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Aequationes Mathematicae

A note on functional equations connected with the Cauchy mean value theorem

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Abstract. The aim of this paper is to describe the solution (f, g) of the equation

$$[f(x) - f(y)]g'(\alpha x + (1-\alpha)y) = [g(x) - g(y)]f'(\alpha x + (1-\alpha)y), \;\; x, y \in I,$$

where $I \subset \mathbb{R}$ is an open interval, $f, g\colon I \to \mathbb{R}$ are differentiable, α is a fixed number from $(0, 1)$.

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1. Introduction

Throughout this paper I is an open interval, $\alpha \in (0, \frac{1}{2}]$ (we can obtain the whole interval $(0, 1)$ because the role of α and $1-\alpha$ is symmetric in (1)). For a differentiable function $f\colon I \to \mathbb{R}$ we define a set $U_f := \{x \in I : f'(x) \neq 0\}$. In view of the Darboux property of f' we can write U_f as a sum of pairwise disjoint open intervals (we denote this family of open intervals by $\mathcal{A}_f$).

We would like to present solutions of the following functional equation

$$[f(x) - f(y)]g'(\alpha x + (1-\alpha)y) = [g(x) - g(y)]f'(\alpha x + (1-\alpha)y), \;\; x, y \in I, \tag{1}$$

where $f, g\colon I \to \mathbb{R}$ are differentiable.

This equation was solved by Balogh et al. [2] for three times differentiable functions on $\mathbb{R}$. For the case of the Lagrange MVT ($g = \mathrm{id}$) with $\alpha = \frac{1}{2}$ this problem was considered by Haruki [3] and Aczél [1]. The generalization of (1) corresponds also to an open problem posed by Sahoo and Riedel [6].

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2. Auxiliary results

We divide our considerations into two cases $\alpha \neq \frac{1}{2}$ or $\alpha = \frac{1}{2}$. We start with two lemmas which we use in both cases.

Lemma 1. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions such that (1) holds. Then for each $x \in \Big((1-\alpha)\inf I + \alpha \inf U_f, (1-\alpha)\sup I + \alpha \sup U_f\Big)$ such that $\exists_{\varepsilon>0}\Big((x-\varepsilon, x] \subset I\backslash U_f$ or $[x, x+\varepsilon) \subset I\backslash U_f\Big)$ we have $g'(x) = 0$.*

Proof. Let $J \subset I\backslash U_f$ be a maximal closed subinterval of $I\backslash U_f$, $a = \inf J$, $b = \sup J$, $a < b$, which means that either $J = [a, b]$ and $a, b \in \operatorname{cl} U_f$ or $J = (\inf I, \inf U_f]$ or $J = [\sup U_f, \sup I)$. We have the following cases:

(a) $\exists_{I_1 \in \mathcal{A}_f} \sup I_1 = a$.
Let $z \in [a, \alpha a + (1-\alpha)b)$, then there exists $x \in I_1$ such that $a - x < \frac{\alpha a + (1-\alpha)b - z}{\alpha}$. Let $y = b - \frac{\alpha x + (1-\alpha)b - z}{1-\alpha}$. Then $z = \alpha x + (1-\alpha)y$. Since

$$\begin{aligned}(1-\alpha)(b-y) &= \alpha x + (1-\alpha)b - z = \alpha(x-a) + \alpha a + (1-\alpha)b - z\\ &> -(\alpha a + (1-\alpha)b - z) + \alpha a + (1-\alpha)b - z = 0,\end{aligned}$$

$$y = \frac{z-\alpha x}{1-\alpha} = \frac{z - \alpha a + \alpha(a-x)}{1-\alpha} > \frac{z-\alpha a}{1-\alpha} \geq \frac{a - \alpha a}{1-\alpha} = a,$$

we have $y \in J$. Hence

$$0 = [g(x) - g(y)]f'(z) = [f(x) - f(y)]g'(z) = [f(x) - f(a)]g'(z). \tag{2}$$

Since $f'|_{I_1} \neq 0$, we have $f(x) \neq f(a)$ and $g'(z) = 0$.

(b) $\exists_{(I_n)_{n\in\mathbb{N}} \subset \mathcal{A}_f}(\forall_{n\in\mathbb{N}} \sup I_n \leq \inf I_{n+1}) \wedge \lim_{n\to\infty} \sup I_n = a$. Let $z \in [a, \alpha a + (1-\alpha)b)$, then there exists $n \in \mathbb{N}$ such that $a - \inf I_n < \frac{\alpha a + (1-\alpha)b - z}{\alpha}$. Let $x \in I_n$ be such that $f(x) \neq f(a)$, $y = b - \frac{\alpha x + (1-\alpha)b - z}{1-\alpha}$. Then $z = \alpha x + (1-\alpha)y$. Similarly as in the previous case we have $y \in J$. Hence (2) holds and $g'(z) = 0$.

