On the output function in a Ginsburg's machine

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Abstract. We give the form of the output function in Ginsburg's machine in which the input and output dictionaries are abelian groups and the transition function is of a special form.

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1. Introduction

By Ginsburg [4] the 5-tuple $(A, B, S, \delta, \lambda)$ is called a machine if

- A (the input dictionary) is a semigroup (with additive notation),
- B (the output dictionary) is a semigroup with additive notation and with the left cancellation property,
- S is an arbitrary set (the set of states), and
- $\delta: S \times A \rightarrow S$ is the transition function such that

$$\delta(\delta(s, x), y) = \delta(s, x + y) \quad \text{for } s \in S, x, y \in A, \quad (1)$$

- $\lambda: S \times A \rightarrow B$ is the output function such that

$$\lambda(s, x + y) = \lambda(s, x) + \lambda(\delta(s, x), y) \quad \text{for } s \in S, x, y \in A. \quad (2)$$

In the paper [4] the notations in the semigroups A and B are multiplicative. We use the additive notation for convenience.

Eq. (1) is called the translation equation and it has many applications [8].

The deterministic automaton [12] is an example of this machine. Here the input and output dictionaries are monoids in which the operation is the concatenation of words.

In this paper the general solutions of Eq. (2) are given if A and B are abelian groups, where $\delta: S \times A \rightarrow S$ is the solution of Eq. (1) for which

$$\{x \in A : \delta(s, x) = s\} \in \{\emptyset, \{0\}, A\} \text{ for every } s \text{ in } S. \quad (3)$$

2. Main result

The general solution $\delta: S \times A \rightarrow S$, where A is a group, of the translation Eq. (1) is of the form [6, 7]

$$\delta(s, x) = \begin{cases} \delta_k^{-1}[\delta_k(s) + x] & \text{for } s \in S_k, x \in A, k \in K, \\ s & \text{for } s \in f(S) \setminus \bigcup S_k, x \in A, \\ \delta_k^{-1}[\delta_k(f(s)) + x] & \text{for } s \in S \setminus f(S), f(s) \in S_k, x \in A, k \in K, \\ f(s) & \text{for } s \in S \setminus f(S), f(s) \in f(S) \setminus \bigcup S_k, x \in A, \end{cases} \quad (4)$$

where $f: S \rightarrow S$ is the idempotent function ($f(f)=f$), S_k for $k \in K$ (possibly, $K = \emptyset$) are disjoint subsets in $f(S)$ such that for every $k \in K$ there exists a subgroup A_k of A for which $1 < \text{card } S_k = \text{card } A/A_k$ and δ_k is a bijection from S_k onto A/A_k (the set of right cosets of the group A for the subgroup A_k - if A is abelian, A/A_k is the quotient group).

In particular, in case (3) holds, for the solution δ of (1) we have $A_k = \{0\}$ for $k \in K$, thus we can consider δ_k as a bijection from S_k onto A . This function δ_k is the inverse function of the bijection $\delta(s, \cdot) : A \rightarrow S_k$ for $s \in S_k$.

Theorem 1. *Let S be an arbitrary set and let A, B be abelian groups. Assume that $\delta: S \times A \rightarrow S$ is a solution of (1) of the form (4), for which (3) is true.*

The function $\lambda: S \times A \rightarrow B$ is a solution of (2) if and only if it is of the form

$$\lambda(s, x) = \begin{cases} \Lambda(\delta(s, x)) - \Lambda(s) & \text{for } s \in \bigcup S_k, x \in A, \\ a(s, x) & \text{for } s \in f(S) \setminus \bigcup S_k, x \in A, \\ g(s) + \Lambda(\delta(f(s), x)) - \Lambda(f(s)) & \text{for } s \in S \setminus f(S), f(s) \in \bigcup S_k, x \in A, \\ g(s) + a(f(s), x) & \text{for } s \in S \setminus f(S), f(s) \in f(S) \setminus \bigcup S_k, x \in A, \end{cases} \quad (5)$$

where Λ is an arbitrary function from $\bigcup S_k$ to B , $a(s, \cdot)$ for $s \in f(S) \setminus \bigcup S_k$ is an arbitrary additive function from A to B and g is an arbitrary function from $S \setminus f(S)$ to B .

