Functions generating (m, M, Ψ) -Schur-convex sums

SILVESTRU SEVER DRAGOMIR AND KAZIMIERZ NIKODEM

Dedicated to Professor Karol Baron on his 70th birthday.

Abstract. The notion of (m, M, Ψ) -Schur-convexity is introduced and functions generating (m, M, Ψ) -Schur-convex sums are investigated. An extension of the Hardy–Littlewood–Pólya majorization theorem is obtained. A counterpart of the result of Ng stating that a function generates (m, M, Ψ) -Schur-convex sums if and only if it is (m, M, ψ) -Wright-convex is proved and a characterization of (m, M, ψ) -Wright-convex functions is given.

Mathematics Subject Classification. Primary 26A51; Secondary 39B62.

Keywords. Strongly convex functions, (m, M, ψ) -convex (Jensen-convex, Wright-convex) functions, (m, M, Ψ) -Schur-convexity.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. Assume that D is a convex subset of X and c is a positive constant. A function $f : D \rightarrow \mathbb{R}$ is called:

- *strongly convex with modulus c* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2 \quad (1)$$

for all $x, y \in D$ and $t \in [0, 1]$;

- *strongly Wright-convex with modulus c* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)\|x - y\|^2 \quad (2)$$

for all $x, y \in D$ and $t \in [0, 1]$;

- *strongly Jensen-convex with modulus c* if (1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2, \quad x, y \in D. \quad (3)$$

The usual concepts of convexity, Wright-convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions was introduced by Polyak [22] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [10, 15, 19, 22–24, 27]). Let us also mention the paper [18] by the second author which is a survey article devoted to strongly convex functions and related classes of functions.

In [1] the first author introduced the following concepts of (m, ψ) -lower convex, (M, ψ) -upper convex and (m, M, ψ) -convex functions (see also [2–4]): Assume that D is a convex subset of a real linear space X , $\psi : D \rightarrow \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ is called (m, ψ) -lower convex ((M, ψ) -upper convex) if the function $f - m\psi$ (the function $M\psi - f$) is convex. We say that $f : D \rightarrow \mathbb{R}$ is (m, M, ψ) -convex if it is (m, ψ) -lower convex and (M, ψ) -upper convex. Denote the above classes of functions by:

$$\begin{aligned}\mathcal{L}(D, m, \psi) &= \{f : D \rightarrow \mathbb{R} \mid f - m\psi \text{ is convex}\}, \\ \mathcal{U}(D, M, \psi) &= \{f : D \rightarrow \mathbb{R} \mid M\psi - f \text{ is convex}\}, \\ \mathcal{B}(D, m, M, \psi) &= \mathcal{L}(D, m, \psi) \cap \mathcal{U}(D, M, \psi).\end{aligned}$$

Let us observe that if $f \in \mathcal{B}(D, m, M, \psi)$ then $f - m\psi$ and $M\psi - f$ are convex and then $(M - m)\psi$ is also convex, implying that $M \geq m$ whenever ψ is not trivial (i.e. is not the zero function).

If $m > 0$ and $(X, \|\cdot\|)$ is an inner product space (that is, the norm $\|\cdot\|$ in X is induced by an inner product: $\|x\| = \sqrt{\langle x, x \rangle}$) the notions of $(m, \|\cdot\|^2)$ -lower convexity and strong convexity with modulus m coincide. Namely, in this case the following characterization was proved in [19]: A function f is strongly convex with modulus c if and only if $f - c\|\cdot\|^2$ is convex (for $X = \mathbb{R}^n$ this result can be also found in [8, Prop. 1.1.2]). However, if $(X, \|\cdot\|)$ is not an inner product space, then the two notions are different. There are functions $f \in \mathcal{L}(D, m, \|\cdot\|^2)$ which are not strongly convex with modulus m , as well as there are functions strongly convex with modulus m which do not belong to $\mathcal{L}(D, m, \|\cdot\|^2)$ (see the examples given in [6]).

If $M > 0$ and $f \in \mathcal{U}(D, M, \psi)$, then f is a difference of two convex functions. Such functions are called *d.c. convex* or δ -convex and play an important role in convex analysis (cf. e.g. [26] and the reference therein). Functions from the class $\mathcal{U}(D, M, \|\cdot\|^2)$ with $M > 0$ were also investigated in [13] under the name *approximately concave functions*.