(c) $\exists_{I_1 \in \mathcal{A}_f} \inf I_1 = b$. Let $z \in (\alpha b + (1-\alpha)a, b]$, then there exists $x \in I_1$ such that $x - b < \frac{z - \alpha b + (1-\alpha)a}{\alpha}$. Let $y = a + \frac{z - \alpha x - (1-\alpha)a}{1-\alpha}$. Then $z = \alpha x + (1-\alpha)y$. Since

$$\begin{aligned}(1-\alpha)(y-a) &= z - \alpha x - (1-\alpha)a = z + \alpha(b-x) - \alpha b - (1-\alpha)a\\ &> z - (z - \alpha b - (1-\alpha)a) - \alpha b - (1-\alpha)a = 0,\end{aligned}$$

$$y = \frac{z-\alpha x}{1-\alpha} = \frac{z - \alpha b + \alpha(b-x)}{1-\alpha} < \frac{z - \alpha b}{1-\alpha} \leq \frac{b - \alpha b}{1-\alpha} = b,$$

we have $y \in J$. Hence (2) holds. Since $f'|_{I_1} \neq 0$, we have $f(x) \neq f(b)$ and $g'(z) = 0$.

(d) $\exists_{(I_n)_{n\in\mathbb{N}} \subset \mathcal{A}_f}(\forall_{n\in\mathbb{N}} \inf I_n \geq \sup I_{n+1}) \wedge \lim_{n\to\infty} \inf I_n = b$. Let $z \in (\alpha b + (1-\alpha)a, b]$, then there exists $n \in \mathbb{N}$ such that $\sup I_n - b < \frac{z - \alpha b + (1-\alpha)a}{\alpha}$.

Let $x \in I_n$ be such that $f(x) \neq f(b)$, $y = a + \frac{z-\alpha x-(1-\alpha)a}{1-\alpha}$. Then $z = \alpha x + (1-\alpha)y$. Similarly as in the previous case we have $y \in J$. Hence (2) holds and $g'(z) = 0$.

Since

$$\alpha b + (1-\alpha)a = a + \alpha(b-a) \leq \frac{a+b}{2} \leq b - \alpha(b-a) = \alpha a + (1-\alpha)b,$$

for $J \subset (\inf U_f, \sup U_f)$ we have $g'(x) = 0$ for $x \in J$.

If $J = (\inf I, \inf U_f]$, then $g'(x) = 0$ for $x \in ((1-\alpha)\inf I + \alpha \inf U_f, \inf U_f]$.

And finally, if $J = [\sup U_f, \sup I)$, then $g'(x) = 0$ for $x \in [\sup U_f, (1-\alpha)\sup I + \alpha \sup U_f)$. □

Lemma 2. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and satisfy Eq. (1). Assume that $\{1, f, g\}$ are linearly dependent on each $J \in \mathcal{A}_f$ and $\inf I = -\infty$ or $\sup I = +\infty$. Then $\{1, f, g\}$ are linearly dependent on I.*

Proof. It is easy to see that if f is constant, then for every g Eq. (1) holds. Assume that f is non-constant and $\sup I = +\infty$.
Let $I_1, I_2 \in \mathcal{A}_f$ satisfy $\sup I_1 \leq \inf I_2$. There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that $g(x) = c_1 f(x) + d_1$ for $x \in I_1$ and $g(x) = c_2 f(x) + d_2$ for $x \in I_2$. For each $x \in I_1$ and $z \in I_2$ we define $y_{x,z} = \frac{z-\alpha x}{1-\alpha} \in I$. Then $z = \alpha x + (1-\alpha)y_{x,z}$. We have

$$\begin{aligned}
&[f(x) - f(y_{x,z})]c_2 f'(z) = [f(x) - f(y_{x,z})]g'(z) \\
&\quad = [g(x) - g(y_{x,z})]f'(z) = [c_1 f(x) + d_1 - g(y_{x,z})]f'(z),\ x \in I_1,\ z \in I_2.
\end{aligned}$$

Hence

$$(c_1 - c_2)f(x)f'(z) = [g(y_{x,z}) - c_2 f(y_{x,z}) - d_1]f'(z),\ x \in I_1,\ z \in I_2,$$

and since $f'(z) \neq 0$, we obtain

$$(c_1 - c_2)f(x) = g\Big(\frac{z-\alpha x}{1-\alpha}\Big) - c_2 f\Big(\frac{z-\alpha x}{1-\alpha}\Big) - d_1,\ x \in I_1,\ z \in I_2. \tag{3}$$

Using the differentiation of the above equation with respect to z we obtain that RHS of (3) is constant and also LHS of (3) is constant. The function $f|_{I_1}$ is injective so we get $c_1 = c_2$.
This shows us that there exists $c \in \mathbb{R}$ such that

$$\forall_{J \in \mathcal{A}_f} \exists_{d_J \in \mathbb{R}} \forall_{x \in J}\, g(x) = cf(x) + d_J.$$

Using this form and Lemma 1, for each $J \in \mathcal{A}_f$ we have the following cases:

(a) $\exists_{I_1 \in \mathcal{A}_f} [\sup J, \inf I_1] \subset I \backslash U_f$.
Let $a = \sup J$, $b = \inf I_1$. We have

$$\begin{aligned}
cf(a) + d_J &= \lim_{x \to a^-} cf(x) + d_J = \lim_{x \to a^-} g(x) = g(a) = g(b) \\
&= \lim_{x \to b^+} g(x) = \lim_{x \to b^+} cf(x) + d_{I_1} = cf(b) + d_{I_1} = cf(a) + d_{I_1}.
\end{aligned}$$

Hence $d_{I_1} = d_J$.

