\alpha: \{1, 2, \dots, \#B\} \rightarrow B$, is well-defined.

Lemma 5. *Let $B \subseteq C \subseteq V$, $\#C < \infty$, and let $(\lambda_b)_{b \in C}$ be such that $\lambda_b = 0$ for all $b \in C \setminus B$. Then $F(C, (\lambda_b)_{b \in C}) = F(B, (\lambda_b)_{b \in B})$.*

Proof. Let $n := \#C$, $m := \#B$ and let, which is always possible, $\alpha: \{1, 2, \dots, n\} \rightarrow C$ be such that $\alpha(\{1, 2, \dots, m\}) = B$. Then

$$\begin{aligned} F(C, (\lambda_b)_{b \in C}) &:= F^*(C, (\lambda_b)_{b \in C}, \alpha) - n\phi(0, 0) \\ &= \sum_{i=1}^{n-1} \phi \left(\lambda_{b_{\alpha(i)}} b_{\alpha(i)}, \sum_{j=i+1}^n \lambda_{b_{\alpha(j)}} b_{\alpha(j)} \right) - n\phi(0, 0) \\ &= \sum_{i=1}^{m-1} \phi \left(\lambda_{b_{\alpha(i)}} b_{\alpha(i)}, \sum_{j=i+1}^n \lambda_{b_{\alpha(j)}} b_{\alpha(j)} \right) \\ &\quad + \sum_{i=m}^{n-1} \phi \left(\lambda_{b_{\alpha(i)}} b_{\alpha(i)}, \sum_{j=i+1}^n \lambda_{b_{\alpha(j)}} b_{\alpha(j)} \right) - n\phi(0, 0) \\ &= F(B, (\lambda_b)_{b \in B}) + m\phi(0, 0) + (n - m)\phi(0, 0) - n\phi(0, 0) \\ &= F(B, (\lambda_b)_{b \in B}), \end{aligned}$$

where we used $\phi(\lambda_{b_{\alpha(m)}} b_{\alpha(m)}, 0) = \phi(0, 0)$. □

Given $b \in V$, $p, q \in \mathbb{Z}$, $q > 0$, we define

$$G^*(b, p, q) := \operatorname{sgn} \left(\frac{p}{q} \right) \left(\sum_{k=0}^{|p|-1} \phi \left(\frac{k}{q} b, \frac{1}{q} b \right) - \left| \frac{p}{q} \right| \sum_{k=1}^{q-1} \phi \left(\frac{k}{q} b, \frac{1}{q} b \right) \right).$$

Lemma 6. *Given $b \in V$, $p, q, r, s \in \mathbb{Z}$, $q, s > 0$ such that $\frac{p}{q} = \frac{r}{s}$, it follows that $G^*(b, p, q) = G^*(b, r, s)$.*

Proof. If $p = 0$ then $r = 0$, too. Thus $G^*(b, p, q) = G^*(b, r, s)$ in this case. If $p \neq 0$ also $r \neq 0$ and we may assume that, say, p and q are coprime. Then, since $\frac{p}{q} = \frac{r}{s}$ there is some $l \in \mathbb{N}$ such that $r = ln$ and $s = lq$. We have to show that

$$\begin{aligned} &\operatorname{sgn}(\lambda) \left(\sum_{k=0}^{|p|-1} \phi \left(\frac{k}{q} b, \frac{1}{q} b \right) - |\lambda| \sum_{k=1}^{q-1} \phi \left(\frac{k}{q} b, \frac{1}{q} b \right) \right) \\ &= \operatorname{sgn}(\lambda) \left(\sum_{k=0}^{|r|-1} \phi \left(\frac{k}{s} b, \frac{1}{s} b \right) - |\lambda| \sum_{k=1}^{s-1} \phi \left(\frac{k}{s} b, \frac{1}{s} b \right) \right). \end{aligned} \tag{9}$$

After division by $\text{sgn}(\lambda)$, observing Lemma 3 and (1), we get for the right hand side of (9) that

$$\begin{aligned} & \sum_{k=0}^{l|p|-1} \phi\left(\frac{k}{lq}b, \frac{1}{lq}b\right) - |\lambda| \sum_{k=0}^{lq-1} \phi\left(\frac{k}{lq}b, \frac{1}{lq}b\right) + |\lambda| \phi\left(0, \frac{1}{lq}b\right) \\ &= \sum_{k=0}^{|p|-1} \phi\left(\frac{k}{q}b, \frac{1}{q}b\right) + |p| \sum_{k=1}^{l-1} \phi\left(\frac{k}{lq}b, \frac{1}{lq}b\right) \\ & \quad - |\lambda| \sum_{k=0}^{q-1} \phi\left(\frac{k}{q}b, \frac{1}{q}b\right) - q|\lambda| \sum_{k=1}^{l-1} \phi\left(\frac{k}{lq}b, \frac{1}{lq}b\right) \\ & \quad + |\lambda| \phi\left(0, \frac{1}{lq}b\right) = \sum_{k=0}^{|p|-1} \phi\left(\frac{k}{q}b, \frac{1}{q}b\right) - |\lambda| \sum_{k=1}^{q-1} \phi\left(\frac{k}{q}b, \frac{1}{q}b\right), \end{aligned}$$

hence (9) is true. □

Thus, given $b \in V$, $\lambda = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q > 0$, the function G , $G(b, \lambda) := G^*(b, p, q)$ is well-defined and $G(b, 0) = 0$.