In [5] Dragomir and Ionescu introduced the concept of g -convex dominated functions, where g is a given convex function. Namely, a function f is called *g -convex dominated*, if the functions $g + f$ and $g - f$ are convex. Note that this concept can be obtained as a particular case of (m, M, ψ) -convexity by choosing $m = -1$, $M = 1$ and $\psi = g$. Observe also (cf. [1]), that in the case

where $I \subset \mathbb{R}$ is an open interval and $f, \psi : I \rightarrow \mathbb{R}$ are twice differentiable, $f \in \mathcal{B}(I, m, M, \psi)$ if and only if

$$m\psi''(t) \leq f''(t) \leq M\psi''(t), \quad \text{for all } t \in I.$$

In particular, if $I \subset (0, \infty)$, $f : I \rightarrow \mathbb{R}$ is twice differentiable and $\psi(t) = -\ln t$, then $f \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$m \leq t^2 f''(t) \leq M, \quad \text{for all } t \in I, \quad (4)$$

which is a convenient condition to verify in applications.

Let $I \subset \mathbb{R}$ be an interval and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in I^n$, where $n \geq 2$. Following I. Schur (cf. e.g. [12, 25]) we say that x is *majorized by* y , and write $x \preceq y$, if there exists a doubly stochastic $n \times n$ matrix P (i.e. a matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x = y \cdot P$. A function $F : I^n \rightarrow \mathbb{R}$ is said to be *Schur-convex* if $F(x) \leq F(y)$ whenever $x \preceq y$, $x, y \in I^n$.

It is known, by the classical works of Schur [25], Hardy et al. [7] and Karata [9] that if a function $f : I \rightarrow \mathbb{R}$ is convex then it *generates Schur-convex sums*, that is the function $F : I^n \rightarrow \mathbb{R}$ defined by

$$F(x) = F(x_1, \dots, x_n) = f(x_1) + \dots + f(x_n)$$

is Schur-convex. It is also known that the convexity of f is a sufficient but not necessary condition under which F is Schur-convex. A full characterization of functions generating Schur-convex sums was given by Ng [16]. Namely, he proved that a function $f : I \rightarrow \mathbb{R}$ generates Schur-convex sums if and only if it is Wright-convex (cf. also [17]). Recently Nikodem et al. [20] obtained similar results in connection with strong convexity in inner product spaces. Let us also mention the paper by Olbryś [21] in which delta Schur-convex mappings are investigated.

The aim of this paper is to present some generalizations and counterparts of the above mentioned results related to (m, ψ) -lower convexity, (M, ψ) -upper convexity and (m, M, ψ) -convexity. We introduce the notion of (m, M, Ψ) -Schur-convex functions and give a sufficient and necessary condition for a function f to generate (m, M, Ψ) -Schur-convex sums. As a corollary we obtain a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem. Finally we introduce the concept of (m, M, ψ) -Wright-convex functions, prove a representation theorem for them and present an Ng-type characterization of functions generating (m, M, Ψ) -Schur-convex sums. It is worth underlining, that our results concern a few different classes of functions related to convexity and are formulated in vector spaces, that is in a much more general setting than the original ones.

2. Main results

Let X be a real vector space. Similarly as in the classical case we define majorization in the product space X^n . Namely, given two n -tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X^n$ we say that x is majorized by y , written $x \preceq y$, if

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix P .

In what follows we will assume that D is a convex subset of a real vector space X , $\psi : D \rightarrow \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. For any $n \geq 2$ define $\Psi_n : D^n \rightarrow \mathbb{R}$ by

$$\Psi_n(x_1, \dots, x_n) = \psi(x_1) + \dots + \psi(x_n), \quad x_1, \dots, x_n \in D. \quad (5)$$

We say that a function $F : D^n \rightarrow \mathbb{R}$ is (m, M, Ψ_n) -Schur-convex if for all $x, y \in D^n$

$$x \preceq y \implies F(x) \leq F(y) - m(\Psi_n(y) - \Psi_n(x)) \quad (6)$$

and

$$x \preceq y \implies F(x) \geq F(y) - M(\Psi_n(y) - \Psi_n(x)). \quad (7)$$

If only condition (6) [condition (7)] is satisfied, we say that F is (m, Ψ_n) -lower $((M, \Psi_n)$ -upper) Schur-convex.