(b) $\exists_{(I_n)_{n\in\mathbb{N}}\subset\mathcal{A}_f}\ (\forall_{n\in\mathbb{N}}\ \inf I_{n+1} \geq \sup I_n) \wedge [\lim_{n\to\infty} \sup I_n, \inf J] \subset I\backslash U_f$.
Let $a = \lim_{n\to\infty} \sup I_n$, $b = \inf J$. In view of the previous case we get $d_{I_n} = d_{I_{n+1}}$. We have

$$\begin{aligned} cf(a) + d_{I_n} &= \lim_{x\to a^-} cf(x) + d_{I_n} = \lim_{x\to a^-} g(x) = g(a) = g(b) = \lim_{x\to b^+} g(x) \\ &= \lim_{x\to b^+} cf(x) + d_J = cf(b) + d_J = cf(a) + d_J,\ n \in \mathbb{N}. \end{aligned}$$

Hence $d_{I_n} = d_J$ for all $n \in \mathbb{N}$.

(c) $\exists_{I_1\in\mathcal{A}_f}\ [\sup I_1, \inf J] \subset I\backslash U_f$.
Let $a = \sup I_1$, $b = \inf J$. We have

$$\begin{aligned} cf(a) + d_{I_1} &= \lim_{x\to a^-} cf(x) + d_{I_1} = \lim_{x\to a^-} g(x) = g(a) = g(b) \\ &= \lim_{x\to b^+} g(x) = \lim_{x\to b^+} cf(x) + d_J = cf(b) + d_J = cf(a) + d_J. \end{aligned}$$

Hence $d_{I_1} = d_J$.

(d) $\exists_{(I_n)_{n\in\mathbb{N}}\subset\mathcal{A}_f}\ (\forall_{n\in\mathbb{N}}\ \sup I_{n+1} \leq \inf I_n) \wedge [\sup J, \lim_{n\to\infty} \inf I_n] \subset I\backslash U_f$.
Let $a = \sup J$, $b = \lim_{n\to\infty} \inf I_n$. In view of the previous case we get $d_{I_n} = d_{I_{n+1}}$. We have

$$\begin{aligned} cf(a) + d_J &= \lim_{x\to a^-} cf(x) + d_J = \lim_{x\to a^-} g(x) = g(a) = g(b) = \lim_{x\to b^+} g(x) \\ &= \lim_{x\to b^+} cf(x) + d_{I_n} = cf(b) + d_{I_n} = cf(a) + d_{I_n},\ n \in \mathbb{N}. \end{aligned}$$

Hence $d_{I_n} = d_J$ for all $n \in \mathbb{N}$.

From the above we obtain that there exist $c, d \in \mathbb{R}$ such that $g(x) = cf(x) + d$ for all $x \in U_f$.
Now we will show that g also has this form on $I\backslash U_f$. We have the following cases:

- Let $J \subset (\inf U_f, \sup U_f)\backslash U_f$ be a closed interval such that $a, b \in \operatorname{cl} U_f$, where $a = \inf J$, $b = \sup J$. Then we have

$$\begin{aligned} g(x) = g(a) &= \lim_{y\to a^-,\ y\in U_f} g(y) = \lim_{y\to a^-,\ y\in U_f} cf(x) + d \\ &= cf(a) + d = cf(x) + d,\ x \in J. \end{aligned}$$

- Assume that $a = \sup U_f < \infty$ and let $J = [\sup U_f, +\infty)$. Then we have

$$\begin{aligned} g(x) = g(a) &= \lim_{y\to a^-,\ y\in U_f} g(y) = \lim_{y\to a^-,\ y\in U_f} cf(x) + d \\ &= cf(a) + d = cf(x) + d,\ x \in [a, +\infty). \end{aligned}$$

- Assume that $a := \inf I < b := \inf U_f$. Let $x \in (a, b]$, $z \in U_f$, $y := \frac{z-\alpha x}{1-\alpha}$. Then $z = \alpha x + (1-\alpha)y$ and from the form of g on U_f and the above two cases we have $g(y) = cf(y) + d$. Hence

$$g(x)f'(z) = [g(x) - g(y)]f'(z) + g(y)f'(z) = [f(x) - f(y)]g'(z) + [cf(y) + d]f'(z) = [f(x) - f(y)]cf'(z) + [cf(y) + d]f'(z) = [cf(x) + d]f'(z),$$

which means that $g(x) = cf(x) + d$.

Assume that f is non-constant and $\inf I = -\infty$. Let $F, G\colon -I \to \mathbb{R}$ be given by the formulas $F(x) = f(-x)$, $G(x) = g(-x)$ for $x \in -I$. Then we have $F'(x) = -f'(-x)$, $G'(x) = -g'(x)$ for $x \in -I$ and F, G satisfy Eq. (1). Since $\sup -I = +\infty$, there exist $c, d \in \mathbb{R}$ such that $G(x) = cF(x) + d$ for $x \in -I$, which means that $g(x) = cf(x) + d$ for $x \in I$. □

3. Main result for the asymmetric case

First, we consider the case when $\alpha \neq \frac{1}{2}$. We start with the following.

