On the special form of integral convolution type inequality due to Walter and Weckesser

TOMASZ MAŁOLEPSZY AND JANUSZ MATKOWSKI

Dedicated to Professor Karol Baron on his 70th birthday.

Abstract. Walter and Weckesser's result (Aequationes Math 46:212–219, 1993), extending the Bushell–Okraśiński convolution type inequality (Bushell and Okraśiński in J Lond Math Soc (2) 41:503–510, 1990), gave some general conditions on the functions $k : [0, d) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ under which, for every increasing function $f : [0, d) \rightarrow [0, \infty)$, the inequality

$$\int_0^x k(x-s)g(f(s))ds \leq g\left(\int_0^x f(s)ds\right), \quad x \in (0, d),$$

is satisfied. Applying the result on a simultaneous system of functional inequalities, we prove that if $d > 1$, then, in general, both k and g must be power functions.

Mathematics Subject Classification. Primary 39B72, 26D15.

Keywords. Bushell–Okraśiński inequality, Walter–Weckesser theorem, System of functional inequalities.

1. Introduction

Inequality

$$\int_0^x (x-s)^{p-1} f(s) ds \leq \left(\int_0^x f(s)^{1/p} ds \right)^p, \quad x \in [0, d], \quad (1.1)$$

where $d \in (0, 1]$, $p \geq 1$ and $f : [0, d] \rightarrow [0, \infty)$ is an arbitrary continuous increasing (throughout the paper we use terms increasing and decreasing in the weak sense) function, is called the classical Bushell–Okraśiński inequality (the non-classical forms of (1.1) include its generalizations to various kinds of integrals—for instance, fuzzy integrals like the Sugeno integral [8] or the

universal integral [1]). It was proved in [3] as an auxiliary result in the study of the existence of solutions of some class of Volterra integral equations and almost immediately questions about an extension of (1.1) arose [9]. The first such extension was given by Walter and Weckesser in [10], but also later many other papers were published, containing results that in particular lead to the Bushell–Okrański inequality (see [4, 5, 7]).

In the aforementioned article [10] Walter and Weckesser proved the following theorem:

Theorem 1. *Let for every $c \in (0, d]$ the function h_c be defined by*

$$h_c(y) := g(cy) - K(c)g(y), \quad y \in [0, \infty), \quad (1.2)$$

where K is given by $K(x) := \int_0^x k(s)ds$ with k being Lebesgue integrable on $[0, d]$. If one of the following two conditions is satisfied:

- (i) $f : [0, d] \rightarrow [0, \infty)$ is increasing, $g : [0, \infty) \rightarrow \mathbb{R}$ is convex and such that for every $c \in (0, d]$ the function h_c is nonnegative and increasing;
- (ii) $f : [0, d] \rightarrow [0, \infty)$ is decreasing, $g : [0, \infty) \rightarrow \mathbb{R}$ is concave and such that for every $c \in (0, d]$ the function h_c is nonnegative and decreasing,

then

$$\int_0^x k(x-s)g(f(s))ds \leq g\left(\int_0^x f(s)ds\right), \quad x \in (0, d].$$

It is easy to observe that the Bushell–Okrański inequality (1.1) can be obtained from Theorem 1 by taking functions g and k , both of which are power functions with the exponent $p \geq 1$ (it is worth noting that now the assumption $d \leq 1$ is no longer needed). Because of the importance of the Bushell–Okrański inequality, the natural question arises: under what additional conditions does an inequality in the Walter–Weckesser theorem reduce to the Bushell–Okrański inequality? In this paper we give a partial answer to this question. We show that such a reduction is enforced if, in essential, $d > 1$ and functions g and K are positive, provided that K also satisfies a certain inequality. The main tool that we use to prove this is a certain result on the solutions of the simultaneous system of functional inequalities stated in the next section as Theorem 2.

2. Auxiliary results

Theorem 2. ([6], Theorem 4) *Let $a, b, \alpha, \beta \in (0, \infty)$ be such that*

$$\begin{aligned} 0 < a < 1 < b, \quad \frac{\log b}{\log a} &\notin \mathbb{Q}, \\ \frac{\log \beta}{\log b} &\leq \frac{\log \alpha}{\log a}, \end{aligned}$$

where $\mathbb{Q}$ denotes the set of rational numbers. Suppose that a function $g : (0, \infty) \rightarrow \mathbb{R}$ satisfies the system of inequalities

$$g(ax) \leq \alpha g(x), \quad g(bx) \leq \beta g(x), \quad x > 0,$$

g is continuous at least at one point, and $g((0, \infty)) \not\subset (-\infty, 0)$. Then either

$$g(x) = 0, \quad x > 0,$$

or

$$p := \frac{\log \beta}{\log b} = \frac{\log \alpha}{\log a}$$

and

$$g(x) = g(1)x^p, \quad x > 0.$$

The suitable result for the reversed inequalities reads as follows.