Note that if $x \preceq y$ then $\Psi_n(x) \leq \Psi_n(y)$. It follows from the fact that the function ψ is convex and so it generates Schur-convex sums Ψ_n .

Given a function $f : D \rightarrow \mathbb{R}$ and an integer $n \geq 2$ we define the function $F_n : D^n \rightarrow \mathbb{R}$ by

$$F_n(x_1, \dots, x_n) = f(x_1) + \dots + f(x_n), \quad x_1, \dots, x_n \in D. \quad (8)$$

Now, let D be a convex subset of a real vector space X , and let $m, M \in \mathbb{R}$. Assume that $\psi : D \rightarrow \mathbb{R}$ is a convex function and $\Psi_n : D^n \rightarrow \mathbb{R}$ is defined by (5). We will prove now that (m, M, ψ) -convex functions generate (m, M, Ψ_n) -Schur-convex sums.

- Theorem 1.** (i) If $f \in \mathcal{L}(D, m, \psi)$, then the function F_n defined by (8) is (m, Ψ_n) -lower Schur-convex;
(ii) If $f \in \mathcal{U}(D, M, \psi)$, then the function F_n defined by (8) is (M, Ψ_n) -upper Schur-convex;
(iii) If $f \in \mathcal{B}(D, m, M, \psi)$, then the function F_n defined by (8) is (m, M, Ψ_n) -Schur-convex.

Proof. To prove (i) fix $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in D^n with $x \preceq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^n t_{ij} y_i, \quad j = 1, \dots, n.$$

Since $f \in \mathcal{L}(D, m, \psi)$, the function $g = f - m\psi$ is convex and hence

$$\begin{aligned} g(x_1) + \cdots + g(x_n) &= \sum_{j=1}^n g\left(\sum_{i=1}^n t_{ij}y_i\right) \leq \sum_{j=1}^n \sum_{i=1}^n t_{ij}g(y_i) \\ &= \sum_{i=1}^n \sum_{j=1}^n t_{ij}g(y_i) = \sum_{i=1}^n g(y_i) \sum_{j=1}^n t_{ij} = g(y_1) + \cdots + g(y_n). \end{aligned}$$

Consequently,

$$\begin{aligned} F_n(x) &= f(x_1) + \cdots + f(x_n) \\ &= g(x_1) + \cdots + g(x_n) + m(\psi(x_1) + \cdots + \psi(x_n)) \\ &\leq g(y_1) + \cdots + g(y_n) + m(\psi(x_1) + \cdots + \psi(x_n)) \\ &= f(y_1) + \cdots + f(y_n) - m(\psi(y_1) + \cdots + \psi(y_n)) \\ &\quad + m(\psi(x_1) + \cdots + \psi(x_n)) \\ &= F_n(y) - m(\Psi_n(y) - \Psi_n(x)). \end{aligned}$$

This shows that F_n satisfies (6), i.e. it is (m, Ψ_n) -lower Schur-convex.

The proof of part (ii) is similar. Since $f \in \mathcal{U}(D, M, \psi)$, the function $h = M\psi - f$ is convex. Hence, for x and y as previously, we have

$$\begin{aligned} F_n(x) &= f(x_1) + \cdots + f(x_n) \\ &= +M(\psi(x_1) + \cdots + \psi(x_n)) - h(x_1) - \cdots - h(x_n) \\ &\geq M(\psi(x_1) + \cdots + \psi(x_n)) - h(y_1) - \cdots - h(y_n) \\ &= M(\psi(x_1) + \cdots + \psi(x_n)) - M(\psi(y_1) + \cdots + \psi(y_n)) \\ &\quad + f(y_1) + \cdots + f(y_n) \\ &= F_n(y) - M(\Psi_n(y) - \Psi_n(x)). \end{aligned}$$

Part (iii) follows from (i) and (ii). $\square$

As an immediate consequence of the above theorem, we obtain the following counterpart of the classical Hardy–Littlewood–Pólya majorization theorem [7].