Lemma 3. *Let $J \subset \mathbb{R}$ be an open interval, $f, g, h\colon J \to \mathbb{R}$ be continuous functions, f be strictly monotone and the following functional equation hold*

$$g(x) - g(y) = h(\alpha x + (1-\alpha)y)[f(x) - f(y)],\ x, y \in J. \tag{4}$$

Then there exist $c, d \in \mathbb{R}$ such that $g(x) = cf(x) + d$, $h(x) = c$ for all $x \in J$.

Proof. Let $u \in J$. We define two numbers

$$a_u := \max\left\{\frac{1-2\alpha}{1-\alpha}\inf J + \frac{\alpha}{1-\alpha}u, \frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}\sup J\right\},$$
$$b_u := \min\left\{\frac{1-2\alpha}{1-\alpha}\sup J + \frac{\alpha}{1-\alpha}u, \frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}\inf J\right\}.$$

We observe that

$$\frac{1-2\alpha}{1-\alpha}\inf J + \frac{\alpha}{1-\alpha}u < \frac{1-2\alpha}{1-\alpha}u + \frac{\alpha}{1-\alpha}u = u,$$
$$\frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}\sup J < \frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}u = u,$$
$$\frac{1-2\alpha}{1-\alpha}\sup J + \frac{\alpha}{1-\alpha}u > \frac{1-2\alpha}{1-\alpha}u + \frac{\alpha}{1-\alpha}u = u,$$
$$\frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}\inf J > \frac{1-\alpha}{\alpha}u - \frac{1-2\alpha}{\alpha}u = u.$$

Hence $a_u < u < b_u$. We define an open interval $I_u := (a_u, b_u) \cap J$. Therefore $u \in I_u$.

Let $v \in I_u$. Then for $x = \frac{1-\alpha}{1-2\alpha}v - \frac{\alpha}{1-2\alpha}u, y = \frac{1-\alpha}{1-2\alpha}u - \frac{\alpha}{1-2\alpha}v$ we have

$$\alpha x + (1-\alpha)y = \frac{(\alpha-\alpha^2)v - \alpha^2 u + (1-2\alpha+\alpha^2)u - (\alpha-\alpha^2)v}{1-2\alpha} = u,$$
$$\alpha y + (1-\alpha)x = \frac{(\alpha-\alpha^2)u - \alpha^2 v + (1-2\alpha+\alpha^2)v - (\alpha-\alpha^2)u}{1-2\alpha} = v.$$

We also have

$$\begin{aligned} x =& \frac{1-\alpha}{1-2\alpha}v - \frac{\alpha}{1-2\alpha}u > \frac{(1-\alpha)a_u - \alpha u}{1-2\alpha} \\ &\geq \frac{(1-2\alpha)\inf J + \alpha u - \alpha u}{1-2\alpha} = \inf J, \\ x =& \frac{1-\alpha}{1-2\alpha}v - \frac{\alpha}{1-2\alpha}u < \frac{(1-\alpha)b_u - \alpha u}{1-2\alpha} \\ &\leq \frac{(1-2\alpha)\sup J + \alpha u - \alpha u}{1-2\alpha} = \sup J, \\ y =& \frac{1-\alpha}{1-2\alpha}u - \frac{\alpha}{1-2\alpha}v < \frac{(1-\alpha)u - \alpha a_u}{1-2\alpha} \\ &\leq \frac{(1-\alpha)u - (1-\alpha)u + (1-2\alpha)\sup J}{1-2\alpha} = \sup J, \\ y =& \frac{1-\alpha}{1-2\alpha}u - \frac{\alpha}{1-2\alpha}v > \frac{(1-\alpha)u - \alpha b_u}{1-2\alpha} \\ &\geq \frac{(1-\alpha)u - (1-\alpha)u + (1-2\alpha)\inf J}{1-2\alpha} = \inf J, \end{aligned}$$

which means that $x, y \in J$.

We observe that

$$\begin{aligned} h(u) = h(\alpha x + (1-\alpha)y) &= \frac{g(x) - g(y)}{f(x) - f(y)} \\ &= \frac{g(y) - g(x)}{f(y) - f(x)} = h(\alpha y + (1-\alpha)x) = h(v). \end{aligned}$$

Hence we have that h is constant on I_u.
Now we show that h is constant. Fix $u, v \in J, u < v$. Let $c_0 = u, c_n = \sup I_{c_{n-1}}$ for $n \in \mathbb{N}$. Since $c_{n-1} \in I_{c_{n-1}}$, $(c_n)_{n\in\mathbb{N}}$ is strictly increasing. We also have $I_{c_{n-1}} \cap I_{c_n} \neq \emptyset$, so h is constant on $(\inf I_u, \lim_{n\to\infty} c_n)$. If $\lim_{n\to\infty} c_n = +\infty$, then we get $h(u) = h(v)$. Assume that $\lim_{n\to\infty} c_n < +\infty$. We have

$$\begin{aligned} c_{n+1} - c_n &\geq \min\left\{\frac{1-2\alpha}{1-\alpha}\sup J + \frac{\alpha}{1-\alpha}c_n, \frac{1-\alpha}{\alpha}c_n - \frac{1-2\alpha}{\alpha}\inf J\right\} - c_n \\ &= \min\left\{\frac{1-2\alpha}{1-\alpha}(\sup J - c_n), \frac{1-2\alpha}{\alpha}(c_n - \inf J)\right\},\ n \in \mathbb{N}. \end{aligned}$$

Since $c_n - \inf J > c_1 - \inf J > 0$ for $n \in \mathbb{N}$,

$$0 = \lim_{n\to\infty}(c_{n+1} - c_n) = \lim_{n\to\infty} \frac{1-2\alpha}{1-\alpha}(\sup J - c_n),$$

which means that $\lim_{n\to\infty} c_n = \sup J$ and we get $h(u) = h(v)$.