Now let \mathcal{B} be a Hamel basis of V , $x \in V$, $B \subseteq \mathcal{B}$ finite and $(\lambda_B)_{b \in B} \in \mathbb{Q}^B$ such that

$$x = \sum_{b \in B} \lambda_b b.$$

Then, using the considerations above, we may define functions $f_1, f_2, f_3: V \rightarrow W$ by

$$\begin{aligned} f_1(x) &:= F(B, (\lambda_b)_{b \in B}) \\ f_2(x) &:= \sum_{b \in B} G(b, \lambda_b) \\ f_3(x) &= \sum_{b \in B} H(b, \lambda_b), \end{aligned}$$

where $H(b, \lambda_b) := \frac{1}{2} ((\text{sgn}(\lambda_b))^2 - \text{sgn}(\lambda_b)) (\phi(0, 0) - \phi(\lambda_b b, -\lambda_b b))$.

Note that also f_3 is well-defined since $((\text{sgn}(\lambda_b))^2 - \text{sgn}(\lambda_b)) = 0$ for $\lambda_b = 0$.

Theorem 1. *Let $\phi : V \times V \rightarrow W$ satisfy (2), (3), let \mathcal{B} be a basis of V over \mathbb{Q} . Then $f : V \rightarrow W$, $f(x) := f_1(x) + f_2(x) + f_3(x)$ for $x \in V$, is a particular solution of (1) and thus necessarily $f(0) = -\phi(0, 0)$.*

A function $g : V \rightarrow W$ solves (1) iff $f - g$ is additive.

Proof. The considerations above show that f is well-defined. Since $f_2(0) = f_3(0) = 0$, we have $f(0) = f_1(0) = -n\phi(0, 0) + (n-1)\phi(0, 0) = -\phi(0, 0)$. Taking for granted that f is a solution of (1), the second part obviously becomes true.

It remains to show that f indeed is a solution. Let $x = \sum_{b \in B} \lambda_b b$, $y = \sum_{b \in B} \mu_b b$, let $n := \#B$ and $\alpha: \{1, 2, \dots, n\} \rightarrow B$ be bijective. With the abbreviations $l_i := \lambda_{\alpha(i)} \beta_{\alpha(i)}$, $m_i := \mu_{\alpha(i)} \beta_{\alpha(i)}$ we get observing (5) that

$$\begin{aligned}
 & f_1(x+y) - f_1(x) - f_1(y) \\
 &= n\phi(0, 0) + \sum_{i=1}^{n-1} \left(\phi \left(l_i + m_i, \sum_{j=i+1}^n (l_j + m_j) \right) \right. \\
 &\quad \left. - \phi \left(l_i, \sum_{j=i+1}^n l_j \right) - \phi(m_i, \sum_{j=i+1}^n m_j) \right) \\
 &= n\phi(0, 0) + \sum_{i=1}^{n-1} \left(\phi \left(\sum_{j=i}^n l_j, \sum_{j=i}^n m_j \right) \right. \\
 &\quad \left. - \phi \left(\sum_{j=i+1}^n l_j, \sum_{j=i+1}^n m_j \right) - \phi(l_i, m_i) \right) \\
 &= n\phi(0, 0) + \phi \left(\sum_{j=1}^n l_j, \sum_{j=1}^n m_j \right) - \sum_{i=1}^n \phi(l_i, m_i) \\
 &= n\phi(0, 0) + \phi(x, y) - \sum_{b \in B} \phi(\lambda_b b, \mu_b b).
 \end{aligned} \tag{10}$$

Now we will show that for all $b \in B$ we have

$$\begin{aligned}
 & G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\
 &= \phi(\lambda_b b, \mu_b b) - \phi(0, 0).
 \end{aligned} \tag{11}$$

We set $\nu_b := \lambda_b + \mu_b = \frac{p_b+r_b}{q}$ and $c_b := \frac{1}{q}b$. Moreover we may assume $\lambda_b = \frac{p_b}{q}$, $\mu_b = \frac{r_b}{q}$ with $p_b, r_b \in \mathbb{Z}$ and some $q \in \mathbb{N}$, a common denominator for all fractions λ_b, μ_b , $b \in B$. Then

$$\begin{aligned}
 & G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) \\
 &= \operatorname{sgn}(\nu_b) \sum_{k=0}^{|\nu_b|^{-1}} \phi(kc_b, c_b) - \frac{p_b+r_b}{q} \sum_{k=1}^{q-1} \phi(kc_b, c_b) \\
 &\quad - \operatorname{sgn}(p_b) \sum_{k=0}^{|\lambda_b|^{-1}} \phi(kc_b, c_b) + \frac{p_b}{q} \sum_{k=1}^{q-1} \phi(kc_b, c_b)
 \end{aligned}$$

$$\begin{aligned}
 & -\operatorname{sgn}(r_b) \sum_{k=0}^{|r_b|-1} \phi(kc_b, c_b) + \frac{r_b}{q} \sum_{k=1}^{q-1} \phi(kc_b, c_b) \tag{12} \\
 & = \operatorname{sgn}(\nu_b) \sum_{k=0}^{|p_b+r_b|-1} \phi(kc_b, c_b) - \operatorname{sgn}(p_b) \sum_{k=0}^{|p_b|-1} \phi(kc_b, c_b) \\
 & \quad - \operatorname{sgn}(r_b) \sum_{k=0}^{|r_b|-1} \phi(kc_b, c_b)
 \end{aligned}$$

and with an analogous application for H

$$\begin{aligned}
 & H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\
 & = \frac{1}{2}(\operatorname{sgn}(\nu_b)^2 - \operatorname{sgn}(\nu_b)) (\phi(0, 0) - \phi(\nu_b b, -\nu_b b)) \\
 & \quad - \frac{1}{2}(\operatorname{sgn}(\lambda_b)^2 - \operatorname{sgn}(\lambda_b)) (\phi(0, 0) - \phi(\lambda_b b, -\lambda_b b)) \tag{13} \\
 & \quad - \frac{1}{2}(\operatorname{sgn}(\mu_b)^2 - \operatorname{sgn}(\mu_b)) (\phi(0, 0) - \phi(\mu_b b, -\mu_b b)).
 \end{aligned}$$

Corresponding to the possible distributions of signs we consider the cases listed in the following table. A column without a numeral contains a case which is obtained from the immediately preceding one, considering (2) by renaming λ_b by μ_b and vice versa. Therefore these cases need not be treated separately.