Corollary 2. *Let $I \subset \mathbb{R}$ be an interval and $n \geq 2$. Assume that $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in I^n$ satisfy:*

- (a) $x_1 \leq \cdots \leq x_n$, $y_1 \leq \cdots \leq y_n$;
- (b) $y_1 + \cdots + y_k \leq x_1 + \cdots + x_k$, $k = 1, \dots, n-1$;
- (c) $y_1 + \cdots + y_n = x_1 + \cdots + x_n$.

Assume also that $f, \psi : I \rightarrow \mathbb{R}$ and ψ is convex.

- (i) *If $f \in \mathcal{L}(D, m, \psi)$, then*

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - m(\Psi_n(y) - \Psi_n(x));$$

(ii) If $f \in \mathcal{U}(D, M, \psi)$, then

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n) - M(\Psi_n(y) - \Psi_n(x));$$

(iii) If $f \in \mathcal{B}(D, m, M, \psi)$, then

$$\begin{aligned} f(y_1) + \cdots + f(y_n) - M(\Psi_n(y) - \Psi_n(x)) &\leq f(x_1) + \cdots + f(x_n) \\ &\leq f(y_1) + \cdots + f(y_n) - m(\Psi_n(y) - \Psi_n(x)). \end{aligned}$$

Proof. Note that assumptions (a)–(c) imply $x \preceq y$ (see e.g. [12]) and apply Theorem 1. $\square$

Remark 3. Specifying the functions ψ and f in Corollary 2 above, one can get various analytic inequalities. For example, if $I \subset (0, \infty)$ and $f \in \mathcal{B}(I, m, M, -\ln)$, then for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$ satisfying conditions (a)–(c), we get

$$m \ln \prod_{i=1}^n \left(\frac{x_i}{y_i} \right) \leq \sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i) \leq M \ln \prod_{i=1}^n \left(\frac{x_i}{y_i} \right),$$

or, equivalently,

$$\prod_{i=1}^n \left(\frac{x_i}{y_i} \right)^m \leq \frac{\exp \left[\sum_{i=1}^n f(y_i) \right]}{\exp \left[\sum_{i=1}^n f(x_i) \right]} \leq \prod_{i=1}^n \left(\frac{x_i}{y_i} \right)^M. \quad (9)$$

If we take, for instance, $I = [k, K] \subset (0, \infty)$ and $f(t) = \frac{1}{p(p-1)} t^p$, with $p > 0$, $p \neq 1$, then $t^2 f''(t) = t^p \in [k^p, K^p]$, which means [cf. (4)] that $f \in \mathcal{B}(I, k^p, K^p, -\ln)$. Therefore, by (9), we then have

$$\prod_{i=1}^n \left(\frac{x_i}{y_i} \right)^{p(p-1)k^p} \leq \frac{\exp \left(\sum_{i=1}^n y_i^p \right)}{\exp \left(\sum_{i=1}^n x_i^p \right)} \leq \prod_{i=1}^n \left(\frac{x_i}{y_i} \right)^{p(p-1)K^p}.$$

One can give other examples by choosing $f(t) = t^q$ with $q < 0$, $f(t) = t \ln t$, etc.

We say that a function $f : D \rightarrow \mathbb{R}$ is (m, ψ) -lower Jensen-convex ((M, ψ) -upper Jensen-convex) if the function $f - m\psi$ (the function $M\psi - f$) is Jensen-convex, i.e. satisfies (3) with $c = 0$. We say that $f : D \rightarrow \mathbb{R}$ is (m, M, ψ) -Jensen-convex if it is (m, ψ) -lower Jensen-convex and (M, ψ) -upper Jensen-convex.

In the next theorem we show that functions generating (m, M, Ψ_n) -Schur-convex sums must be (m, M, ψ) -Jensen-convex.

Theorem 4. Let $f : D \rightarrow \mathbb{R}$.