Let $c := h(u)$ for $u \in J$. Fix $y \in J$ and let $d := g(y) - cf(y)$. We observe that

$$\begin{aligned} g(x) &= g(x) - g(y) + g(y) = [f(x) - f(y)]h(\alpha x + (1-\alpha)y) + g(y) \\ &= cf(x) - cf(y) + g(y) = cf(x) + d,\ x \in J, \end{aligned}$$

which ends the proof. □

Corollary 4. *Let $J \subset \mathbb{R}$ be an open interval, $f, g\colon J \to \mathbb{R}$ be differentiable functions such that (1) holds and $f'(x) \neq 0$ for all $x \in J$. Then there exist $c, d \in \mathbb{R}$ such that $g(x) = cf(x) + d$ for all $x \in J$.*

In view of the above corollary we obtain

Corollary 5. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and satisfy Eq. (1). Then $\{1, f, g\}$ are linearly dependent on each $J \in \mathcal{A}_f$.*

Finally we have the main result for the asymmetric case.

Theorem 6. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and satisfy Eq. (1). Assume that $\inf I = -\infty$ or $\sup I = +\infty$. Then either f is constant and g is an arbitrary function or there exist $c, d \in \mathbb{R}$ such that $g(x) = cf(x) + d$ for $x \in I$.*

Proof. In view of Corollary 5 and Lemma 2 we obtain the thesis of this theorem. □

Remark 7. It is easy to see that for differentiable functions $f, g\colon I \to \mathbb{R}$, if $\{1, f, g\}$ are linearly dependent, then (1) holds.

4. Main result for the symmetric case

Now we consider the case when $\alpha = \frac{1}{2}$.

Lundberg in his papers [4, Table 1], [5, Theorem 1.2] considers the following functional equation

$$\varphi(x+y) = \frac{F(x)G(y) + H(x)L(y)}{m(x) + n(y)}$$

on rectangles in $\mathbb{R}^2$ for continuous functions $F, G, H, L, m, n, \varphi$, which is a generalization of Eq. (1) on each $J \in \mathcal{A}_f$. He presents solutions of this equation but they have indirect forms, so we use only the fact that for differentiable functions $f, g\colon I \to \mathbb{R}$ which satisfy Eq. (1) we have two cases on every open interval $J \in \mathcal{A}_f$: either $\{1, f, g\}$ are linearly dependent on J or f and g are infinitely differentiable. Particularly, on every $J \in \mathcal{A}_f$, if $\{1, f, g\}$ are linearly independent, then they are three times differentiable and we can use the following two facts from [2].

Remark 8 (see [2, Remark 10]). Let $f, g\colon I \to \mathbb{R}$ be differentiable functions which satisfy (1). On every interval $J \in \mathcal{A}_f$ one of the following cases holds:

(a) $\{1, f, g\}$ are linearly dependent;
(b) $f, g \in \mathrm{Lin}\,\{1, \mathrm{id}, \mathrm{id}^2\}$;
(c) $f, g \in \mathrm{Lin}\,\{1, e^{\mu \mathrm{id}}, e^{-\mu \mathrm{id}}\}$ for some $\mu > 0$;
(d) $f, g \in \mathrm{Lin}\,\{1, \sin(\mu \mathrm{id}), \cos(\mu \mathrm{id})\}$ for some $\mu > 0$;

where id is the identity on J.

Lemma 9 (see [2, Lemma 11]). *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions which satisfy (1), $J \in \mathcal{A}_f$ be such that $g'(a) = 0$, where $a = \inf J > \inf I$ or $a = \sup J < \sup I$. Then $\{1, f, g\}$ are linearly dependent on J.*

This lemma is proved in the case $a = \inf J > \inf I$, but the proof in the case $a = \sup J < \sup I$ is analogous.

Now we are ready to prove the main result.

Theorem 10. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and satisfy Eq. (1). Then one of the following possibilities holds:*

(a) $\{1, f, g\}$ *are linearly dependent on each* $J \in \mathcal{A}_f$;
(b) $f, g \in Lin\,\{1, id, id^2\}$;
(c) $f, g \in Lin\,\{1, e^{\mu id}, e^{-\mu id}\}$ *for some* $\mu > 0$;
(d) $f, g \in Lin\,\{1, \sin(\mu id), \cos(\mu id)\}$ *for some* $\mu > 0$.