Proof. of the “if” part. If $s \in \bigcup S_k$, then $\delta(s, x) \in \bigcup S_k$ for every $x \in A$. From here

$$\begin{aligned} \lambda(s, x) + \lambda(\delta(s, x), y) &= \Lambda(\delta(s, x)) - \Lambda(s) + \Lambda[\delta(\delta(s, x), y)] - \Lambda(\delta(s, x)) \\ &= \Lambda(\delta(s, x+y)) - \Lambda(s) = \lambda(s, x+y). \end{aligned}$$

Since $\delta(s, x) = s$ for $s \in f(S) \setminus \bigcup S_k$ we obtain in this case

$$\begin{aligned} \lambda(s, x+y) &= a(s, x+y) \\ &= a(s, x) + a(s, y) = \lambda(s, x) + a(\delta(s, x), y) = \lambda(s, x) + \lambda(\delta(s, x), y). \end{aligned}$$

Case $s \in S \setminus f(S)$, $f(s) \in \bigcup S_k$. Since $f(s) = \delta(s, 0)$, $\delta(s, 0) \in \bigcup S_k$ and $\delta(s, x) = \delta(\delta(s, 0), x) \in \bigcup S_k$. We have from here

$$\begin{aligned}\lambda(s, x+y) &= g(s) + \Lambda(\delta(s, x+y)) - \Lambda(s), \\ \lambda(s, x) &= g(s) + \Lambda(\delta(s, x)) - \Lambda(s), \\ \lambda(\delta(s, x), y) &= \lambda[\delta(\delta(s, x), y)] - \Lambda(\delta(s, x)),\end{aligned}$$

thus the function λ satisfies (2) in this case.

Case $s \in S \setminus f(S)$, $f(s) \in f(S) \setminus \bigcup S_k$. Since $f(s) = \delta(s, 0)$, $\delta(s, x) = \delta(f(s), x) = f(s) \in f(S) \setminus \bigcup S_k$. We have from here

$$\begin{aligned}\lambda(s, x+y) &= g(s) + a(f(s), x+y) \\ &= g(s) + a(f(s), x) + a(f(s), y) \\ &= \lambda(s, x) + a(\delta(s, x), y) \\ &= \lambda(s, x) + \lambda(\delta(s, x), y).\end{aligned}$$

□

Proof. of the “only if” part.

Assume that the function $\lambda: S \times A \rightarrow B$ is a solution of Eq. (2).

Case $s \in \bigcup S_k$. Thus $s \in S_l$ for a $l \in K$. Let s_l be the point of the set S_l such that $\delta_l(s_l) = 0$. We have

$$\lambda(s_l, x+y) = \lambda(s_l, y) + \lambda(\delta(s_l, y), x). \quad (6)$$

By the form of δ there exists y such that $\delta(s_l, y) = \delta_l^{-1}(\delta_l(s_l) + y) = s$, thus $y = \delta_l(s)$. By (6) we have

$$\begin{aligned}\lambda(s, x) &= \lambda(s_l, x + \delta_l(s)) - \lambda(s_l, \delta_l(s)) \\ &= \lambda\{s_l, \delta_l[\delta_l^{-1}(x + \delta_l(s))]\} - \lambda(s_l, \delta_l(s)) \\ &= \lambda\{s_l, \delta_l[\delta(s, x)]\} - \lambda(s_l, \delta_l(s)) = \Lambda_l(\delta(s, x)) - \Lambda_l(s)\end{aligned}$$

for $\Lambda_l(u) = \lambda(s_l, \delta_l(u))$. We have the required form of λ by the function $\Lambda(u) := \Lambda_k(u)$ for $u \in S_k$ and $k \in K$.