	1.	2.	3.	4.	5.	6.	7.	8.
λ_b	> 0	> 0	< 0	> 0	< 0	> 0	< 0	> 0
μ_b	> 0	< 0	> 0	< 0	> 0	< 0	> 0	< 0
ν_b	> 0	> 0	> 0	< 0	< 0	$= 0$	$= 0$	< 0

1. In case $\lambda_b > 0, \mu_b > 0$ (hence $\nu_b > 0$) we have for the sums in (12)

$$\begin{aligned}
 & \sum_{k=0}^{p_b+r_b-1} \phi(kc_b, c_b) - \sum_{k=0}^{p_b-1} \phi(kc_b, c_b) - \sum_{k=0}^{r_b-1} \phi(kc_b, c_b) \\
 & = \sum_{k=r_b}^{r_b+p_b-1} \phi(kc_b, c_b) - \sum_{k=0}^{p_b-1} \phi(kc_b, c_b),
 \end{aligned}$$

therefore

$$\begin{aligned}
 G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) & = \sum_{k=0}^{p_b-1} (\phi((k+r_b)c_b, c_b) - \phi(kc_b, c_b)) \\
 & = \sum_{k=0}^{p_b-1} (\phi(r_b c_b, (k+1)c_b) - \phi(r_b c_b, k c_b)) = \phi(\lambda_b b, \mu_b b) - \phi(0, 0),
 \end{aligned}$$

where we have used (3) and then (2). Obviously $H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) = 0$.

2. In case $\lambda_b > 0, \mu_b < 0, \nu_b > 0$ we have for the sums in (12)

$$\begin{aligned} & \sum_{k=0}^{p_b - |r_b| - 1} \phi(kc_b, c_b) - \sum_{k=0}^{p_b - 1} \phi(kc_b, c_b) + \sum_{k=0}^{|r_b| - 1} \phi(kc_b, c_b) \\ &= \sum_{k=0}^{p_b - |r_b| - 1} \phi(kc_b, c_b) - \sum_{k=|r_b|}^{p_b - 1} \phi(kc_b, c_b). \end{aligned}$$

Therefore

$$\begin{aligned} G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) &= \sum_{k=0}^{p_b - |r_b| - 1} (\phi(kc_b, c_b) - \phi((k + |r_b|)c_b, c_b)) \\ &= \sum_{k=0}^{p_b - |r_b| - 1} (\phi(kc_b, |r_b|c_b) - \phi((k + 1)c_b, |r_b|c_b)) \\ &= \phi(0, |r_b|c_b) - \phi((p_b - |r_b|)c_b, |r_b|c_b) = \phi(0, 0) - \phi(\nu_b b, -\mu_b b), \end{aligned}$$

where we have used (3). This time

$$H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) = -\phi(0, 0) + \phi(\mu_b b, -\mu_b b),$$

such that

$$\begin{aligned} & G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= \phi(\mu_b b, -\mu_b b) - \phi(\lambda_b \pm \mu_b b, -\mu_b b) = \phi(\lambda_b b, \mu_b b) - \phi(0, 0), \end{aligned}$$

where we have used (5) with $\alpha = \lambda_b b, \beta = \mu_b b, \gamma = 0, \delta = -\beta$ and then (1).

3. In case $\lambda_b > 0, \mu_b < 0, \nu_b < 0$ we have for the sums in (12)

$$\begin{aligned} & - \sum_{k=0}^{-p_b + |r_b| - 1} \phi(kc_b, c_b) - \sum_{k=0}^{p_b - 1} \phi(kc_b, c_b) + \sum_{k=0}^{|r_b| - 1} \phi(kc_b, c_b) \\ &= - \sum_{k=0}^{-p_b + |r_b| - 1} \phi(kc_b, c_b) + \sum_{k=p_b}^{|r_b| - 1} \phi(kc_b, c_b), \end{aligned}$$

therefore

$$\begin{aligned} & G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) \\ &= - \sum_{k=0}^{-p_b + |r_b| - 1} (\phi(kc_b, c_b) - \phi((k + p_b)c_b, c_b)) \\ &= - \sum_{k=0}^{-p_b + |r_b| - 1} (\phi(kc_b, p_b c_b) - \phi((k + 1)c_b, p_b c_b)) \end{aligned}$$

$$= -\phi(0, p_b c_b) + \phi((-p_b + |r_b|)c_b, p_b c_b) = -\phi(0, 0) + \phi(-(\lambda_b + \mu_b)b, \lambda_b b),$$

while

$$\begin{aligned} &H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= -\phi((\lambda_b + \mu_b)b, -(\lambda_b + \mu_b)b) + \phi(\mu_b b, -\mu_b b). \end{aligned}$$

It follows that

$$\begin{aligned} &G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= \phi(\lambda_b b, \mu_b b) - \phi(0, 0), \end{aligned}$$

where we have used (3) with $\eta = \lambda_b b$, $\zeta = \mu_b b$ and $\xi = -\eta - \zeta$.

4. In case $\lambda_b > 0$, $\mu_b = -\lambda_b$ (hence $\nu_b = 0$) we get, observing $p_p = |r_b|$ and (2) that

$$\begin{aligned} &G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= 0 + (-\phi(0, 0) + \phi(-\lambda_b b, \lambda_b b)) = \phi(\lambda_b b, \mu_b b) - \phi(0, 0). \end{aligned}$$

5. In case $\lambda_b > 0$, $\mu_b = 0$ (hence $\nu_b = \lambda_b > 0$) we get immediately

$$G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) = H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) = 0, \tag{14}$$

so that

$$\begin{aligned} &G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= \phi(0, 0) - \phi(0, 0) = \phi(\lambda_b b, 0) - \phi(0, 0) = \phi(\lambda_b b, \mu_b b) - \phi(0, 0). \end{aligned} \tag{15}$$

6. In case $\lambda_b < 0$, $\mu_b = 0$ (hence $\nu_b = \lambda_b < 0$) we get $G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) = 0$ immediately and $H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) = 0$ since $\nu_b = \lambda_b$, so that (15) holds in this case also.