Proof. We can split $\mathcal{A}_f$ into disjoint subsets $\mathcal{A}_f = \mathcal{L}_f \cup \mathcal{Q}_f \cup \mathcal{E}_f \cup \mathcal{T}_f$, where

$$\mathcal{L}_f = \{J \in \mathcal{A}_f : \{1, f|_J, g|_J\} \text{ are linearly dependent}\},$$
$$\mathcal{Q}_f = \{J \in \mathcal{A}_f \backslash \mathcal{L}_f : f|_J, g|_J \in \mathrm{Lin}\,\{1, \mathrm{id}, \mathrm{id}^2\}\},$$
$$\mathcal{E}_f = \{J \in \mathcal{A}_f \backslash \mathcal{L}_f : f|_J, g|_J \in \mathrm{Lin}\,\{1, e^{\mu \mathrm{id}}, e^{-\mu \mathrm{id}}\} \text{ for some } \mu > 0\},$$
$$\mathcal{T}_f = \{J \in \mathcal{A}_f \backslash \mathcal{L}_f : f|_J, g|_J \in \mathrm{Lin}\,\{1, \sin(\mu \mathrm{id}), \cos(\mu \mathrm{id})\} \text{ for some } \mu > 0\}.$$

We have the same split for $\mathcal{A}_g$.

If $U_f = U_g = I$, then in view of Remark 8 we have the thesis of this theorem, so we can assume that $U_f \neq I$.

Now we consider four cases:

- Assume that $\mathcal{Q}_f \neq \emptyset$. Let $I \in \mathcal{Q}_f$. We have $f'(p) = 0$, where $p = \inf J > \inf I$ or $p = \sup J < \sup I$. Since there exist $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ such that $f(x) = a_1 x^2 + a_2 x + a_3$, $g(x) = b_1 x^2 + b_2 x + b_3$ for $x \in J$ and f' on $\mathrm{cl}\, J$ has only one root, we have that $0 = f'(p) = 2a_1 p + a_2$ and $J = (p, \sup I)$ or $(\inf I, p)$. In view of Lemma 9 we have $g'(p) \neq 0$. Hence

$$0 = [g(p + h) - g(p - h)]f'(p) = [f(p + h) - f(p - h)]g'(p),$$
$$h \in \mathbb{R}, p + h, p - h \in I,$$

which means that

$$f(p+h)=f(p-h),\ h\in\mathbb{R}, p+h, p-h\in I. \tag{5}$$

We have

$$\begin{aligned} f(p-h)=f(p+h)&=a_1(p+h)^2+a_2(p+h)+a_3\\ &=a_1(p-h)^2+a_2(p-h)+a_3+2h(2a_1p+a_2)\\ &=a_1(p-h)^2+a_2(p-h)+a_3,\ h\in\mathbb{R}, p+h\in J,\ p-h\in I, \end{aligned}$$

so the set $(2p-J)\cap I$ is a subset of some $J_2\in\mathcal{Q}_f$ ($\{1,f,g\}$ are linearly independent on J so also on $(2p-J)\cap I$). Since f' on $\mathrm{cl}\,J_2$ has only one root p, $J_2=I\backslash J$. Hence $f(x)=a_1x^2+a_2x+a_3$ for $x\in I$ and we can also prove in a similar way that $g(x)=b_1x^2+b_2x+b_3$ for $x\in I$.

- Assume that $\mathcal{E}_f\neq\emptyset$. Let $I\in\mathcal{E}_f$. We have $f'(p)=0$, where $p=\inf J>\inf I$ or $p=\sup J<\sup I$. Since there exist $\mu>0$, $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R}$ such that $f(x)=a_1e^{\mu x}+a_2e^{-\mu x}+a_3$, $g(x)=b_1e^{\mu x}+b_2e^{-\mu x}+b_3$ for $x\in J$ and f' on $\mathrm{cl}\,J$ has only one root, we have that $0=f'(p)=\mu(a_1e^{\mu p}-a_2e^{-\mu p})$ and $J=(p,\sup I)$ or $(\inf I,p)$. Similarly as in the first case we obtain (5). We also have

$$\begin{aligned} f(p-h)=f(p+h)&=a_1e^{\mu(p+h)}+a_2e^{-\mu(p+h)}+a_3\\ &=a_1e^{\mu(p-h)}+a_2e^{-\mu(p-h)}+a_3+(e^{\mu h}-e^{-\mu h})(a_1e^{\mu p}-a_2e^{-\mu p})\\ &=a_1e^{\mu(p-h)}+a_2e^{-\mu(p-h)}+a_3,\ h\in\mathbb{R}, p+h\in J,\ p-h\in I, \end{aligned}$$

  so the set $(2p-J)\cap I$ is a subset of some $J_2\in\mathcal{E}_f$ ($\{1,f,g\}$ are linearly independent on J so also on $(2p-J)\cap I$). Since f' on $\mathrm{cl}\,J_2$ has only one root p, $J_2=I\backslash J$. Hence $f(x)=a_1e^{\mu x}+a_2e^{-\mu x}+a_3$ for $x\in I$ and we can also prove in a similar way that $g(x)=b_1e^{\mu x}+b_2e^{-\mu x}+b_3$ for $x\in I$.
- Assume that $\mathcal{T}_f\neq\emptyset$. Let $I\in\mathcal{T}_f$. We have $f'(p)=0$, where $p=\inf J>\inf I$ or $p=\sup J<\sup I$. Since there exist $\mu>0$, $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R}$ such that $f(x)=a_1\cos(\mu x)+a_2\sin(\mu x)+a_3$, $g(x)=b_1\cos(\mu x)+b_2\sin(\mu x)+b_3$ for $x\in J$, we have that $0=f'(p)=\mu(a_2\cos(\mu p)-a_1\sin(\mu p))$. Similarly as in the first case we obtain (5). We also have