Case $s \in f(S) \setminus \bigcup S_k$. In this case $\delta(s, x) = s$ for $x \in A$, thus $\lambda(s, x+y) = \lambda(s, x) + \lambda(s, y)$. There exists an additive function $a(s, \cdot)$ for which $\lambda = a$.

Case $s \in S \setminus f(S)$, $f(s) \in \bigcup S_k$. Since $f(s) = \delta(s, 0)$, we have

$$\Lambda(s, x) = \lambda(s, 0) + \lambda(\delta(s, 0), x) = g(s) + \lambda(f(s), x) = g(s) + \Lambda(\delta(f(s), x)) - \Lambda(f(s))$$

for $g(s) = \lambda(s, 0)$.

Case $s \in S \setminus f(S)$, $f(s) \in f(S) \setminus \bigcup S_k$. Since $f(s) = \delta(s, 0)$, we have by (2) for $x = 0$:

$$\lambda(s, y) = \lambda(s, 0) + \lambda(f(s), y).$$

Since $\delta(f(s), x) = f(x)$ and $\lambda(f(s), x+y) = \lambda(f(s), x) + \lambda(\delta(f(s), x), y) = \lambda(f(s), x) + \lambda(f(s), y)$, the function $\lambda(f(s), \cdot)$ is additive, thus λ has the required form with $g(s) = \lambda(s, 0)$. □

Example 1. Let S be a group and let $\varphi: A \rightarrow S$ be an additive function. Then the function $\delta(s, x) = s + \varphi(x)$ is the solution of (1). If φ is an injection, then $A(s) = \{x \in X : \delta(s, x) = s\} = \{0\}$. The solution λ of (2) is of the form $\lambda(s, x) = \Lambda(\delta(s, x)) - \Lambda(s)$, where Λ is an arbitrary function from S to B . If A is a subgroup of B and φ is not an injection, then $\lambda(s, x) = x$ is the solution of (2) and there does not exist a function Λ such that $\lambda(s, x) = x = \Lambda(s + \varphi(x)) - \Lambda(s)$. Indeed, for $x_1, x_2 \in A, x_1 \neq x_2, \varphi(x_1) = \varphi(x_2)$ we have a contradiction.

Remark. The condition (3) is equivalent to the condition

$$\forall s \in S : [\delta(s, \cdot) : A \rightarrow S \text{ is an injection or } \delta(s, \cdot) \text{ is the constant function}].$$

Remark. In applications it is assumed that $\delta(s, 0) = s$ for $s \in S$. In this case the functions δ and λ in (4) and (5) have the following more simple form

$$\delta(s, x) = \begin{cases} \delta_k^{-1}[\delta_k(s) + x] & \text{for } s \in S_k, x \in A, k \in K, \\ s & \text{for } s \in S \setminus \bigcup S_k, x \in A, \end{cases} \quad (7)$$

$$\lambda(s, x) = \begin{cases} \Lambda(\delta(s, x)) - \Lambda(s) & \text{for } s \in \bigcup S_k, x \in A, \\ a(s, x) & \text{for } s \in f(S) \setminus \bigcup S_k, x \in A. \end{cases} \quad (8)$$

From here $\lambda(s, 0) = 0$ for every s in S .

Remark. Let $\delta \in (s, \cdot) : \mathbb{R} \rightarrow S \subset \mathbb{R}$ for $s \in S$ be continuous functions and let $\delta(s, 0) = s$ for $s \in S$. The solution $\delta : S \times \mathbb{R} \rightarrow S$ of (1), where S is a nondegenerated interval in $\mathbb{R}$, in this case is of the form (7), where S_k are open disjoint subintervals in S and δ_k are homeomorphisms from S_k onto $\mathbb{R}$ [11]. The function δ is then continuous and it is also called a dynamical system. Eq. (2) for $A = (\mathbb{R}, +)$ and $B = (0, +\infty)$ with the usual multiplication has the form

$$\lambda(s, x + y) = \lambda(s, x)\lambda(\delta(s, x), y), \quad (9)$$

thus the function $\ln \lambda(s, x)$ is of the form (8). From here we have the form of the function $\lambda(s, x) = \exp[\ln \lambda(s, x)]$.