7. In case $\lambda_b < 0$, $\mu_b < 0$ (hence $\nu_b < 0$) we have for the sums in (12)

$$\begin{aligned} &-\sum_{k=0}^{|p_b|+|r_b|-1} \phi(kc_b, c_b) + \sum_{k=0}^{|p_b|-1} \phi(kc_b, c_b) + \sum_{k=0}^{|r_b|-1} \phi(kc_b, c_b) \\ &= \sum_{k=0}^{|p_b|-1} \phi(kc_b, c_b) - \sum_{k=|r_b|}^{|p_b|+|r_b|-1} \phi(kc_b, c_b), \end{aligned}$$

therefore

$$\begin{aligned} G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) &= \sum_{k=0}^{|p_b|-1} (\phi(kc_b, c_b) - \phi((k + |r_b|)c_b, c_b)) \\ &= \sum_{k=0}^{|p_b|-1} (\phi(kc_b, |r_b|c_b) - \phi((k + 1)c_b, |r_b|c_b)) = (\phi(|r_b|c_b, 0) - \phi(|r_b|c_b, |p_b|c_b)) \\ &= \phi(0, 0) - \phi(-\lambda_b b, -\mu_b b), \end{aligned}$$

where we have used (3), (1) and (2). Furthermore,

$$\begin{aligned} &H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= -\phi(0, 0) - \phi((\lambda_b + \mu_b)b, -(\lambda_b + \mu_b)b) + \phi(\lambda_b b, -\lambda_b b) + \phi(\mu_b b, -\mu_b b) \\ &= -2\phi(0, 0) + \phi(\lambda_b b, \mu_b b) + \phi(-\lambda_b b, -\mu_b b), \end{aligned}$$

where we have used (5) with $\alpha = \lambda_b b$, $\beta = \mu_b b$, $\gamma = -\alpha$, $\delta = -\beta$, so that

$$\begin{aligned} &G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= \phi(\lambda_b b, \mu_b b) - \phi(0, 0). \end{aligned}$$

8. Finally in case $\lambda_b = \mu_b = \nu_b = 0$, (14) holds trivially and therefore (15) too.

Adding together we thus obtain

$$\begin{aligned} &\sum_{b \in B} G(b, \lambda_b + \mu_b) - G(b, \lambda_b) - G(b, \mu_b) + H(b, \lambda_b + \mu_b) - H(b, \lambda_b) - H(b, \mu_b) \\ &= \sum_{b \in B} \phi(\lambda_b b, \mu_b b) - n\phi(0, 0), \end{aligned}$$

which with (10) yields the result. □

3. Different types of Banach limits

In [1] one may find the following definition of a vector valued Banach limit. It is also shown there that this includes the usual definition of a Banach limit on bounded sequences of reals (see [3]).

Definition 1. Let X be a normed space and let $\ell_\infty(X)$ be the space of bounded sequences on X equipped with the supremum norm. Then $L: \ell_\infty(X) \rightarrow X$ is a *Banach limit* if

- (i) L is linear and continuous
- (ii) $L(x) = \lim_{n \rightarrow \infty} x_n$ for any convergent sequence $x = (x_n)_{n \in \mathbb{N}}$ in X ,
- (iii) (*shift invariance*) $L \circ \sigma = L$, where $\sigma: \ell_\infty(X) \rightarrow \ell_\infty(X)$ is defined by $\sigma((x_n)_{n \in \mathbb{N}}) := (x_{n+1})_{n \in \mathbb{N}}$, and
- (iv) the operator norm of L equals 1: $\|L\| = 1$.

In this paper it was proved that Banach limits exist on the dual X^* of any normed space X . The proof used an ultrafilter \mathcal{U} on \mathbb{N} containing $\{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$ and the definition

$$L((x_n)_{n \in \mathbb{N}}) = \mathcal{U} - \lim \frac{x_1 + x_2 + \dots + x_n}{n} \tag{16}$$

which is meaningful with respect to the w^* -topology since then any ball is compact.

[6, Thm. 5.3.4] contains the result, that $x^* = \mathcal{U} - \lim(x_n^*)$ is contained in

$$\text{cl}_{w^*} \left\{ \frac{1}{n}(x_1^* + x_2^* + \dots + x_n^*) \mid n \in \mathbb{N} \right\},$$

the closure with respect to the w^* -topology of the set $\left\{ \frac{1}{n}(x_1^* + x_2^* + \dots + x_n^*) \mid n \in \mathbb{N} \right\}$.