$$\begin{aligned} &f(p-h)=f(p+h)=a_1\cos(\mu(p+h))+a_2\sin(\mu(p+h))+a_3\\ &=a_1\cos(\mu(p-h))+a_2\sin(\mu(p-h))+a_3\\ &\quad+a_1[\cos(\mu(p+h))-\cos(\mu(p-h))]+a_2[\sin(\mu(p+h))-\sin(\mu(p-h))]\\ &=a_1\cos(\mu(p-h))+a_2\sin(\mu(p-h))+a_3\\ &\quad+2a_2\sin(\mu p)\sin(-\mu h)+2a_1\sin(\mu h)\cos(\mu p)\\ &=a_1\cos(\mu(p-h))+a_2\sin(\mu(p-h))+a_3 \end{aligned}$$

$$+\, 2\sin(\mu h)[a_2\cos(\mu p) - a_1\sin(\mu p)]$$
$$= a_1\cos(\mu(p-h)) + a_2\sin(\mu(p-h)) + a_3,\ \ h\in\mathbb{R}, p+h\in J,\ p-h\in I,$$

so the set $(2p-J)\cap I$ is a subset of some $J_2\in\mathcal{T}_f$ ($\{1,f,g\}$ are linearly independent on J so also on $(2p-J)\cap I$). Using this method we obtain that $f(x)=a_1\cos(\mu x)+a_2\sin(\mu x)+a_3$ for $x\in I$ and we can also prove in a similar way that $g(x)=b_1e^{\mu x}+b_2e^{-\mu x}+b_3$ for $x\in I$.

- Assume that $\mathcal{L}_f\neq\emptyset$. From the three previous cases we get that $\mathcal{A}_f=\mathcal{L}_f$.

□

Theorem 11. *Let $f,g\colon I\to\mathbb{R}$ be differentiable and satisfy Eq.* (1) *and* $\sup I=+\infty$ *or* $\inf I=-\infty$. *Then one of the following possibilities holds:*

(a) $\{1,f,g\}$ *are linearly dependent;*
(b) $f,g\in Lin\{1,id,id^2\}$;
(c) $f,g\in Lin\{1,e^{\mu id},e^{-\mu id}\}$ *for some* $\mu>0$;
(d) $f,g\in Lin\{1,\sin(\mu id),\cos(\mu id)\}$ *for some* $\mu>0$.

Proof. We have only to show that in the first case we have the linear dependence on the whole I, which follows from Lemma 2. □

Remark 12. It is not difficult to check that (1) holds for differentiable functions $f,g\colon I\to\mathbb{R}$ which have one of the forms a – d on (arbitrary) I.

5. Final remarks

If we have an arbitrary open interval we cannot obtain the linear dependency of $\{1,f,g\}$ on the whole I as in Theorems 6 and 11. We have some arbitrariness near the ends of the interval (Example 13) or different coefficients of linear dependence between f and g on the intervals from $\mathcal{A}_f$ (Example 14).

Example 13. Let $h\colon(-1,\alpha-1)\to\mathbb{R}$ be a differentiable function such that $\lim\limits_{x\to(\alpha-1)^-}h(x)=1$ and $\lim\limits_{x\to(\alpha-1)^-}h'(x)=0$, functions $f,g\colon(-1,\alpha)\to\mathbb{R}$ are given by the formulas

$$f(x)=\begin{cases}0, & x\in(-1,0]\\ x^2, & x\in(0,\alpha)\end{cases},$$
$$g(x)=\begin{cases}h(x), & x\in(-1,\alpha-1)\\ 1, & x\in[\alpha-1,0]\\ x^2+1, & x\in(0,\alpha)\end{cases}.$$

Then f and g satisfy Eq. (1). Indeed, we have

- if $x,y\geq\alpha-1$, then $\alpha x+(1-\alpha)y\geq\alpha-1$ and since $g(z)=f(z)+1$ for $z\geq\alpha-1$, we get $f'(\alpha x+(1-\alpha)y)=g'(\alpha x+(1-\alpha)y)$ and $f(x)-f(y)=g(x)-g(y)$.

- if $x, y \le 0$ then $\alpha x + (1-\alpha)y \le 0$, so we get $f'(\alpha x + (1-\alpha)y) = 0$ and $f(x) - f(y) = 0$.
- if $x < \alpha - 1$ and $y > 0$, then
$$\alpha x + (1-\alpha)y > \alpha x > -\alpha \ge \alpha - 1,$$
$$\alpha x + (1-\alpha)y < \alpha(\alpha-1) + (1-\alpha)\alpha = 0,$$
so $\alpha x + (1-\alpha)y \in (\alpha - 1, 0)$, which give us $f'(\alpha x + (1-\alpha)y) = 0$ and $g'(\alpha x + (1-\alpha)y) = 0$.
- if $y < \alpha - 1$ and $x > 0$, then
$$\alpha x + (1-\alpha)y > (1-\alpha)y > 1-\alpha,$$
$$\alpha x + (1-\alpha)y < \alpha\alpha + (1-\alpha)(\alpha-1) = 2\alpha - 1 \le 0,$$
so $\alpha x + (1-\alpha)y \in (\alpha - 1, 0)$, which give us $f'(\alpha x + (1-\alpha)y) = 0$ and $g'(\alpha x + (1-\alpha)y) = 0$.