The function $\lambda(s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for every $s \in I$ if and only if every function Λ_k for $k \in K$ is continuous and $a(s, x) = b(s)x$ for a function $b : S \rightarrow \mathbb{R}$.

Example 2. Let S be a group and let $\varphi: A \rightarrow S$ be an additive function. Then the function $\delta(s, x) = s + \varphi(x)$ is the solution of (1). If φ is a bijection, then $A(s) = \{x \in X : \delta(s, x) = s\} = \{0\}$ and $\delta(s, x) = \varphi(\varphi^{-1}(s) + x)$. The solution λ of (2) is of the form $\lambda(s, x) = \Lambda(\delta(s, x)) - \Lambda(s)$, where Λ is an arbitrary function from S to B .

Example 3. If the function h is a homomorphism from the semigroup A to the semigroup B , then the function $\lambda(s, x) = h(x)$ is the solution of (2) for every δ . If A and B are abelian groups and δ satisfies the condition (3), then this function λ has the form (5).

If the function δ does not satisfy the condition (3) the solution λ of (2) may be not of the form (5). Indeed, the function of the form $\delta(s, x) = g^{-1}(x + g(s))$, where $s \in \{0, 1\}$, $x \in \mathbb{Z}$, $g(0)$ is the set of even integers and $g(1)$ is the set of odd integers, is the solution of (1). The function $\lambda(s, x) = x$ is the output function of the machine $(\mathbb{Z}, \mathbb{Z}, \{0, 1\}, \delta, \lambda)$. Assume that this function is of the form (5). We have thus $x = \Lambda(\delta(s, x)) - \Lambda(s)$ for a function $\Lambda : \{0, 1\} \rightarrow \mathbb{Z}$. Since $\delta(0, 0) = 0 = \delta(0, 2)$, we have $0 = \Lambda(\delta(0, 0)) - \Lambda(0) = \Lambda(\delta(0, 2)) - \Lambda(0) = 2$, thus we obtain a contradiction.

3. Cocycles

The solution λ of Eq. (9) if δ is a dynamical system and $B = [0, +\infty)$ (with 0) with the usual multiplication, i.e., B is a abelian group (with multiplicative notation) with 0, is said to be the cocycle of δ . The 5-tuple $(\mathbb{R}, [0, +\infty), S, \delta, \lambda)$ is not a Ginsburg's machine since the cancellation property is not true in the interval $[0, +\infty)$ with the usual multiplication. We have the following corollary.

Corollary. *Let S, A, B, δ be an arbitrary set, an arbitrary abelian group (with additive notation), an abelian group G (with multiplicative notation) with 0 ($0.a = a.0 = 0$) and the solution of (1) satisfying (3) and $\delta(s, 0) = s$ for s in S , respectively. The function $\lambda : S \times A \rightarrow B$ is the solution of (9) if and only if it is of the form*

$$\lambda(s, x) = \begin{cases} 0 & \text{for } s \in S_0, x \in A \\ \Lambda(\delta(s, x))[\Lambda(s)]^{-1} & \text{for } s \in \bigcup S_k \setminus S_0, x \in A, \\ E(s, x) & \text{for } s \in S \setminus [\bigcup S_k \setminus S_0], x \in A, \end{cases} \quad (10)$$

where S_k are as in Theorem 1, $\Lambda : \bigcup S_k \setminus S_0 \rightarrow G$ is an arbitrary function, $E(s, \cdot)$ for $s \in S \setminus [\bigcup S_k \setminus S_0]$ is an arbitrary exponential function from A to G ($E(s, x+y) = E(s, x)E(s, y)$) and $S_0 = \emptyset$ or it is the union of transitive fibres $\delta(s, A)$ of δ (not necessarily all).

Proof. of the “if” part. If $S_0 = \emptyset$, then we have $\delta(s, x) \in S_k$ for $s \in S_k$. From here

$$\lambda(\delta(s, x), y) = \Lambda(\delta(\delta(s, x), y))[\Lambda(\delta(s, x))]^{-1} = \Lambda(\delta(s, x + y))[\Lambda(\delta(s, x))]^{-1}.$$

We obtain (10) by the form of $\lambda(s, x + y)$ and $\lambda(s, x)$ in this case.