Theorem 2. *Let X be a normed space and X^* its dual. Then the Banach limit L defined by (16) has the property that for any sequence $\xi = (x_n^*)_{n \in \mathbb{N}} \in \ell_\infty(X^*)$ the value $L(\xi)$ is contained in $\text{cl}_{w^*} \text{conv}\{x_n^* \mid n \in \mathbb{N}\}$, the closure with respect to the w^* -topology of the convex hull of the elements in the sequence. But there are Banach limits which do not fulfill this property.*

Proof. The first part is clear by the lines above if one takes into account that

$$\left\{ \frac{1}{n}(x_1^* + x_2^* + \dots + x_n^*) \mid n \in \mathbb{N} \right\} \subseteq \text{conv}\{x_n^* \mid n \in \mathbb{N}\}.$$

The second part follows from the last item of the following remark, since by [2] there are different Banach limits on the reals. □

Corollary 1. *Since all Banach limits are shift invariant it follows that $L(\xi) \in \bigcap_{m \in \mathbb{N}} \text{cl}_{w^*} \text{conv}\{x_n^* \mid n \in \mathbb{N}, n \geq m\}$, which in the original (real) case reflects the property that for any Banach limit λ on \mathbb{R}*

$$\liminf_{n \rightarrow \infty} x_n \leq \lambda((x_n)_{n \in \mathbb{N}}) \leq \limsup_{n \rightarrow \infty} x_n.$$

Remark 1. Let $X := \mathbb{R}^2$ be endowed with the ℓ_1 -norm $\|(x, y)\| := |x| + |y|$, let $e_1 = (1, 0), e_2 = (0, 1)$.

1. The operator norm on X^* is given by $\|f\| = \|(f(e_1), f(e_2))\|_\infty := \max\{|f(e_1)|, |f(e_2)|\}$.
2. A sequence $(f_n)_{n \in \mathbb{N}}$ is contained in $\ell_\infty(X^*)$ iff the sequences $(f_n(e_i))_{n \in \mathbb{N}}$ are contained in $\ell_\infty(\mathbb{R})$ for $i = 1, 2$. More exactly: $\|f_n\| \leq M$ for all n iff $|f_n(e_i)| \leq M$ for all $n \in \mathbb{N}$ and for $i = 1, 2$.
3. The w^* -topology and the norm-topology on X^* coincide, since X has finite dimension.
4. Let L_1, L_2 be Banach limits on \mathbb{R} . Then $L: \ell_\infty(X^*) \rightarrow X^*, L((f_n)_{n \in \mathbb{N}}) := L_1((f_n(e_1))_{n \in \mathbb{N}})\pi_1 + L_2((f_n(e_2))_{n \in \mathbb{N}})\pi_2$ is a Banach limit on X^* , where π_1, π_2 denote the projections from X to the first and second component respectively.
5. If $L_1 \neq L_2$ there is some sequence $(f_n)_{n \in \mathbb{N}} \in \ell_\infty(X^*)$ such that $L((f_n)_{n \in \mathbb{N}}) \notin \text{cl}_{w^*} \text{conv}\{f_n \mid n \in \mathbb{N}\}$.

Proof of the remark above. The first part follows immediately from the definition. This implies the second item. A proof of the third one is contained in [11].

Regarding the fourth item one may proceed in the following manner. Let $(f_n)_{n \in \mathbb{N}}$ be any sequence in $\ell_\infty(X^*)$. Then the sequences $f_n(e_1)$ and $f_n(e_2)$ are bounded real sequences. Thus their Banach limits exist for all Banach limits on \mathbb{R} . Accordingly $L((f_n)_{n \in \mathbb{N}}) \in X^*$. Obviously L is shift invariant since L_1, L_2 are. If the sequence f_n converges to f , the sequences $f_n(e_i)$ will converge to $f(e_i)$, $i = 1, 2$. Thus $L_i((f_n(e_i))) = f(e_i)$ for $i = 1, 2$. Since obviously L is also linear, it remains to show that $\|L\| = 1$. Note that $\|L((f_n)_{n \in \mathbb{N}})\| = \max\{|L_1((f_n(e_1))_{n \in \mathbb{N}})|, |L_2((f_n(e_2))_{n \in \mathbb{N}})|\}$. Note also that $\|L_1\| = \|L_2\| = 1$. This implies $|L_1((f_n(e_1))_{n \in \mathbb{N}})| \leq 1 \cdot \|(f_n(e_1))_{n \in \mathbb{N}}\|_\infty$ and $|L_2((f_n(e_2))_{n \in \mathbb{N}})| \leq 1 \cdot \|(f_n(e_2))_{n \in \mathbb{N}}\|_\infty$. Using 2. this also gives $\|L((f_n)_{n \in \mathbb{N}})\| \leq \|(f_n)_{n \in \mathbb{N}}\|_\infty = \sup\{\|f_n\|_\infty \mid n \in \mathbb{N}\}$. So $\|L\| \leq 1$. Let $f_n(e_1) = f_n(e_2) = 1$, i.e., $f_n(x, y) = x + y$. Then $L((f_n)_{n \in \mathbb{N}})(x, y) = x + y$ for all x, y since $L_i((1)_{n \in \mathbb{N}}) = 1$. Note finally that $\|(f_n)_{n \in \mathbb{N}}\|_\infty = 1 = \|L(f_n)_{n \in \mathbb{N}}\|$. This implies $\|L\| \geq 1$ and altogether $\|L\| = 1$.

Now we deal with the final part. Let $c = (c_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathbb{R})$ be such that $L_1(c) \neq L_2(c)$ and define $f_n := c_n(\pi_1 - \pi_2)$. Then $\|f_n\| = |c_n|$ for all n . For all $x \in \mathbb{R}$ the set $A_x := \{f \in X^* \mid f(x, x) = 0\}$ is closed in the w^* -topology as it is the complement of the w^* -open set $\{f \in X^* \mid f(x, x) \neq 0\}$. Let $A := A_x$ for some $x \neq 0$. By 2. this set is also closed with respect to the norm topology. It is also convex since it is a linear subspace of X^* . Assume now that $f := L((f_n)_{n \in \mathbb{N}}) \in \text{cl}_{w^*} \text{conv} \{f_n \mid n \in \mathbb{N}\}$. Then $f \in A$ since A is convex and closed. But $f = L_1(c)\pi_1 - L_2(c)\pi_2$ and thus $f(x, x) = (L_1(c) - L_2(c))x \neq 0$ as $x \neq 0$. This yields $f \notin A$. □

Using Banach limits L_1, L_2, \dots, L_m it is easy to construct Banach limits on \mathbb{R}^n . The following theorem answers the question whether a result similar to that of Theorem 2 holds true.