Theorem 3. Let $a, b, \alpha, \beta \in (0, \infty)$ be such that

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q},$$

$$\frac{\log \beta}{\log b} \geq \frac{\log \alpha}{\log a}.$$

Suppose that a function $g : (0, \infty) \rightarrow \mathbb{R}$ satisfies the system of inequalities

$$g(ax) \geq \alpha g(x), \quad g(bx) \geq \beta g(x), \quad x > 0,$$

g is continuous at least at one point, and $g((0, \infty)) \not\subset (-\infty, 0)$. Then either

$$g(x) = 0, \quad x > 0,$$

or

$$p := \frac{\log \beta}{\log b} = \frac{\log \alpha}{\log a}$$

and

$$g(x) = g(1)x^p, \quad x > 0.$$

Let $d > 0$ be arbitrarily fixed. In order to directly use Theorems 2 and 3, we must relax some of the assumptions about the domains g and K made in Theorem 1, so in this section let $g : (0, \infty) \rightarrow [0, \infty)$, k be a Lebesgue integrable function such that $K : (0, d) \rightarrow (0, \infty)$, where

$$K(x) := \int_0^x k(s) ds, \quad (2.1)$$

and let a bivariable function $h : (0, d) \times (0, \infty) \rightarrow \mathbb{R}$ be defined by the following formula

$$h(x, y) := g(xy) - K(x)g(y), \quad x \in (0, d), y \in (0, \infty). \quad (2.2)$$

Applying Theorems 2 and 3, we show now that the constant sign of h implies that g must be a power function. Namely, we have the following

Theorem 4. *Let a bivariable function $h : (0, d) \times (0, \infty) \rightarrow \mathbb{R}$ be given by (2.2), where $g : (0, \infty) \rightarrow [0, \infty)$ is continuous at a point and $K : (0, d) \rightarrow (0, \infty)$.*

If $1 < d \leq \infty$, and one of the following two conditions is satisfied:

- (i) *the function h is nonnegative and there exist $a, b \in (0, d)$ such that*

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\log K(b)}{\log b} \geq \frac{\log K(a)}{\log a},$$

- (ii) *the function h is nonpositive and there exist $a, b \in (0, d)$ such that*

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\log K(b)}{\log b} \leq \frac{\log K(a)}{\log a},$$

then either

$$g(y) = 0, \quad y > 0,$$

or g is positive,

$$\frac{\log K(b)}{\log b} = \frac{\log K(a)}{\log a},$$

and

$$g(y) = g(1) y^p, \quad y \in (0, \infty),$$

where

$$p := \frac{\log K(a)}{\log a}.$$

Proof. Assume that (i) holds true. Putting

$$\alpha := K(a), \quad \beta := K(b)$$

and using the nonnegativity of h , we notice that g satisfies the following system of inequalities

$$g(ay) \geq \alpha g(y), \quad g(by) \geq \beta g(y), \quad y > 0.$$

Because g is continuous at a point, Theorem 3 implies the result.

In case (ii), applying Theorem 2, we argue analogously. □

This theorem does not give any specific information about K . It turns out, however, that if g is positive and K satisfies an additional condition, which is quite natural in the context of the inequality $\frac{\log K(b)}{\log b} \geq \frac{\log K(a)}{\log a}$ in (i) or its converse in (ii), then both g and K must be power functions of the same exponent. Namely, we have the following

Theorem 5. *Let a bivariable function $h : (0, d) \times (0, \infty) \rightarrow \mathbb{R}$ be given by (2.2), where $g : (0, \infty) \rightarrow [0, \infty)$ is continuous at a point and $K : (0, d) \rightarrow (0, \infty)$ is continuous.*

If $1 < d \leq \infty$, and one of the following two conditions is satisfied:

(i) the function h is nonnegative and

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\};$$

(ii) the function h is non positive and

$$\inf \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \geq \sup \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\};$$

then either

$$g(y) = 0, \quad y > 0,$$

or g is positive, for some real p

$$g(y) = g(1)y^p \text{ for all } y \in (0, \infty); \quad K(x) = x^p \text{ for all } x \in (0, d),$$

and h is equal to zero, i.e.,

$$g(xy) = K(x)g(y), \quad x \in (0, d), \quad y \in (0, \infty).$$

Proof. Assume that condition (i) holds true. For an arbitrarily fixed number $a \in (0, 1)$, choose $b \in (1, d)$ such that $\log a$ and $\log b$ are incommensurable and put

$$\alpha := K(a), \quad \beta := K(b).$$

The assumed nonnegativity of h implies that g satisfies the simultaneous system of inequalities

$$g(ay) \geq \alpha g(y), \quad g(by) \geq \beta g(y), \quad y > 0.$$

Moreover, since $0 < a < 1 < b < d$, we have

$$\frac{\log \alpha}{\log a} = \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} = \frac{\log \beta}{\log b}.$$

Since g is continuous at a point, in view of Theorem 3, either $g \equiv 0$ or

$$g(y) = g(1)y^p, \quad y > 0,$$

where

$$p := \frac{\log \alpha}{\log a} = \frac{\log \beta}{\log b}.$$

To find the form of K in the latter case, notice that from the definitions of α , β and p , we have

$$\frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} = p,$$

whence

$$K(a) = a^p \quad \text{and} \quad K(b) = b^p.$$

Since the definition of p does not depend on b , and the set of the numbers b such that $\frac{\log b}{\log a} \notin \mathbb{Q}$ is dense in $(1, d)$, the continuity of K implies that $K(x) = x^p$ for

all $x \in [1, d]$. Interchanging the roles of a and b in this reasoning we conclude that $K(x) = x^p$ for all $x \in (0, 1)$. Thus we have shown that

$$K(x) = x^p, \quad x \in (0, d).$$

Now, for all $x \in (0, d)$ and $y > 0$, making use of the definition of h , we get

$$h(x, y) = g(xy) - K(x)g(y) = q(xy)^p - x^p qy^p = 0,$$

which shows that

$$g(xy) = K(x)g(y), \quad x \in (0, d), y > 0.$$

In case (ii), the simultaneous system of inequalities is reversed, so applying Theorem 2, we can argue similarly. This completes the proof. $\square$

Remark 1. In Theorem 5 condition (i), which is equivalent to the implication: for all s, t ,

$$0 < s < 1 < t < d \implies \frac{\log K(t)}{\log t} \geq \frac{\log K(s)}{\log s},$$

is satisfied, if the function

$$[(-\infty, \log d) \setminus \{0\}] \ni t \longmapsto [K(\exp t)]^{1/t}$$

is increasing. Similarly, condition (ii), which is equivalent to the implication: for all s, t ,

$$0 < s < 1 < t < d \implies \frac{\log K(t)}{\log t} \leq \frac{\log K(s)}{\log s},$$

is satisfied, if the function

$$[(-\infty, \log d) \setminus \{0\}] \ni t \longmapsto [K(\exp t)]^{1/t}$$

is decreasing.

3. Main results

From now on, following [10], we assume that functions g and K are defined on larger intervals, i.e. on $[0, \infty)$ and $(0, d]$, respectively. Based on Theorems 4 and 5, we obtain certain refinements of Theorem 1. To be more precise, with some additional assumptions about K (the same ones as in Theorems 4 and 5, respectively) we find all admissible forms of the nonnegative function g in Theorem 1, provided that $d > 1$.

Corollary 1. *Assume that $1 < d < \infty$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be convex, $K : (0, d] \rightarrow (0, \infty)$, and for every $c \in (0, d]$ let the function h_c defined by (1.2) be nonnegative and increasing.*