If $S_0 = \emptyset$ and $s \in S \setminus \bigcup S_k$ we obtain (10) since here $\delta(s, x) = s$.

Assume that $S_0 \neq \emptyset$. The equality (10) is evident for $s \in S_0$.

If $s \in S_k \setminus S_0 \subset S \setminus S_0$, then $S \setminus S_0 \neq \emptyset$, thus it is the union of transitive fibres of δ , too. From here $\delta(s, x) \in S_k \setminus S_0$ and the proof of (10) is as above.

For $s \in S \setminus (\bigcup S_k \cup S_0)$ we have $\delta(s, x) = s$ thus

$$\lambda(s, x + y) = E(s, x + y) = E(s, x)E(s, y) = \lambda(s, x)\lambda(s, y) = \lambda(s, x)\lambda(\delta(s, x), y).$$

□

Proof. of the “only if” part. Assume that the function $\lambda : S \times A \rightarrow B$ is the solution of (9). Let S_0 be the set $\{s \in S : \exists x \in A : \lambda(s, x) = 0\}$. This set is equal to the set $\{s \in S : \forall x \in A : \lambda(s, x) = 0\}$ since if $\lambda(s, x_0) = 0$ for a x_0 in A , then $\lambda(s, x_0 + x) = 0$ for every x in A by (9). We prove that if $S_0 \neq \emptyset$, then it is the union of transitive fibres of δ , i.e., we prove that if $S_0 \cap S_k \neq \emptyset$, then $S_k \subset S_0$. Assume for the indirect proof that there exists $s_0 \in S_k$ which is not in S_0 . From here $\lambda(s_0, x) \neq 0$ for every x in A . There exists $s_1 \in S_0 \cap S_k$, thus $\lambda(s_1, x) = 0$ for every x in A . Since $s_0, s_1 \in S_k$, there exists x_0 such that $s_1 = \delta(s_0, x_0)$. By (9) we have

$$0 \neq \lambda(s_0, x_0 + x) = \lambda(s_0, x_0)\lambda(\delta(s_0, x_0), x) = \lambda(s_0, x_0)\lambda(s_1, x) = \lambda(s_0, x_0) \cdot 0 = 0.$$

By the above $\lambda(s, x) = 0$ for s in S_0 and x in A . For $s \in S \setminus S_0$ the function $\lambda(s, \cdot)$ has its values in the abelian group G (with multiplicative notation!) thus we have the conclusion by Remark 2. $\square$

Example 4. Let δ be the solution of (1) as in Example 1. Then the solution of Eq. (9) is of the form $\lambda(s, x) = 0$ or $\lambda(s, x) = \Lambda(x + \varphi^{-1}(s))[\Lambda(\varphi^{-1}(s))]^{-1}$, where $\Lambda : \bigcup S_k \setminus S_0 \rightarrow G$ is an arbitrary function.

Remark. The cocycle λ for which $S_0 \neq \emptyset$ and $S \neq S_0$ is not continuous since $\lambda(s, 0)$ is not continuous in this case. Indeed, for $s \in S_0$ we have $\lambda(s, 0) = 0$. For $s \in S \setminus S_0$ we obtain $\lambda(s, 0) = 1$ since $\lambda(s, 0) = \lambda(s, 0 + 0)\lambda(\delta(s, 0), 0) = \lambda(s, 0)\lambda(s, 0)$. If $S_0 = S$, then the cocycle λ is evidently continuous since $\lambda(s, x) = 0$ for $(s, x) \in S \times \mathbb{R}$.