- (i) If for some $n \geq 2$ the function F_n defined by (8) is (m, Ψ_n) -lower Schur-convex, then f is (m, ψ) -lower Jensen-convex;
- (ii) If for some $n \geq 2$ the function F_n defined by (8) is (M, Ψ_n) -upper Schur-convex, then f is (M, ψ) -upper Jensen-convex;
- (iii) If for some $n \geq 2$ the function F_n defined by (8) is (m, M, Ψ_n) -Schur-convex, then f is (m, M, ψ) -Jensen-convex.

Proof. To prove (i) take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \dots, y_2), \quad x = (x_1, x_2, y_2, \dots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2)$, $x = (x_1, x_2)$). One can check easily that $x \preceq y$. Therefore, by (6),

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is

$$2f\left(\frac{y_1 + y_2}{2}\right) \leq f(y_1) + f(y_2) - m\left(\psi(y_1) + \psi(y_2) - 2\psi\left(\frac{y_1 + y_2}{2}\right)\right).$$

Hence, for $g = f - m\psi$ we have

$$\begin{aligned} 2g\left(\frac{y_1 + y_2}{2}\right) &= 2f\left(\frac{y_1 + y_2}{2}\right) - 2m\psi\left(\frac{y_1 + y_2}{2}\right) \\ &\leq f(y_1) + f(y_2) - m(\psi(y_1) + \psi(y_2)) = g(y_1) + g(y_2), \end{aligned}$$

which means that f is (m, ψ) -lower Jensen-convex.

The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). $\square$

We say that a function $f : D \rightarrow \mathbb{R}$ is (m, ψ) -lower Wright-convex ((M, ψ) -upper Wright-convex) if the function $f - m\psi$ (the function $M\psi - f$) is Wright-convex, i.e. satisfies (2) with $c = 0$. We say that $f : D \rightarrow \mathbb{R}$ is (m, M, ψ) -Wright-convex if it is (m, ψ) -lower Wright-convex and (M, ψ) -upper Wright-convex.

As was shown above in Theorems 1 and 2, if a function $f : D \rightarrow \mathbb{R}$ is (m, M, ψ) -convex, then for every $n \geq 2$ the corresponding function F_n defined by (8) is (m, M, Ψ_n) -Schur-convex and if for some $n \geq 2$ the function F_n is (m, M, Ψ_n) -Schur-convex, then f is (m, M, ψ) -Jensen-convex. The next theorem characterizes all the functions f for which F_n are (m, M, Ψ_n) -Schur-convex. It is a counterpart of the result of Ng [16] on functions generating Schur-convex sums.

Recall also that a subset D of a vector space X is said to be *algebraically open* if for every $x \in D$ and for every $y \in X$ there exists $\varepsilon > 0$ such that

$$\{ty + (1 - t)x \mid t \in (-\varepsilon, \varepsilon)\} \subset D.$$

Theorem 5. Let $f : D \rightarrow \mathbb{R}$, where D is an algebraically open convex subset of a vector space X . Then:

- (i) If f is (m, ψ) -lower Wright-convex, then for every $n \geq 2$ the function F_n defined by (8) is (m, Ψ_n) -lower Schur-convex. Conversely, if for some $n \geq 2$ the function F_n is (m, Ψ_n) -lower Schur-convex, then f is (m, ψ) -lower Wright-convex;
- (ii) If f is (M, ψ) -upper Wright-convex, then for every $n \geq 2$ the function F_n defined by (8) is (M, Ψ_n) -upper Schur-convex. Conversely, if for some $n \geq 2$ the function F_n is (M, Ψ_n) -upper Schur-convex, then f is (M, ψ) -upper Wright-convex;

- (iii) If f is (m, M, ψ) -Wright-convex, then for every $n \geq 2$ the function F_n defined by (8) is (m, M, Ψ_n) -Schur-convex. Conversely, if for some $n \geq 2$ the function F_n is (m, M, Ψ_n) -Schur-convex, then f is (m, M, ψ) -Wright-convex.