Theorem 3. *Let $m \geq 2$ and let $L_1, L_2, \dots, L_m: \ell_\infty(\mathbb{R}) \rightarrow \mathbb{R}$ be Banach limits. Then $L: \ell_\infty(\mathbb{R}^m) \rightarrow \mathbb{R}^m$, $L((x_n)_{n \in \mathbb{N}}) := (L_i((x_n^{(i)})_{n \in \mathbb{N}}))_{1 \leq i \leq m}$, where $x_m = (x_m^{(i)})_{1 \leq i \leq m}$, is a Banach limit for $(\mathbb{R}^m, \|\cdot\|_\infty)$.*

This Banach limit has the property that $L((x_n)_{n \in \mathbb{N}}) \in \text{cl}_{\|\cdot\|} \text{conv} \{n_n \mid n \in \mathbb{N}\}$ for all $(x_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathbb{R}^m)$ iff $L_1 = L_2 = \dots = L_m$.

Proof. The properties (i)–(iii) for L immediately follow from the corresponding properties of L_i . Since $\|L_i\| = 1$ and since we use the supremum norm on \mathbb{R}^m it follows that $\|L\| \leq 1$. Now we consider the constant sequence determined by $1_m := (1, 1, \dots, 1)$. L_i are Banach limits. Thus $L_i((1, 1, \dots)) = 1$. Therefore $L((1_m, 1_m, \dots)) = 1_m$. Thus $\|L\| \geq \|1_m\|_\infty = 1$ and also $\|L\| = 1$, as desired.

Suppose that $L_1 = L_2 = \dots = L_m$ and assume that

$$L((x_n)_{n \in \mathbb{N}}) \notin K := \text{cl}_{\|\cdot\|} \text{conv} \{x_n \mid n \in \mathbb{N}\}.$$

Then by the separation theorem for closed convex sets (see [4, Théorème 1.7]) there is some $a = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ and some $\alpha \in \mathbb{R}$ such that

$\sum_{i=1}^m a_i x_n^{(i)} < \alpha$ for all n and such that $\sum_{i=1}^m a_i L_i((x_n^{(i)})_{n \in \mathbb{N}}) > \alpha$. But $\sum_{i=1}^m a_i L_i((x_n^{(i)})_{n \in \mathbb{N}}) = L_1((\sum_{i=1}^m a_i x_n^{(i)})_{n \in \mathbb{N}})$ and by the property of a Banach limit this value must be $\leq \alpha$ since $\sum_{i=1}^m a_i x_n^{(i)} < \alpha$ for all n . Thus $L((x_n)_{n \in \mathbb{N}}) \in K$, as desired.

Finally assume (without loss of generality) that $L_1 \neq L_2$ and that $L_1((z_n)_{n \in \mathbb{N}}) \neq L_2((z_n)_{n \in \mathbb{N}})$ for some $(z_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathbb{R}^m)$. Then the sequence $x_n, x_n := (z_n, z_n, 0, \dots, 0)$ is contained in $\ell_\infty(\mathbb{R}^m)$. Moreover all x_n are contained in the (closed) linear subspace $E := \{x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m \mid x_1 = x_2, x_3 = \dots = x_m = 0\}$ of \mathbb{R}^m . Assume that $L((x_n)_{n \in \mathbb{N}}) \in \text{cl}_{\|\cdot\|} \text{conv} \{x_n \mid n \in \mathbb{N}\} \subseteq E$. But

$$L((x_n)_{n \in \mathbb{N}}) = (L_1((z_n)_{n \in \mathbb{N}}), L_2((z_n)_{n \in \mathbb{N}}), 0, \dots, 0)$$

and $L_1((z_n)_{n \in \mathbb{N}}) \neq L_2((z_n)_{n \in \mathbb{N}})$, a contradiction. □

4. Solutions of the inhomogeneous Cauchy equation expressed in terms of Banach limits

In [10] it was shown that for $V = W = \mathbb{R}$ the relations (2) and (3) imply

$$\begin{aligned} &\phi(n(x + y), x + y) - \phi(nx, x) - \phi(ny, y) \\ &= \phi((n + 1)x, (n + 1)y) - \phi(nx, ny) - \phi(x, y), \quad x, y \in V \end{aligned} \tag{17}$$

provided that (1) has a solution at all. The proof for arbitrary rational vector spaces V, W is the same. Thus (17) holds true in view of Theorem 1. The following generalizes Theorem 2 of [10].

Theorem 4. *Let V be a rational vector space and X a normed space which admits a Banach limit $L: \ell_\infty(X) \rightarrow X$. Assume that $\phi: V \times V \rightarrow X$ satisfies (2) and (3) and that the sequences $\phi(nx, x)$ and $\phi(nx, ny)$ are contained in $\ell_\infty(X)$ for all $x, y \in V$. Then $f: V \rightarrow X, f(x) := -L((\phi(nx, x))_{n \in \mathbb{N}})$ is a solution of (1).*

Proof. Note that L may be applied to all sequences generated by (17) separately. Then the result follows from the fact that L is shift invariant and that L maps constant functions to the corresponding constant. □

Example 1 (Unbounded ϕ). Let the rational vector space V as above be of infinite dimension and let X admit a Banach limit $L: \ell_\infty(X) \rightarrow X$. For $x \in V$ let $\mathcal{C}_x := \mathbb{Q}_{>0}x$, where $\mathbb{Q}_{>0}$ denotes the set of all positive rational numbers. Then the set of all \mathcal{C}_x gives a partition of V . Let $R \subseteq V$ be a set of representatives for this partition, i.e., $V = \bigcup_{r \in R} \mathcal{C}_r$ and $\mathcal{C}_r \cap \mathcal{C}_s = \emptyset$ for $r, s \in R, r \neq s$.