The function

$$\lambda(s, x) = \begin{cases} \exp[x + \tans][\exp(\tans)]^{-1} = \exp x & \text{for } s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \in \mathbb{R}, \\ 1 & \text{for } s \in \left[\frac{\pi}{2}, 2\right), x \in \mathbb{R} \end{cases}$$

is the cocycle for the dynamical system

$$\delta(s, x) = \begin{cases} \tan^{-1}[x + \tans] & \text{for } s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x \in \mathbb{R}, \\ s & \text{for } s \in \left[\frac{\pi}{2}, 2\right), x \in \mathbb{R}. \end{cases}$$

This cocycle is not continuous (e.g., at the point $(\frac{\pi}{2}, 1)$) though the functions $\Lambda(u) = \exp u$ on $\mathbb{R}$ and $E(s, x) = 1$ on $[\frac{\pi}{2}, 2) \times \mathbb{R}$ are continuous [see (10)].

Problem. For which functions Λ and E in (10) is the cocycle continuous if $S_0 = \emptyset$?

Remark. I announced the above results for $A=B=\mathbb{R}$ in [9] (for continuous cocycles see [1, 5] too).

4. Stability

Let the output dictionary B in the machine be a metric space with metric d .

Definition 1. (*b-stability*) Let δ be the fixed input function in the machine $(S, A, B, \delta, \lambda)$. Eq. (2) for the output function is said to be *b-stable* if for every function $\Lambda : S \times A \rightarrow B$ for which the function $d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)]$ is bounded there exists a solution λ of (2) such that the function $d[\Lambda(s, x), \lambda(s, x)]$ is bounded.

Definition 2. (*stability*) Let δ be the fixed input function in the machine $(S, A, B, \delta, \lambda)$. Eq. (2) for the output function is said to be *stable* if for every $\varepsilon > 0$ there exists $\beta > 0$ such that for every function $\Lambda : S \times A \rightarrow B$ for which

$$d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)] \leq \beta \text{ for } (s, x, y) \in S \times A \times A \quad (11)$$

there exists a solution λ of (2) such that

$$d[\Lambda(s, x), \lambda(s, x)] \leq \varepsilon \text{ for } (s, x) \in S \times A. \quad (12)$$

Definition 3. (*inverse stability*) Let δ be the fixed input function in the machine $(A, B, S, \delta, \lambda)$. Eq. (2) for the output function is said to be *inversely stable* if for every $\beta > 0$ there exists $\varepsilon > 0$ such that for every function $\Lambda : S \times A \rightarrow B$ for which the condition (12) is true for a solution λ of (2) we have (11).

Remark. These definitions are not equivalent. Indeed, e.g., let $S = \{0\}$ and let A be free group generated by two elements, $B = \mathbb{Z}$ with the usual addition and usual metric and $\delta(0, x) = 0$ for $x \in A$. Eq. (2) has in this case the form $\lambda(0, x + y) = \lambda(0, x) + \lambda(0, y)$, thus this equation is the equation of a homomorphism from A to B . Such an equation is not *b-stable* [2], thus Eq. (2) is not *b-stable* either. This equation is *stable* since it is sufficient to put $\beta < 1$ for every ε .

Example 5. Eq. (2) is *stable* with $\delta = \frac{\varepsilon}{7}$ if A is the interval $(0, +\infty)$ with the usual addition, B is a Banach space, S is a closed interval in $\mathbb{R}$ and $\delta: S \times A \rightarrow A$ is a continuous solution of (1) [1].

We have here the following.

Theorem 2. Let S be an arbitrary set, $(A, +)$ a groupoid, $(B, +)$ an abelian semigroup divisible by $n > 1$ and with the metric d such that $n^k d(u, v) = d(n^k u, n^k v)$ for $k \in \mathbb{Z}, u, v \in B$. Let δ be an arbitrary function from $S \times A$ to S . If Eq. (2) is *stable*, then it is *b-stable*.