Proof. To prove (i) assume that f is (m, ψ) -lower Wright-convex and fix an $n \geq 2$. Since the function $g = f - m\psi$ is Wright-convex, it is of the form $g = g_1 + a$, where g_1 is convex and a is additive (cf. [11]; here the assumption that D is algebraically open is needed). Therefore it generates Schur-convex sums. Thus, for $x = (x_1, \dots, x_n) \preceq y = (y_1, \dots, y_n)$, we have

$$g(x_1) + \dots + g(x_n) \leq g(y_1) + \dots + g(y_n).$$

Hence

$$\begin{aligned} f(x_1) + \dots + f(x_n) - m(\psi(x_1) + \dots + \psi(x_n)) \\ \leq g(y_1) + \dots + g(y_n) - m(\psi(y_1) + \dots + \psi(y_n)), \end{aligned}$$

which means that

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is F_n is (m, Ψ_n) -lower Schur-convex. Now, assume that for some $n \geq 2$ the function F_n is (m, Ψ_n) -lower Schur-convex. Take $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \dots, n$. Then $x = (x_1, \dots, x_n) \preceq y = (y_1, \dots, y_n)$. Therefore, by (6),

$$F_n(x) \leq F_n(y) - m(\Psi_n(y) - \Psi_n(x)),$$

that is

$$\begin{aligned} f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) \leq f(y_1) + f(y_2) - m(\psi(y_1) \\ + \psi(y_2) - \psi(x_1) - \psi(x_2)). \end{aligned}$$

Hence, for $g = f - m\psi$ we get

$$\begin{aligned} g(ty_1 + (1-t)y_2) + g((1-t)y_1 + ty_2) \\ = f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) - m\psi(ty_1 + (1-t)y_2) \\ - m\psi((1-t)y_1 + ty_2) \\ \leq f(y_1) + f(y_2) - m\psi(y_1) - m\psi(y_2) = g(y_1) + g(y_2). \end{aligned}$$

Thus g is Wright-convex, which means that f is (m, ψ) -lower Wright-convex. The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). $\square$

Remark 6. In the special case where $(X, \|\cdot\|)$ is an inner product space, $\psi = \|\cdot\|^2$ and $m = c > 0$, parts (i) of the above Theorems 1, 4, 5 reduce to the results obtained in [20] for strong Schur-convexity. For $m = 0$ and $X = \mathbb{R}^n$ they coincide with the Ng theorem [16].

Finally, we give a representation theorem for (m, M, ψ) -Wright-convex functions. It is known (and easy to check) that every convex function is Wright-convex, and every Wright-convex function is Jensen-convex, but not the converse (some examples can be found in [18]). In [16] Ng proved that a function f defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = f_1 + a$, where f_1 is a convex function and a is an additive function (see also [18]). Kominek [11] extended that result to functions defined on algebraically open subset of a vector space. An analogous result for strongly Wright-convex functions was obtained in [14]. In the next theorem we give a similar representation for (m, M, ψ) -Wright-convex functions. In the proof we will use the following fact:

Lemma 7. *Assume that $f, g : D \rightarrow \mathbb{R}$ are convex functions, $a : X \rightarrow \mathbb{R}$ is additive and $a(x) = f(x) - g(x)$ for all $x \in D$. Then a is an affine function on D .*

Proof. Fix $x, y \in D$ and consider the function $\varphi : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\varphi(s) = a(sx + (1-s)y) = f(sx + (1-s)y) - g(sx + (1-s)y), \quad s \in [0, 1].$$

As a difference of convex functions on $[0, 1]$, φ is continuous on $(0, 1)$. Fix any $t \in (0, 1)$ and take a sequence (q_n) of rational numbers in $(0, 1)$ tending to t . By the additivity of a we have

$$a(q_n x + (1 - q_n)y) = q_n a(x) + (1 - q_n)a(y),$$

whence

$$\varphi(q_n) = q_n a(x) + (1 - q_n)a(y).$$

Going to the limit we get

$$\varphi(t) = ta(x) + (1 - t)a(y).$$

Hence

$$a(tx + (1 - t)y) = ta(x) + (1 - t)a(y),$$

which proves that a is affine on D . □

Theorem 8. *Let $f : D \rightarrow \mathbb{R}$, where D is an algebraically open convex subset of a vector space X . Then:*