Take $g: R \rightarrow X$ arbitrary and $h: V \rightarrow X$ bounded and define $f: V \rightarrow X$ by $f(x) := g(r) + h(x)$, where $x \in \mathcal{C}_r$. Moreover let $\phi: V \times V \rightarrow X$ be defined by $\phi(x, y) := f(x + y) - f(x) - f(y)$.

Then Theorem 4 works for ϕ : If $x \in \mathcal{C}_r, y \in \mathcal{C}_s, x + y \in \mathcal{C}_t$ the Cauchy difference for f is given by $f(x + y) - f(x) - f(y) = g(t) - g(r) - g(s) + h(x + y) - h(x) - h(y)$. Since $nx, (n + 1)x \in \mathcal{C}_r$, we get $\phi(nx, x) = h((n + 1)x) - h(nx) - h(x)$. A similar calculation shows that $\phi(nx, ny) = g(t) - g(r) - g(s) + h(n(x + y)) - h(nx) - h(ny)$. Thus the hypotheses of the theorem are satisfied.

g may be chosen unbounded since by the assumption on V the set R of representatives has to be infinite.

Example 2 ($L^2([0, 1])$). By [6, Lemma 4.2.2] any Hilbert space, in particular the space of square integrable functions on $[0, 1]$ admit a Banach limit. Define $\phi: L^2([0, 1]) \times L^2([0, 1]) \rightarrow L^2([0, 1])$ by $\phi(f, g) := \sin \circ (f + g) - \sin \circ f - \sin \circ g$. This is well defined, since $\sin \circ f$ is measurable and even square integrable since \sin is bounded. Moreover ϕ itself is bounded because $\int_0^1 |\sin^2(f(x))| dx \leq 1$. Thus a fortiori ϕ satisfies the hypotheses of the theorem above.

Example 3 ($\ell_2(\mathbb{R})$). $\ell_2(\mathbb{R})$ is a Hilbert space. Thus also in this case Banach limits exist. For $x = (x_n)_{n \in \mathbb{N}}$ put $f(x) := (\sin(x_n)/n^2)_{n \in \mathbb{N}}$. Then $f(x) \in \ell_2(\mathbb{R})$ and $\|f(x)\|_2^2 \leq \sum_{n=1}^\infty \frac{1}{n^2}$. Accordingly the theorem may be applied to ϕ with $\phi(x, y) = f(x + y) - f(x) - f(y)$.

Example 4. Consider \mathbb{R}^m with the maximum norm. In view of Theorem 3 we may choose real Banach limits L_1, L_2, \dots, L_m , such that L with $L((x_n)_{n \in \mathbb{N}}) := (L_i((x_n^{(i)})_{n \in \mathbb{N}}))_{1 \leq i \leq m}$ is a Banach limit for \mathbb{R}^m . Then we may construct some $\phi: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ by taking any $\phi_i: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the hypotheses of Theorem 4 and by defining $\phi(x, y) := (\phi_i(x_i, y_i))_{1 \leq i \leq m}$. Concrete examples for ϕ_i may be found in [10].

5. Abstract Banach limits and solutions of the inhomogeneous Cauchy equation

Analyzing the considerations of the previous section it turns out that condition (iv) of a Banach limit is not used when proving Theorem 4. Here we want to discuss a more general situation. Throughout we consider an Abelian semigroup S , an Abelian group $G \neq \{0\}$ and an inhomogeneity $\phi: S \times S \rightarrow G$, which satisfies (2) and (3) so that (1) has a solution $f: S \rightarrow G$.

Theorem 5. *Let \mathcal{S} be a subgroup of the group $G^{\mathbb{N}}$ of all sequences $(z_n)_{n \in \mathbb{N}}$ with $z_n \in G$ for all n . Let us suppose that all sequences $\underline{c} := (c)_{n \in \mathbb{N}}, c \in G$, are contained in \mathcal{S} and that $\sigma(\mathcal{S}) \subseteq \mathcal{S}$, where σ is defined as in Definition 1, (iii). Let furthermore $L: \mathcal{S} \rightarrow G$ be a homomorphism of groups such that $L(\underline{c}) = c$ for all $c \in G$ and $L \circ \sigma = L$.*

Then $g, g(x) := -L((\phi(nx, x)_{n \in \mathbb{N}}))$, is a solution of (1) provided that all sequences $(\phi(nx, x))_{n \in \mathbb{N}}$ and $(\phi(nx, ny))_{n \in \mathbb{N}}, x, y \in G$, are contained in \mathcal{S} .

Proof. Put $g_n(x) := \phi(nx, x)$ and $h_n(x, y) := \phi(nx, ny)$ and let $G(x) := (g_n(x))_{n \in \mathbb{N}}, H(x, y) := (h_n(x, y))_{n \in \mathbb{N}}$. Because 1 is solvable by assumption, relation (17) holds true, i.e.,

$$G(x + y) - G(x) - G(y) = \sigma(H(x, y)) - H(x, y) - \phi(x, y).$$

By assumption all sequences involved are contained in \mathcal{S} . Thus applying L and using the properties thereof we get $-g(x + y) + g(x) + g(y) = -\phi(x, y)$, as desired. \square

Now we discuss some examples of G, \mathcal{S}, L .

Example 5. G arbitrary, $\mathcal{S} = \underline{C} := \{c \mid c \in G\}, L(c) := c$.

Example 6. (Generalization of the example above) Let $n \in \mathbb{N}$, assume that $x \mapsto nx =: \mu_n(x) \in G$ is bijective and write $\frac{1}{n}x := \mu_n^{-1}(x)$. Let $\mathcal{S} := \underline{C}_n := \{x = (x_m)_{m \in \mathbb{N}} \in G^{\mathbb{N}} \mid x_{m+n} = x_m \text{ for all } m \in \mathbb{N}\}$ and $L(x) := \frac{1}{n}(x_1 + \dots + x_n)$.