(i) *If there exist $a, b \in (0, d]$ such that*

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 1 \leq \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b},$$

then either

$$g \equiv 0,$$

or g is positive on $(0, \infty)$ and

$$g(y) = g(1)y^p, \quad y \in [0, \infty),$$

where

$$p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \geq 1.$$

(ii) If there exist $a, b \in (0, d]$ such that

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 0 \leq \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} < 1,$$

then

$$g \equiv 0.$$

Proof. (i) The function g , being a convex function, is in particular continuous in $(0, \infty)$. The nonnegativity of h_c and Theorem 4 imply that either g is zero in $(0, \infty)$ or $g(y) = g(1)y^p$ for all $y > 0$, with $p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \geq 1$. Moreover, the extension of g to the interval $[0, \infty)$ by putting $g(0) := \alpha$, $\alpha \geq 0$, preserves its convexity. To finish the proof, we have to show that such extensions preserve the nonnegativity and the increasingness of h_c . When $g \equiv 0$ on $(0, \infty)$, then, for all $c \in (0, d]$,

$$h_c(0) = g(0) - K(c)g(0) = \alpha(1 - K(c)) \quad \text{and} \quad h_c(y) = 0, \quad y > 0,$$

hence $\alpha = 0$, as $h_c(0) \leq h_c(y)$ for all $y > 0$. We obtain the same value of α for the second type of function g , i.e. $g(y) = g(1)y^p$ for $y \in (0, \infty)$. In this case we have

$$h_c(0) = \alpha(1 - K(c)) \quad \text{and} \quad h_c(y) = g(1)y^p(c^p - K(c)), \quad c \in (0, d], \quad y > 0.$$

As $\lim_{y \rightarrow 0^+} h_c(y) = 0$, the increasingness of h_c implies that $\alpha = 0$.

(ii) In this case a similar argument works. The difference in the thesis comes from the fact that the convexity of power functions fails when $p \in (0, 1)$. $\square$

A similar argument in combination with Theorem 5 gives

Corollary 2. Assume that $1 < d < \infty$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be convex, $K : (0, d] \rightarrow (0, \infty)$ be continuous and such that

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\},$$

and for every $c \in (0, d]$ let the function h_c be nonnegative and increasing.

(i) If

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} > 1,$$

then either

$$g \equiv 0,$$

or g is positive on $(0, \infty)$ and for some real $p > 1$

$$g(y) = g(1) y^p \text{ for all } y \in (0, \infty), \quad \text{and} \quad K(x) = x^p \text{ for all } x \in (0, d].$$

(ii) If

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} < 1,$$

then

$$g \equiv 0.$$

If g is concave, the corresponding results read as follows:

Corollary 3. Assume that $1 < d < \infty$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be concave, $K : (0, d] \rightarrow (0, \infty)$, and for every $c \in (0, d]$ let the function h_c be nonnegative and decreasing.

(i) If there exist $a, b \in (0, d]$ such that

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 1 < \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b},$$

then

$$g \equiv 0.$$

(ii) If there exist $a, b \in (0, d]$ such that

$$0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 0 < \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} \leq 1,$$

then either

$$g \equiv 0,$$

or g is positive on $(0, \infty)$ and

$$g(y) = g(1)y^p, \quad y \in [0, \infty),$$

where

$$p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \leq 1.$$

Corollary 4. Assume that $1 < d < \infty$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be concave, $K : (0, d] \rightarrow (0, \infty)$ be continuous and such that

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\},$$

and for every $c \in (0, d]$ let the function h_c be nonnegative and decreasing.

(i) If

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} > 1,$$

then

$$g \equiv 0.$$

(ii) If

$$\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} < 1,$$

then either

$$g \equiv 0,$$

or g is positive on $(0, \infty)$ and for some real $p < 1$

$$g(y) = g(1)y^p \text{ for all } y \in (0, \infty), \quad \text{and} \quad K(x) = x^p \text{ for all } x \in (0, d).$$

An immediate consequence of Corollary 1 is the following useful result showing that if in the Walter–Weckesser theorem $d > 1$ and k is taken in such a way that K is a power function, then the number of possible convex functions g is quite limited.

Corollary 5. Assume that $1 < d < \infty$. Let $K(x) = x^p$ for $x \in (0, d]$, $g : [0, \infty) \rightarrow [0, \infty)$ be convex, and for every $c \in (0, d]$ let the function h_c be nonnegative and increasing.