Proof. Assume that Eq. (2) is *stable*, thus for $\varepsilon = 1$ there exists $\beta > 0$ such that for every function $\Lambda: S \times A \rightarrow B$ if $d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)] \leq \beta$, then there exists a solution λ of (2) for which $d[\Lambda(s, x), \lambda(s, x)] \leq 1$. Let $l \in \mathbb{Z}$ be such that $n^l \leq \beta$, thus if

$$d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)] \leq n^l,$$

then $d[\Lambda(s, x), \lambda(s, x)] \leq 1$, too. Suppose that for the function $\Lambda: S \times A \rightarrow B$ the function $d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)]$ is bounded by n^k for $k \in \mathbb{Z}$. From here

$$d[n^{1-k}\Lambda(s, x + y), n^{1-k}\Lambda(s, x) + n^{1-k}\Lambda(\delta(s, x), y)] \leq n^1,$$

thus there exists a solution λ of (2) such that $d[n^{1-k}\Lambda(s, x), \lambda(s, x)] \leq 1$. We have thus $d[\Lambda(s, x), n^{k-1}\lambda(s, x)] \leq n^{k-1}$ and since the function $n^{k-1}\lambda(s, x)$ is a solution of (2) the proof is finished. $\square$

Remark. This stability depends on the function δ . Indeed, let N be a non-complete normed space and let $S=A=B=N$. Assume that there exists s_0 such that $\delta(s_0, x) = x$ for $x \in A$, e.g., $\delta(s, x) = s+x$, and that for the function $\Lambda: N \times N \rightarrow N$ we have $|\Lambda(s, x + y) - \Lambda(s, x) - \Lambda(s + x, y)| \leq \varepsilon$ for $\varepsilon > 0$. From here for $s = s_0$ we obtain $|\Lambda(s_0, x + y) - \Lambda(s_0, x) - \Lambda(x, y)| \leq \varepsilon$. Since the function $\lambda(s, y) = \Lambda(s_0, s+y) + \Lambda(s_0, s)$ is the solution of (2), Eq. (2) is b-stable and stable with $\beta = \varepsilon$. Assume at present there exists s_1 such that $\delta(s_1, x) = s_1$ for $x \in A$ and suppose that Eq. (2): $\lambda(s_1, x + y) = \lambda(s_1, x) + \lambda(s_1, y)$ is b-stable. This equation is the equation of homomorphism from A to B and it is b-stable too. Since the space B is not complete we have a contradiction with the following theorem [3]:

- Let A be an abelian group with an element of infinite order. Assume that the homomorphism equation from A to a normed space B is b-stable. Then B is complete.

The last example is good for stability too since the above theorem is true for stability in place of b-stability on the basis of Theorem 2.

Problem. Characterize the input function for which the output function is stable.

We have for the inverse stability the following.

Theorem 3. *Let S be an arbitrary set, $(A, +)$ a groupoid, $(B, +)$ a groupoid with the metric d such that $d(u, v) = d(u+w, v+w) = d(w+u, w+v)$ for $u, v, w \in B$. Let δ be an arbitrary function from $S \times A$ to S . In this case Eq. (2) is inversely stable.*

Proof. We have

$$\begin{aligned} d[\Lambda(s, x + y), \Lambda(s, x) + \Lambda(\delta(s, x), y)] &\leq d[\Lambda(s, x + y), \lambda(s, x + y)] \\ &\quad + d[\lambda(s, x + y), \lambda(s, x) + \lambda(\delta(s, x)y)] \\ &\quad + d[\lambda(s, x) + \lambda(\delta(s, x), y), \Lambda(s, x) + \lambda(\delta(s, x), y)] \\ &\quad + d[\Lambda(s, x) + \lambda(\delta(s, x), y), \Lambda(s, x) + \Lambda(\delta(s, x), y)] \\ &= d[\Lambda(s, x + y), \lambda(s, x + y)] \\ &\quad + d[\lambda(s, x + y), \lambda(s, x) + \lambda(\delta(s, x)y)] \\ &\quad + d[\lambda(s, x), \Lambda(s, x)] + d[\lambda(\delta(s, x)y), \Lambda(\delta(s, x), y)]. \end{aligned}$$

Let $\varepsilon = \frac{\beta}{3}$ for $\beta > 0$. If for a function Λ there exists a solution λ of (2) such that (12) is true, then (11) is true, too. $\square$

Remark. It is possible to consider other stabilities (see [10]).

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