Note that $L(c) = \frac{1}{n}(nc) = c$ and that $L(x) = \frac{1}{n} \sum_{i=m}^{m-1+n} x_i$ for all $m \in \mathbb{N}$, since $\sum_{i=m}^{m+n} x_i = \sum_{i=m}^{m-1+n} x_i - x_m + x_{m+n} = \sum_{i=m}^{m-1+n} x_i$.

Example 7. Let G be an Abelian group such that μ_n is bijective for all $n \in \mathbb{N}$, i.e., a uniquely divisible Abelian group and thus a rational vector space. Define $\mathcal{S} := \underline{C}_\infty := \bigcup_{n \in \mathbb{N}} \underline{C}_n$. This in fact is a linear subspace of G since $z \in \underline{C}_n, w \in \underline{C}_m, r, s \in \mathbb{Q}$ imply that $rz + sw \in \underline{C}_{nm}$. From Example 6 it follows that $\frac{1}{n}(x_1 + \dots + x_n) = \frac{1}{n} \sum_{i=m}^{m-1+n} x_i$ for all $x \in \underline{C}_n$. In fact

$$\frac{1}{n}(x_1 + \dots + x_n) = \frac{1}{m}(x_1 + \dots + x_m) \quad \text{for all } x \in \underline{C}_n \cap \underline{C}_m.$$

Since $\underline{C}_n, \underline{C}_m \subseteq \underline{C}_{nm}$, it is enough to show that $\frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{nm} \sum_{i=1}^{nm} x_i$, which follows from

$$\sum_{i=1}^{nm} x_i = \sum_{k=1}^m \sum_{i=(k-1)n+1}^{kn} x_i = \sum_{k=1}^m n \left(\frac{1}{n} \sum_{i=1}^n x_i \right) = mn \left(\frac{1}{n} \sum_{i=1}^n x_i \right).$$

This implies that $L: \underline{C}_\infty \rightarrow G, L(x) := \frac{1}{n} \sum_{i=1}^n x_i$ for $x \in \underline{C}_n$ is well-defined. Moreover L is linear and obviously has all the remaining properties to be satisfied.

Example 8. Let G be a normed space over K, K be a subfield of the field \mathbb{C} of complex numbers and let

$$\mathcal{S} := \{x \in G^{\mathbb{N}} \mid \mu(x) \text{ converges to some } \xi \in G\},$$

where $\mu(x) := (\frac{1}{n}(x_1 + x_2 + \dots + x_n))_{n \in \mathbb{N}}$. The function $L: \mathcal{S} \rightarrow G$ is defined by $L(x) := \lim_{n \rightarrow \infty} \mu(x) = \lim_{n \rightarrow \infty} \frac{1}{n}(x_1 + x_2 + \dots + x_n)$. Obviously L is linear and $L(c) = c$. Moreover $L(\sigma(x)) = L(x)$ since

$$\frac{1}{n}(x_2 + x_3 + \dots + x_{n+1}) = \frac{n+1}{n} \cdot \frac{1}{n+1}(x_1 + x_2 + \dots + x_{n+1}) - \frac{x_1}{n}.$$

In this example $\underline{C}_\infty \subseteq \mathcal{S}$ and $L(x) = \frac{1}{m}(x_1 + x_2 + \dots + x_m)$ for all $x \in \underline{C}_m \subseteq \underline{C}_\infty$:

If $x \in \underline{C}_m$ and $n \in \mathbb{N}$ let $k := [n/m]$, i.e., $mk \leq n < (k+1)m$. Then

$$\frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} \sum_{j=1}^k \sum_{i=(j-1)m+1}^{jm} x_i + \frac{1}{n} \sum_{i=mk+1}^n x_i = \frac{1}{n} km \frac{1}{m} \sum_{i=1}^m x_i + \frac{1}{n} \sum_{i=1}^{n-mk} x_i. \tag{18}$$

$\frac{km}{n} \leq 1 < \frac{km}{n} + \frac{m}{n}$ implies $1 - \frac{m}{n} < \frac{km}{n} \leq 1$. Thus the first term in the last part of (18) tends to $\frac{1}{n} \sum_{i=1}^m x_i$ for $n \rightarrow \infty$. The corresponding second part tends to zero since there are only finitely many terms of the form $\sum_{i=1}^{n-mk} x_i$.

Finally we note that in general $\mathcal{S} \not\subseteq \ell_\infty(G)$.

Let us assume that $\sqrt{k} \in K$ for all $k \in \mathbb{N}$. We fix some $c \in G \setminus \{0\}$ and put $x_{2k+\varepsilon} := (-1)^{2k+\varepsilon-1} \sqrt{k}c$ for $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}$. Then obviously $x \notin \ell_\infty(G)$ but $x \in \mathcal{S}$ since $\sum_{i=1}^{2k} x_i = 0$ and $\sum_{i=1}^{2k+1} x_i = \sqrt{k+1}c$.

Of course any normed space X admitting a Banach limit L is also an example when we take $\mathcal{S} = \ell_\infty(X)$.

The following remark however demonstrates that there must be some restrictions on the space \mathcal{S} .

Remark 2. For $\mathcal{S} = G^{\mathbb{N}}$ no suitable choice is possible: Assume that $L: G^{\mathbb{N}} \rightarrow G$ is a homomorphism and satisfies $L(\underline{c}) = c$ for all $c \in G$ and $L \circ \sigma = L$. Choose $c \in G \setminus \{0\}$ and define $x = (c, 2c, \dots, nc, \dots)$. Then $\sigma(x) - x = \underline{c}$ and $L(\underline{c}) = 0$, a contradiction.

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