(i) If $p \geq 1$, then either

$$g(y) = g(1)y^p, \quad g(1) > 0,$$

or

$$g \equiv 0.$$

(ii) If $p \in (0, 1)$, then

$$g \equiv 0.$$

Proof. Choose a, b such that $0 < a < 1 < b$ and for which $\log a$ and $\log b$ are incommensurable. The form of K implies that $\frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} = p$, so the application of Corollary 1 ends the proof. $\square$

Remark 2. The assumption $d > 1$ in Corollary 5 is essential. If $d \in (0, 1]$, then for $K(x) = x^p$, there exist non-power convex functions g , for which h_c is nonnegative and increasing. An example of such a function is $g(y) = \frac{y^p}{y+1}$, where $p \geq 2$ is arbitrarily fixed. Indeed, we have

$$h_c(y) = \frac{y^{p+1}(c^p - c^{p+1})}{(cy+1)(y+1)}, \quad y \in [0, \infty), \quad c \in (0, d],$$

and, as $c^p > c^{p+1}$, the function h_c is nonnegative. To check its increasingness, notice that

$$h'_c(y) = \frac{(1-c)c^p y^p (p(y+1)(cy+1) - cy^2 + 1)}{(cy+1)^2 (y+1)^2} \geq 0,$$

iff

$$p(y+1)(cy+1) - cy^2 + 1 \geq 0, \quad y \in [0, \infty), \quad c \in (0, d]. \quad (3.1)$$

But inequality (3.1) is equivalent to

$$(pc - c)y^2 + (p + pc)y + p + 1 \geq 0, \quad y \in [0, \infty), \quad c \in (0, d],$$

and its validity can be easily deduced from the fact that for $y \in [0, \infty)$, $c \in (0, d]$ the inequality $y^2(pc - c) + y(p + pc) \geq 0$ holds true.

An analogous result for concave functions follows from Corollary 3.

Corollary 6. *Assume that $1 < d < \infty$. Let $K(x) = x^p$ for $x \in (0, d]$, $g : [0, \infty) \rightarrow [0, \infty)$ be concave, and for every $c \in (0, d]$ let the function h_c be nonnegative and decreasing.*

(i) *If $p > 1$, then*

$$g \equiv 0.$$

(ii) *If $p \in (0, 1]$, then either*

$$g(y) = g(1)y^p, \quad g(1) > 0,$$

or

$$g \equiv 0.$$

Acknowledgements

The authors are indebted to Dr. Dorota Krassowska for her important remark. The authors would also like to thank the reviewer for his/her valuable comments and helpful suggestions.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Agahi, H., Mohammadpour, A., Mesiar, R., Mansour Vaezpour, S.: Liapunov-type inequality for universal integral. *Int. J. Intell. Syst.* **27**, 908–925 (2012)
- [2] Bullen, P.S.: *A Dictionary of Inequalities*. Addison-Wesley Longman, London (1998)
- [3] Bushell, P.J., Okrański, W.: Nonlinear Volterra equations with convolution kernel. *J. Lond. Math. Soc. (2)* **41**, 503–510 (1990)
- [4] Egorov, Y.: On the best constant in a Poincaré–Sobolev inequality. *Oper. Theory Adv. Appl. Birkhäuser Basel* **117**, 106–109 (2000)
- [5] Heinig, H.P., Maligranda, L.: Weighted inequalities for monotone and concave functions. *Stud. Math.* **116**, 133–165 (1995)

- [6] Krassowska, D., Matkowski, J.: A pair of linear inequalities and a characterization of L^p -norm. *Ann. Polon. Math.* **85**(1), 1–11 (2005)
- [7] Malamud, S.M.: Some complements to the Jensen and Chebyshev inequalities and a problem of W. Walter. *Proc. Am. Math. Soc.* **129**, 2671–2678 (2001)
- [8] Romn-Flores, H., Flores-Franuli, A., Chalco-Cano, Y.: A convolution type inequality for fuzzy integrals. *Appl. Math. Comput.* **195**, 94–99 (2008)
- [9] Walter, W.: Problem: an integral inequality by Bushell and Okraśiński. In: *General Inequalities 6, Proceedings of the 6th International Conference on General Inequalities, Oberwolfach, Dec. 9–15, 1990, International Series on Numerical Mathematics, Birkhäuser, Basel, vol. 103, pp. 495–496* (1992)
- [10] Walter, W., Weckesser, V.: An integral inequality of convolution type. *Aequationes Math.* **46**, 212–219 (1993)

Tomasz Małolepszy and Janusz Matkowski
Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
ul. prof. Z. Szafrana 4a
65-516 Zielona Góra,
Poland
e-mail: T.Malolepszy@wmie.uz.zgora.pl

Janusz Matkowski
e-mail: J.Matkowski@wmie.uz.zgora.pl

Received: October 8, 2017

Revised: May 6, 2018