- (i) *f is (m, ψ) -lower Wright-convex if and only if $f = g_1 + a_1$, where $g_1 \in \mathcal{L}(D, m, \psi)$ and $a_1 : X \rightarrow \mathbb{R}$ is additive;*
- (ii) *f is (M, ψ) -upper Wright-convex if and only if $f = g_2 + a_2$, where $g_2 \in \mathcal{U}(D, M, \psi)$ and $a_2 : X \rightarrow \mathbb{R}$ is additive;*
- (iii) *f is (m, M, ψ) -Wright-convex if and only if $f = g + a$, where $g \in \mathcal{B}(D, m, M, \psi)$ and $a : X \rightarrow \mathbb{R}$ is additive.*

Proof. To prove (i) assume first that f is (m, ψ) -lower Wright-convex, that is $h = f - m\psi$ is Wright-convex. By the Ng representation theorem [16] (extended by Kominek [11] to functions defined on algebraically open domains), there exist a convex function $h_1 : D \rightarrow \mathbb{R}$ and an additive function $a_1 : X \rightarrow \mathbb{R}$ such that $h = h_1 + a_1$ on D . Then $g_1 = h_1 + m\psi$ belongs to $\mathcal{L}(D, m, \psi)$ and

$$f = h + m\psi = h_1 + a_1 + m\psi = g_1 + a_1,$$

which was to be proved. Conversely, if $f = g_1 + a_1$ with some $g_1 \in \mathcal{L}(D, m, \psi)$ and a_1 additive, then $f - m\psi = g_1 - m\psi + a_1$ is Wright-convex as a sum of a convex function and an additive function. This shows that f is (m, ψ) -lower Wright-convex.

The proof of part (ii) is analogous.

Part (iii). If $f = g + a$, where $g \in \mathcal{B}(D, m, M, \psi)$ and $a : X \rightarrow \mathbb{R}$ is additive, then, by (i) and (ii) f is (m, ψ) -lower Wright-convex and (M, ψ) -upper Wright-convex. Consequently, it is (m, M, ψ) -Wright-convex.

The proof in the opposite direction is more delicate. If f is (m, M, ψ) -Wright-convex, then $f - m\psi$ and $M\psi - f$ are Wright-convex. Then

$$f - m\psi = h_1 + a_1 \quad \text{and} \quad M\psi - f = h_2 + a_2$$

with some convex functions h_1, h_2 and additive functions a_1, a_2 . Hence

$$a_1 + a_2 = (M - m)\psi - (h_1 + h_2)$$

which, by Lemma 5, implies that $A = a_1 + a_2$ is affine. Denote $a = a_1$ and $g = f - a$. Then

$$g - m\psi = f - a - m\psi = h_1,$$

which implies that $g \in \mathcal{L}(D, m, \psi)$ because h_1 is convex. Also

$$M\psi - g = M\psi - f + a = h_2 + a_2 + a = h_2 + A,$$

which implies that $g \in \mathcal{U}(D, m, \psi)$ because $h_2 + A$ is convex. Thus $g \in \mathcal{B}(D, m, \psi)$ and $f = g + a$, which finishes the proof. $\square$

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Dragomir, S.S.: On the reverse of Jensen's inequality for isotonic linear functionals. J. Inequal. Pure Appl. Math. **2**(3), 1–13 (2001). (Art. 36)
- [2] Dragomir, S.S.: Some inequalities for (m, M) -convex mappings and applications for Csiszár Φ -divergence in information theory. Math. J. Ibaraki Univ. **33**, 35–50 (2001)