Example 14. Let $c \in \mathbb{R}$, functions $f, g\colon (-\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha}) \to \mathbb{R}$ be given by the formulas
$$f(x) = \begin{cases} -x^2, & x \in (-\frac{\alpha}{1-\alpha}, 0) \\ 0, & x \in [0,1] \\ (x-1)^2, & x \in (1, \frac{1}{1-\alpha}) \end{cases},$$
$$g(x) = \begin{cases} -cx^2, & x \in (-\frac{\alpha}{1-\alpha}, 0) \\ 0, & x \in [0,1] \\ (x-1)^2, & x \in (1, \frac{1}{1-\alpha}) \end{cases}.$$
Then f and g satisfy Eq. (1). Indeed, we have

- if $x, y \in (-\frac{\alpha}{1-\alpha}, 0)$ or $x, y \in [0,1]$ or $x, y \in (1, \frac{1}{1-\alpha})$, then it is obvious.
- if $x < 0$ and $y > 1$, then
$$0 \le 1 - 2\alpha \le -\alpha + (1-\alpha) \le \alpha x + (1-\alpha)y \le (1-\alpha)y \le 1,$$
so $f'(\alpha x + (1-\alpha)y) = g'(\alpha x + (1-\alpha)y) = 0$.
- if $y < 0$ and $x > 1$, then
$$0 = \alpha - (1-\alpha)\frac{\alpha}{1-\alpha} \le \alpha x + (1-\alpha)y \le \alpha x \le \frac{\alpha}{1-\alpha} \le 1,$$
so $f'(\alpha x + (1-\alpha)y) = g'(\alpha x + (1-\alpha)y) = 0$.
- if $x < 0 \le y \le 1$ then either $\alpha x + (1-\alpha)y \in [0,1]$ (and we get $f'(\alpha x + (1-\alpha)y) = g'(\alpha x + (1-\alpha)y) = 0$) or $\alpha x + (1-\alpha)y < 0$ (and we obtain $f(x)g'(\alpha x + (1-\alpha)y) = g(x)f'(\alpha x + (1-\alpha)y)$).
Analogously we have the case when $y < 0 \le x \le 1$.
- if $0 \le x \le 1 < y$ then either $\alpha x + (1-\alpha)y \in [0,1]$ (and we get $f'(\alpha x + (1-\alpha)y) = g'(\alpha x + (1-\alpha)y) = 0$) or $\alpha x + (1-\alpha)y > 1$ (and we obtain $f(x)g'(\alpha x + (1-\alpha)y) = g(x)f'(\alpha x + (1-\alpha)y)$).
Analogously we have the case when $0 \le y \le 1 < x$.

Sahoo and Riedel [6, 9, Sect. 2.7] posed the following

Problem 15. Find all functions $f, g, \varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$[f(x) - f(y)]\varphi\left(\frac{x+y}{2}\right) = [g(x) - g(y)]\psi\left(\frac{x+y}{2}\right), \quad x, y \in \mathbb{R}. \tag{6}$$

This problem was solved by Balogh et al. [2, Theorem 12] for three times differentiable functions f, g.

Using Theorem 11 we can extend their solutions for differentiable functions (the proof is similar).

Theorem 16. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and $\varphi, \psi\colon I \to \mathbb{R}$ be arbitrary functions satisfying Eq. (6), $\sup I = +\infty$ or $\inf I = -\infty$. If $\forall_{x\in I}\ [\varphi(x) \neq 0 \vee \psi(x) \neq 0]$, then one of the following possibilities holds:*

(a) *there exist constants $A_0, A_1, A_2 \in \mathbb{R}$ such that for all $s \in I$ we have $A_0 + A_1 f(s) + A_2 g(s) = 0$ and $[A_1\psi(s) + A_2\varphi(s)]g'(s) = 0$;*

(b) *there exist constants $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathbb{R}$ such that for all $s \in I$ we have $f(s) = A_0 + A_1 s + A_2 s^2$, $g(s) = B_0 + B_1 s + B_2 s^2$ and*

$$[A_1 + 2A_2 s]\varphi(s) = [B_1 + 2B_2 s]\psi(s);$$

(c) *there exist $\mu > 0$ and constants $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathbb{R}$ such that for all $s \in I$ we have $f(s) = A_0 + A_1 e^{\mu s} + A_2 e^{-\mu s}$, $g(s) = B_0 + B_1 e^{\mu s} + B_2 e^{-\mu s}$ and*

$$[A_1 e^{\mu s} - A_2 e^{-\mu s}]\varphi(s) = [B_1 e^{\mu s} - B_2 e^{-\mu s}]\psi(s);$$

(d) *there exist $\mu > 0$ and constants $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathbb{R}$ such that for all $s \in I$ we have $f(s) = A_0 + A_1 \sin(\mu s) + A_2 \cos(\mu s)$, $g(s) = B_0 + B_1 \sin(\mu s) + B_2 \cos(\mu s)$ and*

$$[A_1 \cos(\mu s) - A_2 \sin(\mu s)]\varphi(s) = [B_1 \cos(\mu s) - B_2 \sin(\mu s)]\psi(s).$$

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