- [3] Dragomir, S.S.: On the Jensen's inequality for isotonic linear functionals. *Nonlinear Anal. Forum* **7**(2), 139–151 (2002)
- [4] Dragomir, S.S.: A survey on Jensen's type inequalities for positive functionals. In: Pardalos, P.M., et al. (eds.) *Nonlinear Analysis. Springer Optimization and Its Applications* 68, pp. 177–232. Springer, New York (2012)
- [5] Dragomir, S.S., Ionescu, N.M.: On some inequalities for convex-dominated functions. *L'Anal. Num. Théor. L'Approx.* **19**(1), 21–27 (1990)
- [6] Dragomir, S.S., Nikodem, K.: Jensen's and Hermite–Hadamard's inequalities for lower and strongly convex functions on normed spaces. *Bull. Iranian Math. Soc.* (**in press**)
- [7] Hardy, G.H., Littlewood, J.E., Pólya, G.: *Inequalities*. Cambridge University Press, Cambridge (1952)
- [8] Hiriart-Urruty, J.B., Lemaréchal, C.: *Fundamentals of Convex Analysis*. Springer, Berlin (2001)
- [9] Karamata, J.: Sur une inégalité relative aux fonctions convexes. *Publ. Math. Univ. Belgrade* **1**, 145–148 (1932)
- [10] Klaričić, Bakula M., Nikodem, K.: On the converse Jensen inequality for strongly convex functions. *J. Math. Anal. Appl.* **434**, 516–522 (2016)
- [11] Kominek, Z.: On additive and convex functionals. *Radovi Mat.* **3**, 267–279 (1987)
- [12] Marshall, A.W., Olkin, I.: *Inequalities: Theory of Majorization and Its Applications*, Mathematics in Science and Engineering 143. Academic Press Inc., New York (1979)
- [13] Merentes, N., Nikodem, K.: Strong convexity and separation theorems. *Aequat. Math.* **90**, 47–55 (2016)
- [14] Merentes, N., Nikodem, K., Rivas, S.: Remarks on strongly Wright-convex functions. *Ann. Polon. Math.* **103**(3), 271–278 (2011)
- [15] Montrucchio, L.: Lipschitz continuous policy functions for strongly concave optimization problems. *J. Math. Econ.* **16**, 259–273 (1987)
- [16] Ng C. T.: Functions generating Schur-convex sums, In: Walter, W. (ed.) *General Inequalities 5* (Oberwolfach, 1986), *Internat. Ser. Numer. Math.* vol. 80, pp. 433–438. Birkhäuser Verlag, Basel–Boston (1987)
- [17] Nikodem, K.: On some class of midconvex functions. *Ann. Polon. Math.* **72**, 145–151 (1989)
- [18] Nikodem, K.: On strongly convex functions and related classes of functions. In: Rassias, T.M. (ed.) *Handbook of Functional Equations. Functional Inequalities. Springer Optimization and Its Application* 95, pp. 365–405. Springer, New York (2014).
- [19] Nikodem, K., Páles, Zs: Characterizations of inner product spaces by strongly convex functions. *Banach J. Math. Anal.* **5**(1), 83–87 (2011)
- [20] Nikodem, K., Rajba, T., Wąsowicz, Sz: Functions generating strongly Schur-convex sums. In: Bandle, C., et al. (eds.) *Inequalities and Applications 2010*, *International Series of Numerical Mathematics* 161, pp. 175–182. Springer, New York (2012)
- [21] Olbryś, A.: On delta Schur-convex mappings. *Publ. Math. Debrecen* **86**, 313–323 (2015)
- [22] Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. *Sov. Math. Dokl.* **7**, 72–75 (1966)
- [23] Rajba, T., Wąsowicz, Sz: Probabilistic characterization of strong convexity. *Opusc. Math.* **31**(1), 97–103 (2011)
- [24] Roberts, A.W., Varberg, D.E.: *Convex Functions*. Academic Press, New York (1973)
- [25] Schur, I.: Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie. *Sitzungsber. Berl. Math. Ges.* **22**, 9–20 (1923)
- [26] Veselý, L., Zajíček, L.: *Delta-convex mappings between Banach spaces and applications*. *Dissert. Math.* **289**, PWN, Warszawa (1989)
- [27] Vial, J.P.: Strong convexity of sets and functions. *J. Math. Econ.* **9**, 187–205 (1982)

Silvestru Sever Dragomir
Mathematics, College of Engineering and Science
Victoria University
P.O. Box 14428 Melbourne VIC 8001
Australia
e-mail: sever.dragomir@vu.edu.au

and

DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of
Computer Science and Applied Mathematics
University of the Witwatersrand (Wits)
Private Bag 3 Johannesburg 2050
South Africa

Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2
43-309 Bielsko-Biala
Poland
e-mail: knikodem@ath.bielsko.pl

Received: January 20, 2018

Revised: April 17, 2018